LMU LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN

MATHEMATISCHES INSTITUT

Prof, Dr. Ivo Sachs, Prof. Dr. Thomas Vogel
Dr. Tomáš Procházka, Dr. Stephan Stadler

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Differentiable manifolds

Exam - Solution

What to do first:

- Please stow away your mobile phone in a place where it is not distracting you and turn it off. No cheat sheets, pocket calculators etc. are permitted.
- Check that your copy of this exam contains **6 exercises**.
- Do not use the colors red or green. Write with a pen and not with a pencil.
- Put your name on top of **every** sheet of paper you hand in. Make an extra effort so that we can decipher it.
- Different exercises are to be solved on different sheets of paper. Make sure we know what you are solving.
- Hand in only one solution per exercise. If you hand in several solutions for one question we will invariably grade the worst version and ignore all others. Cross out clearly whatever you want us to ignore.
- Fill out the following:

Name: ______________________ Surname: ______________________

Matrikelnr.: ______________________ Semester: ______________________

Subject: Bachelor Mathematics ❑ Master Mathematics ❑

Bachelor Physics ❑ Master Physics ❑

TMP ❑

Other :____

You have **120 Minutes**.

Good luck!

	1	2	3	4	5	6	$\sum$
	/6	/11	/6	/7	/10	/10	/50

Warning: Some of the following solutions of exercises contain much more detail than was expected from you in 120 minutes.

Name: ____________________

Exercise 1. [6 Points]

Show that the map

$$\begin{aligned} S : \mathbb{CP}^1 \times \mathbb{CP}^1 &\longrightarrow \mathbb{CP}^3 \\ ([w_0 : w_1], [z_0 : z_1]) &\longmapsto [w_0 z_0 : w_0 z_1 : w_1 z_0 : w_1 z_1] \end{aligned}$$

is well defined and smooth. It is enough verify smoothness on a neighbourhood of one point, for example $p = ([1:0],[1:0])$.

By definition of $\mathbb{CP}^1$ at least one of the numbers w_0, w_1, say w_0 is not zero. We may also assume that $z_0 \neq 0$. Then $w_0 z_0 \neq 0$ and $(w_0 z_0, w_0 z_1, w_1 z_0, w_1 z_1)$ represents an element of $\mathbb{CP}^3$.
$S([w_0 : w_1], [z_0 : z_1])$ does not depend on the choice of representative (w_0, w_1) of $[w_0 : w_1]$: If $(\lambda w_0, \lambda w_1)$ with $\lambda \neq 0$ is another representative, then

$$(\lambda w_0 z_0, \lambda w_0 z_1, \lambda w_1 z_0, \lambda w_1 z_1) = \lambda (w_0 z_0, w_0 z_1, w_1 z_0, w_1 z_1).$$

The left hand side and the right hand side represent the same element of $\mathbb{CP}^3$.
The same argument shows that the choice of a representative (z_0, z_1) of $[z_0 : z_1]$ does not matter.
We fix the chart for $\mathbb{CP}^1$

$$\begin{aligned} \varphi : \{[w_0 : w_1] \in \mathbb{CP}^1 \mid w_0 \neq 0\} &\longrightarrow \mathbb{C} \simeq \mathbb{R}^2 \\ [w_0 : w_1] &\longmapsto w_1 / w_0. \end{aligned}$$

and the corresponding product chart

$$\begin{aligned} \hat{\varphi} : \{([w_0 : w_1], [z_0 : z_1]) \in \mathbb{CP}^1 \times \mathbb{CP}^1 \mid w_0 \neq 0 \neq z_0\} &\longrightarrow \mathbb{C}^2 \simeq \mathbb{R}^2 \times \mathbb{R}^2 = \mathbb{R}^4 \\ ([w_0 : w_1], [z_0 : z_1]) &\longmapsto (\varphi([w_0 : w_1]), \varphi([z_0 : z_1])). \end{aligned}$$

This is surjective and the inverse map is

$$\hat{\varphi}^{-1}(\xi, \eta) = ([1 : \xi], [1 : \eta]).$$

The point p has the coordinates $\hat{\varphi}(p) = (0,0) \in \mathbb{C}^2$ (in terms of $\hat{\varphi}$. The image of p is $S(p) = [1:0:0:0]$, i.e. it is contained in the chart domain of

$$\begin{aligned} \psi : \{[a_0 : a_1 : a_2 : a_3] \in \mathbb{CP}^3 \mid a_0 \neq 0\} &\longrightarrow \mathbb{C}^3 \simeq \mathbb{R}^6 \\ [a_0 : a_1 : a_2 : a_3] &\longmapsto (a_1/a_0, a_2/a_0, a_3/a_0) \end{aligned}$$

Then near $(0,0) \in \mathbb{C}^2$

$$\begin{aligned} (\psi \circ S \circ \hat{\varphi}^{-1})(\xi, \eta) &= (\psi \circ S)([1:\xi],[1:\eta]) \\ &= \psi([1 : \eta : \xi : \xi\eta]) \\ &= (\eta, \xi, \xi\eta) \in \mathbb{C}^3. \end{aligned}$$

This is smooth (and holomorphic) near $(0,0)$. Thus S is smooth near p.
Remark: The map S is a Segre embedding which is an embedding

$$\mathbb{CP}^n \times \mathbb{CP}^m \longrightarrow \mathbb{CP}^{(n+1)(m+1)-1}.$$

The definition is analogous the the one above. If you know what a projective manifold/variety is, then this shows that the product of projective manifolds/varieties is again projective.

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Exercise 2. [11 Punkte]

We consider a smooth map $f : S^3 \longrightarrow S^2$ and a volume form ω on S^2. Recall that $H^1(S^3) = \{0\}$.

1. Formulate Stokes theorem for a general compact, orientable manifold with boundary.
2. Show (using methods from the lecture, for example using Hodge $*$) that $H^2(S^3) = \{0\}$.
3. Conclude that there is a 1-form λ on S^3 such that $f^*\omega = d\lambda$.
4. Let $\lambda, \widehat{\lambda}$ be two 1-forms on S^3 such that $d\lambda = f^*\omega = d\widehat{\lambda}$. Prove

$$\int_{S^3} \lambda \wedge d\lambda = \int_{S^3} \widehat{\lambda} \wedge d\widehat{\lambda}.$$

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1. Theorem (Stokes): Let M be an oriented, compact, smooth manifold of dimension n with boundary and $\omega \in \Omega^{n-1}(M)$ a smooth form. Then

$$\int_{\partial M} \omega = \int_M d\omega.$$

The boundary is oriented by the outward normal first convention.

2. We equip M with a Riemannian metric. It was explained in the lecture (January 10, 2017), that $*$ defines an isomorphism

$$* : \mathit{Harm}^k(M) \longrightarrow \mathit{Harm}^{n-k}(M)$$

between spaces of harmonic forms when M is closed, smooth and oriented for all k. By the Hodge decomposition theorem this corresponds to an isomorphism

$$\mathrm{PD} : H^k(M) \longrightarrow H^{n-k}(M).$$

In the case $M = S^3$ and $k = 1$ we obtain $H^2(S^3) = \{0\}$ from $H^1(S^3) = \{0\}$.

3. $f^*\omega$ is closed since $df^*\omega = F^*d\omega$ and every 2-form on a 2-manifold is closed. Since

$$H^2(S^3) = \frac{\ker(d : \Omega^2(S^3) \longrightarrow \Omega^3(S^3))}{\operatorname{im}(d : \Omega^1(S^3) \longrightarrow \Omega^2(S^3))} = \{0\}$$

it follows that $f^*\omega$ is exact, i.e. there is a 1-form λ on S^3 such that $d\lambda = F^*\omega$.

4. If $d\lambda = F^*\omega = d\widehat{\lambda}$, then the difference $\widehat{\lambda} - \lambda$ is a closed 1-form. Because of $H^1(S^3) = \{0\}$ it is exact, i.e. there is a smooth function g such that $dg = \widehat{\lambda} - \lambda$. Then by Stokes theorem and $\partial S^3 = \emptyset$

$$\begin{aligned}
\int_{S^3} \widehat{\lambda} \wedge d\widehat{\lambda} - \int_{S^3} \lambda \wedge d\lambda &= \int_S^3 (\widehat{\lambda} - \lambda) \wedge F^*\omega \\
&= \int_{S^3} dg \wedge F^*\omega \\
&= \int_{S^3} d(g \cdot F^*\omega) = 0.
\end{aligned}$$

Name: ____________________

Exercise 3. [6 Points]

1. Let M, N be closed, oriented, connected manifold of the same dimension n. Define the mapping degree of a smooth map $f : M \longrightarrow N$.

2. Consider the case $M = N = T^2 = S^1 \times S^1$. As oriented coordinates on T^2 we use pairs of numbers $\begin{pmatrix} \varphi \\ \psi \end{pmatrix} \in \mathbb{R}^2$. Two such pairs represent the same point in T^2 if and only if their difference lies in $2\pi\mathbb{Z}^2$. Finally, note $\int_{S^1} d\varphi = 2\pi = \int_{S^1} d\psi$.

   Compute the mapping degree of the map

$$\begin{aligned} f : T^2 &\longrightarrow T^2 \\ \begin{pmatrix} \varphi \\ \psi \end{pmatrix} &\longmapsto \begin{pmatrix} 3 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \varphi \\ \psi \end{pmatrix}. \end{aligned}$$

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1. According to the lecture $H^n(M) \simeq \mathbb{R}$ for closed, connected and oriented manifolds of dimension n. Let $\omega \in \Omega^n(N)$ represent a non-trivial cohomology class (i.e. $\int_N \omega \neq 0$). Then the degree of f is

$$\deg(f) = \frac{\int_M f^*\omega}{\int_N \omega}.$$

2. We pick $\omega = d\varphi \wedge d\psi$. Then $\int_T^2 \omega = \int_{S^1} d\varphi \int_{S^1} d\psi = 4\pi^2 \neq 0$. Moreover

$$f^*\omega = d(3\varphi - \psi) \wedge d\psi = 3d\varphi \wedge \psi.$$

   Hence

$$\deg(f) = \frac{\int_{T^2} f^*\omega}{\int_{T^2} \omega} = 3.$$

   Alternatively one can argue as follows: Since $T^2 \times S^1 \times S^1$ and a basis of $T_{(\varphi,\psi)}T^2$ is given by $\frac{\partial}{\partial\varphi}, \frac{\partial}{\partial\psi}$. With respect to this basis the differential of f is

$$Df_{(\varphi,\psi)} : T_{(\varphi,\psi)}T^2 = \mathbb{R} \oplus \mathbb{R} \longrightarrow T_{f(\varphi,\psi)}T^2 = \mathbb{R} \oplus \mathbb{R}$$

   is represented by the matrix $\begin{pmatrix} 3 & 1 \\ 0 & 1 \end{pmatrix}$. This matrix is non-degenerate and has positive determinant, i.e. every point is a regular value. Therefore Df is orientation preserving everywhere. Finally, $f^{-1}(0,0) = \{(0,0), (2\pi/3, 0), (4\pi/3, 0)\}$ has three elements. Hence $\deg(f) = 3$ by a theorem in the lecture.

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Exercise 4. [7 points]

Let M be a smooth manifold and ∇ a connection on TM.

1. Define the curvature tensor R of ∇.
2. Assume that M has dimension 1. Prove that $R \equiv 0$.
3. How does the Levi-Civita connection and the associated curvature tensor of a semi-Riemannian metric (M, g) change when g is replaced by $c \cdot g$ with $0 < c \in \mathbb{R}$ a constant. What happens to the scalar curvature?

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1. $R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$ where X, Y, Z are smooth vector fields on M.
2. R is a tensor, i.e. the value of $R(X,Y)Z$ at $p \in M$ depends only on the values of X, Y, Z at that point. It is antisymmetric in the first two variables $R(X,Y)Z = -R(Y,X)Z$. If M is one dimensional then for all $X, Y \in T_pM$ there is $\lambda \in \mathbb{R}$ such that $\lambda X(p) = Y(p)$. Then at p and by linearity and antisymmetry

$$R(X,Y)Z = R(X(p), Y(p))Z(p) = \lambda R(X(p), X(p))Z(p) = 0.$$

3. The Levi-Civita connection ∇ is uniquely determined by the requirements

   (i) ∇ is torsion free, i.e. $\nabla_X Y - \nabla_Y X - [X,Y] \equiv 0$ for all vector fields X, Y on M, and

   (ii) ∇ is metric, i.e. $L_X(g(Y,Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$.

   The first condition is verified when g is multiplied with a constant and the same is true for the second. Thus the Levi-Civita connection does not change when g is replaced by $c \cdot g$. By definition, the same is true for the associated curvature tensor.

   Finally, let e_i be an orthonormal basis of T_pM with respect to g. Then

$$\mathrm{scal}(p) = \sum_{i,j} g(R(e_i, e_j)e_j, e_i).$$

   An orthonormal basis with respect to $c \cdot g$ is $e_i/\sqrt{c}$. The scalar curvature $\widehat{\mathrm{scal}}$ of $c \cdot g$ is then

$$\begin{aligned}
\widehat{\mathrm{scal}}(p) &= \sum_{i,j} (c \cdot g) \left(R(e_i/\sqrt{c}, e_j/\sqrt{c}) e_j/\sqrt{c}, e_i/\sqrt{c} \right) \\
&= \frac{\mathrm{scal}(p)}{c}
\end{aligned}$$

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Exercise 5. [10 Points]

1. Let G be a group and a smooth manifold at the same time. When is G a Lie-group? What is the Lie-algebra $\mathfrak{g}$ of a Lie-group G?
2. Consider the group
$$G = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \;\middle|\; a > 0 \text{ and } b \in \mathbb{R} \right\}.$$
Define a smooth structure on G so that G is a Lie-group (with respect to matrix multiplication).
3. Choose a basis X, Y of $\mathfrak{g}$ and compute $[X, Y]$.

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1. G is a Lie-group when the multiplication $\mu : G \times G \longrightarrow G$ and the inversion $\mathrm{inv} : G \longrightarrow G$ are smooth maps. The Lie-algebra is the space of left invariant vector fields on G, the commutator of two left invariant vector fields is again left invariant. This determines the Lie-algebra structure on $\mathfrak{g}$.
2. G can be covered with one single chart. Define
$$\begin{aligned} \varphi : G &\longrightarrow \{(x, y) \in \mathbb{R}^2 \mid x > 0\} \\ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} &\longmapsto (a, b). \end{aligned}$$
We write $\varphi \times \varphi$ for the product chart on $G \times G$. Then
$$\begin{aligned} \varphi \circ \mu \circ (\varphi \times \varphi)^{-1}(x_1, y_1, x_2, y_2) &= \varphi \circ \mu \left(\begin{pmatrix} x_1 & y_1 \\ 0 & x_1^{-1} \end{pmatrix}, \begin{pmatrix} x_2 & y_2 \\ 0 & x_2^{-1} \end{pmatrix} \right) \\ &= \varphi \left(\begin{pmatrix} x_1 x_2 & x_1 y_2 + y_1 x_2^{-1} \\ 0 & x_1^{-1} x_2^{-1} \end{pmatrix} \right) \\ &= (x_1 y_1, x_1 y_2 + y_1 x_2^{-1}) \\ \varphi \circ \mathrm{inv} \circ \varphi^{-1}(x, y) &= \varphi \circ \mathrm{inv} \left(\begin{pmatrix} x & y \\ 0 & x^{-1} \end{pmatrix} \right) \\ &= \varphi \left(\begin{pmatrix} x^{-1} & -y \\ 0 & x \end{pmatrix} \right) \\ &= (x^{-1}, -y). \end{aligned}$$
Thus μ and inv are both smooth, G is a Lie-group.
3. G is a subgroup of $\mathrm{Gl}(2, \mathbb{R})$. A basis X, Y of $\mathfrak{g}$ is
$$X = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
$$Y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

(see below for a more pedestrian way). The commutator is

$$\begin{aligned}
[X,Y] &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} = 2Y.
\end{aligned}$$

The more pedestrian way in terms of coordinates: A path in G at the unit matrix representing the tangent vector $\frac{\partial}{\partial x}$ in terms of the above coordinates is

$$\begin{aligned}
\gamma : (-1,1) &\longrightarrow G \\
t &\longmapsto \begin{pmatrix} 1+t & 0 \\ 0 & (1+t)^{-1} \end{pmatrix}.
\end{aligned}$$

Let X be the left-invariant vector field X which equals $\frac{\partial}{\partial x}$. Then $X(g)$ is represented by $g \cdot \gamma$ for $g = \varphi^{-1}(x,y)$, hence

$$\begin{aligned}
X(x,y) &= \left.\frac{d}{dt}\right|_{t=0} \varphi\left(\left(\begin{pmatrix} x & y \\ 0 & x^{-1} \end{pmatrix}\begin{pmatrix} 1+t & 0 \\ 0 & (1+t)^{-1} \end{pmatrix}\right)\right) \\
&= \left.\frac{d}{dt}\right|_{t=0} \varphi\left(\begin{pmatrix} x(1+t) & y(1+t)^{-1} \\ 0 & x^{-1}(1+t)^{-1} \end{pmatrix}\right) \\
&= \left.\frac{d}{dt}\right|_{t=0} ((1+t)x, y/(1+t)) \\
&= x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}
\end{aligned}$$

We now compute $[X,Y]$

$$\begin{aligned}
[X,Y] &= \left[x\frac{\partial}{\partial x} - y\frac{y}{\partial y}, x\frac{\partial}{\partial y}\right] \\
&= x\frac{\partial}{\partial y} - \left(-x\frac{\partial}{\partial y}\right) \\
&= 2x\frac{\partial}{\partial y} = 2Y.
\end{aligned}$$

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Exercise 6. [10 Points]
On $\mathbb{R}^3 \setminus \{0\}$ consider the 2-form

$$\omega = \frac{\frac{1}{2}\varepsilon_{ijk}x^i dx^j \wedge dx^k}{\|x\|^3}$$

and the 2-sphere $S^2 = \{(x^1, x^2, x^3 \,|\, (x^1)^2 + (x^2)^2 + (x^3)^2 = 1\}$ in $\mathbb{R}^3$.

1. Verify that ω is closed.
2. Let η denote the restriction of ω to S^2. Compute $\int_{S^2} \eta$.
3. Let Σ be a sphere of radius 3 around the point $(3, 2, 5)$. Compute $\int_\Sigma \eta'$ where η' is the restriction of ω to Σ.

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1. For closedness, we can first write

$$\omega = \frac{xdy \wedge dz + ydz \wedge dz + zdx \wedge dy}{(x^2 + y^2 + z^2)^{3/2}}$$

and compute

$$\begin{aligned} d\omega &= \frac{3dx \wedge dy \wedge dz}{(x^2+y^2+z^2)^{3/2}} - \frac{3 \cdot 2}{2} \frac{(xdx + ydy + zdz) \wedge (xdy \wedge dz + ydz \wedge dx + zdx \wedge dy)}{(x^2+y^2+z^2)^{5/2}} \\ &= \frac{3dx \wedge dy \wedge dz}{(x^2+y^2+z^2)^{3/2}} - \frac{3(x^2+y^2+z^2)(dx \wedge dy \wedge dz)}{(x^2+y^2+z^2)^{5/2}} = 0. \end{aligned}$$

2. For this computation, it is useful to parametrize the S^2 using the spherical coordinates

$$\begin{aligned} x &= \sin\vartheta \cos\varphi \\ y &= \sin\vartheta \sin\varphi \\ z &= \cos\vartheta \end{aligned}$$

where $\vartheta \in (0, \pi)$ and $\varphi \in (0, 2\pi)$. These coordinates do not cover the whole S^2 but the subset that is not covered by these coordinates is measure zero, so for the integration we can still work only in this patch. To express $\eta = i^*\omega$ where i is the embedding $S^2 \to \mathbb{R}^3$ we compute

$$\begin{aligned} i^*dx &= \cos\vartheta \cos\varphi d\vartheta - \sin\vartheta \sin\varphi d\varphi \\ i^*dy &= \cos\vartheta \sin\varphi d\vartheta + \sin\vartheta \cos\varphi d\varphi \\ i^*dz &= -\sin\vartheta d\vartheta \end{aligned}$$

and

$$\begin{aligned} i^*(xdy \wedge dz) &= \sin\vartheta \cos\varphi(\sin^2\vartheta \cos\varphi d\vartheta \wedge d\varphi) \\ i^*(ydz \wedge dx) &= \sin\vartheta \sin\varphi(\sin^2\vartheta \sin\varphi d\vartheta \wedge d\varphi) \\ i^*(zdx \wedge dy) &= \cos\vartheta(\cos\vartheta \sin\vartheta \cos^2\varphi + \cos\vartheta \sin\vartheta \sin^2\varphi)d\vartheta \wedge d\varphi \end{aligned}$$

so

$$\eta = i^*\omega = (\sin^3\vartheta + \sin\vartheta \cos^2\vartheta)d\vartheta \wedge d\varphi = \sin\vartheta d\vartheta \wedge d\varphi$$

which is the standard volume form on S^2. The integral is now

$$\int_{S^2} \eta = \int_0^{\pi} d\vartheta \int_0^{2\pi} d\varphi \sin\vartheta = 2\pi \int_0^{\pi} \sin\vartheta d\vartheta = 2\pi \int_{-1}^{1} dy = 4\pi.$$

where we changed the variable to $y = -\cos\vartheta$. The result agrees with the area of S^2 of unit radius in $\mathbb{R}^3$.

3. We can use the Stokes theorem. ω is well-defined smooth 2-form in the closed ball B around the point $(3, 2, 5)$ with radius 3 (because the origin of $\mathbb{R}^3$ where ω is not well-defined is outside of this ball). We thus have

$$\int_{\Sigma} \eta' = \int_{\partial B} \eta' = \int_{\partial B} i'^{*}\omega = \int_{B} di'^{*}\omega = \int_{B} i'^{*} d\omega = 0$$