Additional exercises

The exercises below are similar in style/difficulty to those which will come up in the exam. It is advisable to review the entire lecture.

Exercise 1. On $\mathbb{R}^4$ with the standard Riemannian metric and kartesian coordinates (x, y, z, w) consider

$$\omega = x dx \wedge dy + z dz \wedge dw \in \Omega^1(\mathbb{R}^4)$$
$$X = \cos(w)\frac{\partial}{\partial x} + \sin(y)\frac{\partial}{\partial y}$$

and compute $d\omega, i_X\omega, L_X\omega, *\omega$ and $\omega \wedge *\omega$.

Exercise 2. On $\mathbb{R}^4$ with cartesian coordinates consider the 1-forms $\alpha = dz + x dy$ and $\beta = dx + w dy$ and the vector fields

$$W = \frac{\partial}{\partial w} \qquad\qquad X = \frac{\partial}{\partial y} - w\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}$$

Compute $\alpha \wedge \beta \wedge d\beta, \alpha \wedge \beta \wedge d\alpha$ and $[W, X], [X, [W, X]], [W, [W, X]]$.

Exercise 3. Let (M, g) be a semi-Riemannian manifold. Let λ be a smooth function on M and $h = e^{2f} g$ be the rescaled metric. Let ∇ respectively $\widehat{\nabla}$ be the Levi-Civita connection of (M, g) respectively (M, h). Show that

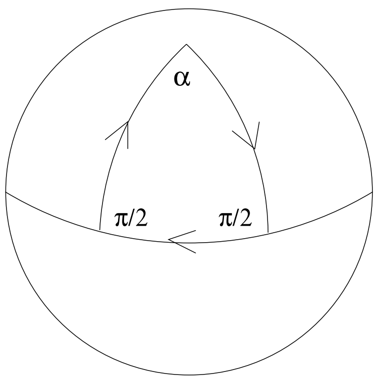
$$\widehat{\nabla}_X Y = \nabla_X Y + (L_X f)Y + (L_Y f)X - g(X, Y)\mathrm{grad}_g(f).$$

Here grad_g denotes the gradient vector field of f with respect to g.

Exercise 4. Let M be a smooth manifold an $U \subset M$ be an open subset of M. Show that there is a smooth atlas on U such that the inclusion $i : U \longrightarrow M$ is smooth.

Exercise 5. Let M, N be smooth manifolds. Define a smooth atlas on $M \times N$ such that the projection maps $M \times N \longrightarrow M$ and $M \times N \longrightarrow N$ are smooth.

Exercise 6. Consider $S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$ with standard Riemannian metric and the Levi-Civita connection. For $\alpha \in (0, 2\pi)$ consider a geodesic triangle with two angles equal to $\pi/2$ at the equator and angle α at the north pole.
Determine the parallel transport P_γ along γ.



Exercise 7. Let M be a smooth manifold and α a 1-form.

1. Let ∇ be a torsion free connection. Let $\widehat{\nabla}_X Y = \nabla_X Y + \alpha(X)Y$. Check that $\widehat{\nabla}$ is a connection and compute its torsion.

2. For which α does $\widehat{\nabla}$ have vanishing torsion?

Exercise 8. Let M be a smooth manifold. Starting from a smooth atlas on M define a smooth structure on T^*M. Show that T^*M is orientable.

Exercise 9. Let M be a smooth manifold, g a semi-Riemannian metric on M and ∇ a connection on TM. Assume that $\nabla, \widehat{\nabla}$ are both metric connections on TM with respect to g. Show that for all $X, Y \in \chi(M)$

$$g(\nabla_X Y - \widehat{\nabla}_X Y, Y) = 0.$$