Ludwig-Maximilians-Universität München
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Lecture in the summer term 2017/18

Topology 2

Please note: These notes summarize the content of the lecture, many details and examples are omitted. Sometimes, but not always, we provide a reference for proofs, examples or further reading. I will not attempt to give the first reference where a theorem appeared. Some proofs might take two lectures although they appear in a single lecture in these notes. Changes to this script are made without further notice at unpredictable times. If you find any typos or errors, please let me know.

1. Lecture on April, 12 – Hurewicz theorem on π_1, H_1

- **Reference:** [Ha], p.166–168.
- **Setup:** Let X be a path connected topological space, x_0 a base point. If
$$\gamma : ([0,1],\{0,1\}) \longrightarrow (X, x_0)$$
represents an element $[\gamma] \in \pi_1(X, x_0)$, then after identifying $[0,1]$ with Δ^1 (the standard 1-simplex) by an affine map ψ so that $0v_0$ goes to 0 and v_1 to 1 we can view $\gamma \circ \psi$ as a singular 1-simplex. Since $\gamma \circ \psi(v_0) = \gamma \circ \psi(v_1)$ we have $\partial\gamma \circ \psi = 0$, i.e. $\gamma \circ \psi$ is a 1-cycle and represents a homology class.
- **Theorem (Hurewicz):** The map
$$\begin{aligned} \mathrm{hur} : \pi_1(X, x_0) &\longrightarrow H_1(X; \mathbb{Z}) \\ [\gamma] &\longmapsto [\gamma \circ \psi] \end{aligned}$$
is well defined, a group homomorphism, surjective and $\ker(\mathrm{hur})) = [\pi_1(X, x_0), \pi_1(X, x_0)]$ is the commutator subgroup.
- **Warning:** The equivalence relations using to the define the two groups above are quite different (homotopy versus homology).
- **Reminder:** The commutator subgroup of a group G is the smallest subgroup containing the set
$$\{ghg^{-1}h^{-1} \mid g, h \in G\}.$$
It is normal and if $G \longrightarrow A$ is a group homomorphism with A Abelian, then the kernel of this map contains $[G, G]$. We write $G_{ab} = G/[G, G]$, this is an Abelian group.
- **Remark:** The equivalence relation used to define $\pi_1(X, x_0)_{ab}$ is generated by the following two operations:
  - Two curves which are homotopic relative endpoints are equivalent.
  - If $\gamma = \alpha * \beta$ for closed loops α, β based at x_0, then $\gamma \sim \beta * \alpha$.
- **Proof:** For well-definedness, group homorphisms, and surjectivity, see [Ha]. The kernel has to contain $[\pi_1, \pi_1]$ since H_1 is Abelian.

  We show that the induced map
$$\mathrm{hur} : (\pi_1(X, x_0))_{ab} \longrightarrow H_1(X; \mathbb{Z})$$
is injective. This then implies the Theorem.

Let $\gamma : [0,1] \longrightarrow X$ such that $[\gamma] = 0 \in H_1(X;\mathbb{Z})$. Then there is a 2-chain σ such that $\partial\sigma = \gamma$. Let $\sigma = \pm\sigma_1 \ldots \pm \sigma_m$. We write $\tau^i_{kl}, i = 1, \ldots, m$ for the summands of $\partial\sigma_i$. In $C_*(X;\mathbb{Z})$

$$\begin{aligned} \partial\sigma &= \pm(\tau^1_{01} - \tau^1_{02} + \tau^1_{12}) \ldots \pm (\tau^m_{01} - \tau^m_{02} + \tau^m_{12}) \\ &= \gamma \end{aligned} \tag{1}$$

where we omit identification maps. The summands in (1), with the exception of one summand γ, come in pairs so that a summand and its partner have opposite signs. We form a CW-complex K as follows:

- K^0 consists of vertices of σ_i, these points are identified when their images under the singular 2-simplices in X coincide.
- K^1 is formed by closed 1-cell for each pair of edges above. These cells are attached to K^0 using the end points of τ^i_{kl} from the pair. In addition we add one 1-cell corresponding to γ.
- We add one 2-cell for each σ_i above. The are attached to the one cells corresponding to their edges.

This is a finite CW-complex which comes with a map

$$F : K \longrightarrow X$$

which is given by σ^i one each two cell when the 2-cells are attached carefully (so that the restriction of σ^i to an edge is independent from the 2-cell attached to that edge).

The vertex k_0 corresponding to the endpoints of γ is mapped to x_0. We homotope F as follows: Since X is path connected, we may homotope $F|_{K^0}$ so that the homotopy is constant on k_0 and after the homotopy K^0 is mapped to $\{x_0\}$. By the homotopy extension theorem this homotopy can be extended to a homotopy of F defined on the entire complex K.

We obtain new data homotopic to the old data: a curve γ' homotopic to γ rel. endpoint, a map $F' : K \longrightarrow X$, singular 2-simplices σ'_i (as restrictions of F' to closed 2-cells), and restrictions τ^i_{kl} of F' to boundary segments. The singular simplex σ'_i maps vertices to x_0. The edge maps corresponding to adjacent 2-simplicies coincide. In particular, the restrictions of F' to edges of K^1 represent elements in $\pi_1(X, x_0)$ (!).

The complex K is no longer useful, we only need to remember the homotoped singular 2-simplices. We forget all primes in the notation.

Up to homology, we may replace every summand τ^i_{kl} above which has a $-$-sign by the loop parametrized in the opposite sense (γ^{-1} is homologous to $-\tau$ for every singular 1-simplex). Except for γ every summand in the above sum has a partner describing the same path parametrized backwards.

In $\pi_1(X, x_0)_{ab}$ (with $+$ denoting the concatination of paths).

$$\gamma = \gamma + path_1 + path_1^{-1} + \ldots + path_n + path_n^{-1}.$$

were $path_j$ denotes one of the summands of (1) which appear in pairs. In the **Abelian** group $\pi_1(X, x_0)_{ab}$ (but **not** in $\pi_1(X, x_0)$) this sum can be reordered so that

$$\gamma = \left(\tau^1_{01} + \left(\tau^1_{02}\right)^{-1} + \tau^1_{12}\right)^{\pm 1} \ldots \pm \left(\tau^m_{01} + (\tau^m_{02})^{-1} + \tau^m_{12}\right)^{\pm 1}.$$

The singular 2-simplex σ_1 provides a nullhomotopy for the fist bracket, hence the first bracket vanishes in $\pi_1(X,x_0)_{ab}$. The same is true for all other brackets. Thus $\gamma \sim 0$ in $\pi_1(X,x_0)$, i.e. $\gamma \in [\pi_1(X,x_0,\pi_1(X,x_0)]$.

- **Remark:** This implies that if $hur(\gamma : S^1 \longrightarrow X) = 0 \in H_1(X;\mathbb{Z})$, then there is an oriented surface Σ with one boundary component such that the map $\gamma : \partial\Sigma \simeq S^1 \longrightarrow X$ extends to a map $\Gamma : \Sigma \longrightarrow X$.

  Recall that if γ is nullhomotopic, then Σ can be chosen to be a disc.
- **Remark:** The Hurewicz theorem can be generalized to higher homotopy groups as follows: If X is $k-1$-connected, then $\pi_k(X,x_0) \longrightarrow H_k(X;\mathbb{Z})$ is an isomorphism.

  The hypothesis $k-1$-connected cannot be discarded. For example, we know that $\pi_2(T^2) = \{0\}$ (since T^2 is covered by a contractible space). Using cellular homology is quite easy to see that $H_2(T^2;\mathbb{Z}) \simeq \mathbb{Z}$.

2. Lecture on April, 16 – Coefficients, Borsuk-Ulam

- We defined $C_*(X;Z)$ as the free Abelian group generated by singular simplices. Free Abelian groups are automatically $\mathbb{Z}$-modules via $n\cdot\sigma = \sigma + \ldots + \sigma$ (n summands if $n > 0$) and $n\cdot\sigma = (-\sigma) + \ldots (-\sigma)$ ($-n$ summands if $n < 0$).
- If A is an Abelian group we can define $C_*(X;A)$ as follows: elements are formal sums $\sum_\sigma a_\sigma\sigma$ over all singular simplices and $a_i \in A$ with $a_i = 0_A$ for almost all i. The group law is

$$\left(\sum_\sigma a_\sigma \cdot \sigma\right) + \left(\sum_\sigma b_\sigma \cdot \sigma\right) = \sum_\sigma (a_\sigma + b_\sigma)\cdot\sigma.$$

If σ is a singular simplex, this is not an element of $C_*(X;A)$. By an unspoken convention $\sigma \in C_*(X;\mathbb{Z})$ is understood as $1\cdot\sigma$.

For $n \in \mathbb{Z}, a \in A$ and σ a singular simplex $(na)\cdot\sigma \in C_*(X;A)$ and

$$na\cdot\sigma = a\cdot n\sigma. \tag{2}$$

Thus we can define the boundary operator $\partial : C_*(X;A) \longrightarrow C_{k*-1}(X;A)$ as follows:

$$\partial\left(\sum_\sigma a_\sigma\cdot\sigma\right) := \sum_\sigma a_\sigma\cdot(\partial\sigma).$$

This turns $(C_*(X;A),\partial)$ into a chain complex and one can go through all definitions/general theorems which were discussed last semester. All of them hold except the following:

- **Theorem (coefficients):** If X is a one-point space, then

$$H_k(X;A) \simeq \begin{cases} 0 & k \neq 0 \\ A & k = 0. \end{cases}$$

- **Remark:** For this and the axioms one obtains as in the case $A = \mathbb{Z}$ and $n \geq 1$:

$$H_k(S^n;A) \simeq \begin{cases} A & k = 0, n \\ 0 & \text{otherwise} \end{cases}$$

- **Remark:** Let $\varphi : A_1 \longrightarrow A_2$ be a group homomorphism. Then φ induces a chain map $\varphi_* : C_*(X;A_1) \longrightarrow C_*(X;A_2)$ via

$$\varphi_*\left(\sum_\sigma a_\sigma\cdot\sigma\right) = \sum_\sigma \varphi_*(a_\sigma)\cdot\sigma.$$

Moreover, if $f : X \longrightarrow Y$ is continuous, then $f_* \circ \varphi_* = \varphi_* \circ f_*$.

- **Lemma:** Let $f : S^k \longrightarrow S^k, k \geq 1$, be continuous of degree m and A Abelian. Then then $f_* : H_k(S^k, A) \longrightarrow H_k(S^k; A)$ is multiplication by m, i.e.

$$f_*(X) = \begin{cases} \overbrace{X + \ldots + X}^{m \text{ times}} & m \geq 0 \\ \underbrace{-X + \ldots + (-X)}_{m \text{ times}} & m < 0 \end{cases}$$

- **Proof:** Let $a \in A$ and

$$\varphi : \mathbb{Z} \longrightarrow A$$
$$m \longmapsto m \cdot a.$$

Then the diagram

$$\begin{array}{ccc} H_n(S^n; Z) & \xrightarrow[m\cdot]{f_*} & H_n(S^n; \mathbb{Z}) \\ \downarrow{\scriptstyle \varphi_*} & & \downarrow{\scriptstyle \varphi_*} \\ H_n(S^n; A) & \xrightarrow{f_*} & H_n(S^n; A) \end{array}$$

commutes. This proves the claim.
- The Hurewicz theorem has no direct analogue for $H_1(X; A)$ (because $\pi_1(X, x_0)_{ab}$ is a $\mathbb{Z}$-module, but not an A-module).
- **Terminology:** $H_*(X; A)$ is called homology with coefficients in A etc.
- The cellular chain complex with coefficients in A is defined in the obvious way. It computes $H_*(X; A)$.
- **Example:** The cellular chain complex of $\mathbb{RP}^n$ with coefficients in $\mathbb{Z}_2$: For the standard CW-structure $C_k(\mathbb{RP}^n; \mathbb{Z}_2) = \mathbb{Z}_2$ iff $0 \leq k \leq n$ and 0 otherwise. All differentials are zero. Then

$$H_k(\mathbb{RP}^n; \mathbb{Z}_2) \simeq \begin{cases} \mathbb{Z}_2 & 0 \leq k \leq n \\ 0 & \text{otherwise} \end{cases}$$

- **Proposition:** Let $f : S^n \longrightarrow S^n$ be continuous such that $f(-x) = -f(x)$. Then the degree of f is odd.
- **Preliminaries:** Let $\mathrm{pr} : \widehat{X} \longrightarrow X$ be a 2-sheeted covering. This is automatically normal, we denote the non-trivial deck transformation by φ. Consider the short exact sequence

$$0 \longrightarrow C_n(X; \mathbb{Z}_2) \xrightarrow{\tau} C_n(\widehat{X}; \mathbb{Z}_2) \xrightarrow{\mathrm{pr}_*} C_n(X; \mathbb{Z}_2) \longrightarrow 0$$

where τ is defined as follows. For a singular n-simplex in X we choose a lift $\widehat{\sigma} : \Delta^n \longrightarrow \widehat{X}$ such that $\mathrm{pr} \circ \widehat{\sigma} = \sigma$. Then

$$\tau(\sigma) = \widehat{\sigma} + \varphi_*(\widehat{\sigma}).$$

This determines τ on general chains by linearity and τ is a chain map. Thus there is an associated long exact sequence, the map induced by τ is a *transfer map*, the associated long exact sequence is a *transfer sequence.*
- **Proof of the Proposition:** Consider the covering $\mathrm{pr} : S^n \longrightarrow \mathbb{RP}^n$. Since $f(-x) = -f(x)$, f induces a map $\widehat{f} : \mathbb{RP}^n \longrightarrow \mathbb{RP}^n$. We need to understand how $\widehat{f}, f$ interact with the short exact sequence: Using the definition of τ, one

can check that the diagram

$$\begin{tikzcd}
0 \arrow[r] & C_i(\mathbb{RP}^n;\mathbb{Z}_2) \arrow[r, "\tau"] \arrow[d, "\widehat{f}_*"] & C_i(S^n;\mathbb{Z}_2) \arrow[r, "\mathrm{pr}_*"] \arrow[d, "f_*"] & C_i(\mathbb{RP}^n;\mathbb{Z}_2) \arrow[r] \arrow[d, "\widehat{f}_*"] & 0 \\
0 \arrow[r] & C_i(\mathbb{RP}^n;\mathbb{Z}_2) \arrow[r, "\tau"] & C_i(S^n;\mathbb{Z}_2) \arrow[r, "\mathrm{pr}_*"] & C_i(\mathbb{RP}^n;\mathbb{Z}_2) \arrow[r] & 0
\end{tikzcd}$$

commutes. By naturality of the long exact sequence there is map from the transfer sequence to itself:

$$\begin{tikzcd}
H_1(S^n;\mathbb{Z}_2) \arrow[r] \arrow[d, "f_*"] & H_1(\mathbb{RP}^n;\mathbb{Z}_2) \arrow[r, "\partial_*"] \arrow[d, "\widehat{f}_*"] & H_0(\mathbb{RP}^n;\mathbb{Z}_2) \arrow[r, "\tau_*"] \arrow[d, "\widehat{f}_*"] & H_0(S^n;\mathbb{Z}_2) \arrow[r, "\mathrm{pr}_*"] \arrow[d, "f_*"] & H_0(\mathbb{RP}^n;\mathbb{Z}_2) \arrow[d, "\widehat{f}_*"] \\
H_1(S^n;\mathbb{Z}_2) \arrow[r] & H_1(\mathbb{RP}^n;\mathbb{Z}_2) \arrow[r, "\partial_*"] & H_0(\mathbb{RP}^n;\mathbb{Z}_2) \arrow[r, "\tau_*"] & H_0(S^n;\mathbb{Z}_2) \arrow[r, "\mathrm{pr}_*"] & H_0(\mathbb{RP}^n;\mathbb{Z}_2)
\end{tikzcd}$$

The right-most map pr_* is an isomorphism, so the map preceding it must be zero. Therefore, the connecting morphism $H_1(\mathbb{RP}^n;\mathbb{Z}_2) \longrightarrow H_0(\mathbb{RP}^n;\mathbb{Z}_2)$ is sujective, hence it is an isomorphism. Moreover, the map $\widehat{f}_* : H_0(\mathbb{RP}^n;\mathbb{Z}_2) \longrightarrow H_0(\mathbb{RP}^n;\mathbb{Z}_2)$ is an isomorphism. Hence $\widehat{f}_* : H_1(\mathbb{RP}^n;\mathbb{Z}_2) \longrightarrow H_1(\mathbb{RP}^n;\mathbb{Z}_2)$ is an isomorphism.

Inductively, one obtains that $\widehat{f}_* : H_n(\mathbb{RP}^n;\mathbb{Z}_2) \longrightarrow H_n(\mathbb{RP}^n;\mathbb{Z}_2)$ is an isomorphism. Finally, look at

$$\begin{tikzcd}
H_{n+1}(\mathbb{RP}^n;\mathbb{Z}_2) = 0 \arrow[r] \arrow[d] & H_n(\mathbb{RP}^n;\mathbb{Z}_2) \arrow[r] \arrow[d, "\widehat{f}_*"] & H_n(S^n;\mathbb{Z}_2) = \mathbb{Z}_2 \arrow[r] \arrow[d, "f_*"] & H_n(\mathbb{RP}^n;\mathbb{Z}_2) \arrow[d, "\widehat{f}_*"] \\
H_{n+1}(\mathbb{RP}^n;\mathbb{Z}_2) = 0 \arrow[r] & H_n(\mathbb{RP}^n;\mathbb{Z}_2) \arrow[r] & \mathbb{H}_n(S^n;\mathbb{Z}_2) = \mathbb{Z}_2 \arrow[r] & H_n(\mathbb{RP}^n;\mathbb{Z}_2)
\end{tikzcd}$$

The horizontal maps in the middle are injective, hence they are isomorphisms. Hence, f_* is an isomorphism. We have shown that it is also the multiplication with an integer, which must be odd.

Using the morphism $\varphi : \mathbb{Z} \longrightarrow \mathbb{Z}_2$ of coefficient groups we conclude that the degree of f is odd.

- **Corollary (Borsuk-Ulam):** Let $g : S^n \longrightarrow \mathbb{R}^n$ be a continuous map. Then there is a point x such that $f(x) = f(-x)$.
- **Proof:** Assume not and consider

$$\begin{aligned} f : S^n &\longrightarrow S^{n-1} \\ x &\longmapsto \frac{g(x) - g(-x)}{\|g(x) - g(-x)\|}. \end{aligned}$$

The restriction of this map to the equator (yet another copy of S^{n-1}) has the symmetry property needed for the previous proposition. Therefore, it has odd degree. However, this restriction is null-homotopic (shrink S^{n-1} in one of the hemispheres in S^n), so the degree would be zero. This is a contradiction.

3. Lecture on April, 19 – Coefficients

- Recall that the singular chain complex with coefficients in an Abelain group A was defined last time.
- (2) implies, that $C_*(X;A)$ is isomorphic to the tensor product $C_*(X) \otimes_{\mathbb{Z}} A = C_*(X) \otimes A$.

- **Reminder:** Let G, H be Abelian groups. Then there is an Abelian group $G \otimes H$ and a bilinear map $i : G \times H \longrightarrow G \otimes H$ such that for each Abelian group B and each bilinear map $f : G \times H \longrightarrow B$ there is a *unique* group homomorphism $F : G \otimes H \longrightarrow B$ such that $F \circ i = f$.

$$\begin{tikzcd} G \times H \arrow[r, "f"] \arrow[d, "i"'] & B \\ G \otimes H \arrow[ur, dashed, "\exists! F"'] & \end{tikzcd} \tag{3}$$

This determines $G \otimes H$ up to canonical isomorphism. Existence of $G \otimes H$: Let Z be the free Abelian group generated by elements of $G \times H$ and $I \subset Z$ the subgroup generated by elements $(g, h + h') - (g, h') - (g, h)$ and $(g + g', h) - (g, h) - (g', h)$. Then $G \otimes H = Z/I$ has the desired properties.

In particular, $g \otimes (nh) = (ng) \otimes h = n(g \otimes h)$ for $n \in \mathbb{Z}$.

- **Remark:** $i(g, h) =: g \otimes h$. Not every element of $G \otimes H$ is of this form.
- **Examples:**
  - $A \otimes \mathbb{Z} = A$, $\mathbb{Z}_q \otimes \mathbb{Z}_p = \{0\}$ when $gcd(p, q) = 1$. ($1 = mp + nq$ for suitable integers p, q. Then q is invertible in the multiplicative group $\mathbb{Z}_p$.)
  - $A \otimes B$ and $B \otimes A$ are canonically isomorphic.
  - Assume that every element in A has finite order. Then $A \otimes \mathbb{Q} = \{0\}$.
  - $(A_1 \oplus A_2) \otimes B \simeq (A_1 \otimes B) \oplus (A_2 \otimes B)$, the isomorphism is canonical/natural. The same holds for arbitrary direct sums.
- **Reminder:** An Abelian group A is free if it has a basis (i.e. there is a set $B \subset A$ such for every $a \in A$ there are uniquely determined $n_b \in \mathbb{Z}, b \in B$, such that almost all $n_b = 0$ and $a = \sum n_b \cdot b$.

  Alternatively, for each map $f : B \longrightarrow A'$ into an Abelian group A' there is a unique homomorphism $F : A \longrightarrow A'$ such that

$$\begin{tikzcd} B \arrow[r, "f"] \arrow[d, "i"'] & A' \\ A \arrow[ur, dashed, "\exists! F"'] & \end{tikzcd}$$

commutes. The set B is then a basis. Its cardinality is the rank of A.

To see that the rank well defined, note that $A/2A$ is a $\mathbb{Z}_2$ vector space such that B induces a basis.

- **Fact:** If G is free Abelian and H a subgroup, then H is free and rank$(G) \geq$ rank(H). This is non-trivial, a reference is Theorem III.B.3 in [ScS].
- **Reminder:** The following theorem from algebra is useful to compute tensor products.
- **Theorem:** Let A be a finitely generated Abelian group. Then there are $y_1, \dots, y_r, z_1, \dots, z_p \in A$ with the following properties.
  - A is the (internal) direct sum of the cyclic subgroups generated by these elements.
  - The y_i have finite order $t_i \geq 2$ such that t_{i+1} is a multiple of t_i. The order of z_i is infinite. Every set of elements generating A has at least $r + p$ elements. The numbers p and the torsion coefficients) $t_1, \dots, t_p$ are

independent from the choice of generators. In particular

$$A \simeq \mathbb{Z}^p \oplus \underbrace{\bigoplus_i \mathbb{Z}/(t_i\mathbb{Z})}_{\simeq T(A)} .$$

Here the torsion subgroup $T(A) \subset A$ is the subgroup (!) consisting of all elements of finite order in A.

- Consider the exact sequence $0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}_m \longrightarrow 0$. Tensoring with $\mathbb{Z}_m$ we obtain a sequence $0 \longrightarrow \mathbb{Z}_m \longrightarrow \mathbb{Z}_m \longrightarrow \mathbb{Z}_m \longrightarrow 0$ which is no longer exact. Since homology measures the failure of a chain complex of being exact we need to better understand the effect of tensoring with a group G on exact sequences.
- A useful Lemma when dealing with exact sequences is the following:
- **Five Lemma:** Assume that the rows in the following commutative diagram of Abelian groups and homomorphisms are exact, and that f_2, f_4 are isomorphism while f_1 is surjective and f_5 is injective. Then f_3 is an isomorphism.

$$\begin{tikzcd} A_1 \arrow[r] \arrow[d, "f_1"'] & A_2 \arrow[r] \arrow[d, "f_2"'] & A_3 \arrow[r] \arrow[d, "f_3"'] & A_4 \arrow[r] \arrow[d, "f_4"'] & A_5 \arrow[d, "f_5"'] \\ B_1 \arrow[r] & B_2 \arrow[r] & B_3 \arrow[r] & B_4 \arrow[r] & B_5 \end{tikzcd}$$

- **Definition:** A short exact sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{h} C \longrightarrow 0 \tag{4}$$

splits if there is homomorphism $r : C \longrightarrow B$ such that $h \circ r = \mathrm{id}_C$. ($r$ is a right inverse of h).

- **Lemma:** The short exact sequence (4) splits if and only if either of the following conditions hold.
  - There is an isomorphism $\varphi : A \oplus C \longrightarrow B$ such that $f(a) = \varphi(a, 0)$ and $g(\varphi(0, c)) = c$.
  - f has a left inverse, i.e. there is a homomorphism $l : B \longrightarrow A$ such that $l \circ f = \mathrm{id}_A$.
- **Example:** When C is free, then the exact sequence (4) splits. The sequence $0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}_m \longrightarrow 0$ does not split.

4. Lecture on April, 23 – Coefficients, Tor

- **Theorem:** If the sequence

$$A \xrightarrow{f} B \xrightarrow{h} C \longrightarrow 0$$

is exact, then the same is true after tensoring with an Abelian group G. When

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{h} C \longrightarrow 0$$

is exact and splits, then the same is true after tensoring with an Abelian group G.

- **Proof:** The second part is obvious: If r is a right inverse of h, then $r \otimes \mathrm{id}_G : C \otimes G \longrightarrow B \otimes G$ is a right inverse of $h \otimes \mathrm{id}_G$. The exactness then follows from general properties of $\oplus, \otimes$ (distributive).

Let U be the image of $f \otimes \mathrm{id}$. This is contained in $\ker(h \otimes \mathrm{id})$. Therefore $h \otimes \mathrm{id}$ descends to a map

$$h' : \frac{B \otimes G}{U} \longrightarrow C \otimes G.$$

In order to prove that $U = \ker(h \otimes \mathrm{id})$ we define an inverse for h' using (3) and the surjectivity of h: Let

$$\begin{aligned} \varphi : C \times G &\longrightarrow \frac{B \otimes G}{U} \\ (c, g) &\longmapsto [b \otimes g] \text{ if } h(b) = c. \end{aligned}$$

This is a well defined bilinear map. There is a unique morphism

(5) $$\Phi : C \otimes G \longrightarrow \frac{B \otimes G}{U}$$

such that $\Phi \circ i = \varphi$. The composition $h' \circ \varphi$ is the standard map $C \times G \longrightarrow C \otimes G$. Therefore (by uniqueness in (3)), $h' \circ \Phi = \mathrm{id}_{C \otimes G}$. Moreover,

$$\Phi \circ h'([b \otimes g]) = \Phi(h(b) \otimes g) = [b \otimes g].$$

- **Terminology:** $\otimes G$ is said to be *right-exact.*
- **Remark:** If (4) is exact, and we tensor with G, then the result is not necessarily exact. The potential non-exactness is due to the possibility that $f \times \mathrm{id} : A \otimes G \longrightarrow B \otimes G$ has a non-trivial kernel. The smaller the kernel, the better.
- **Definition:** Let A be Abelian. An exact sequence $0 \longrightarrow R \longrightarrow F \longrightarrow A \longrightarrow 0$ is a *free resolution* of A if F is free and Abelian.
- **Remark:** Subgroups of free Abelian groups are free, hence R is also free when $R \longrightarrow F \longrightarrow A$ is a free resolution of A.
- **Examples:**
  - $0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}_m \longrightarrow 0$ is a free resolution of $\mathbb{Z}_m$.
  - Let $F(A)$ be the (abstract) free group generated by the elements of A (if $A = \mathbb{Z}_m$, then $F(A) \simeq \mathbb{Z}^m$). There is a canonical map $p : F(A) \longrightarrow A$. Let $R(A) = \ker(p)$. Then

(6) $$0 \longrightarrow R(A) \xrightarrow{i} F(A) \longrightarrow A \longrightarrow 0$$

  is a free resolution of A. This shows that every Abelian group has a free resolution.
- **Definition:** Let $\mathrm{Tor}(A, G) = \ker(i \times \mathrm{id} : R(A) \otimes G \longrightarrow F(A) \otimes G)$.
- Up to isomorphism $Tor(A, G)$ does not depend on the choice of the free resolution above. Moreover, $Tor(A, G)$ is functorial in A for fixed G (more is true), i.e. from $A \longrightarrow \tilde{A}$ we obtain a map $\mathrm{Tor}(A, G) \longrightarrow \mathrm{Tor}(A', G)$ in a way that is consistent with composition of maps.
- **Lemma:** Let $A, \tilde{A}$ be Abelian groups, $f : A \longrightarrow \tilde{A}$ a homomorphism and two free resolutions:

(7) $$\begin{tikzcd} \mathcal{S} : 0 \arrow[r] & R \arrow[r, "i"] \arrow[d, "f''"'] & F \arrow[r, "p"] \arrow[d, "f'"] \arrow[dl, dotted, "\alpha"'] & A \arrow[r] \arrow[d, "f"'] & 0 \\ \widetilde{\mathcal{S}} : 0 \arrow[r] & \tilde{R} \arrow[r, "\tilde{i}"'] & \tilde{F} \arrow[r, "\tilde{p}"'] & \tilde{A} \arrow[r] & 0 \end{tikzcd}$$

Then the following holds:

1. There are homomorphisms f', f'' such that the diagram commutes. If f_1', f_1'' are two other such homomorphisms, then there is a map $\alpha : F \longrightarrow \tilde{R}$ such that $f' - f_1' = \tilde{i} \circ \alpha$ and $f'' - f_1'' = \alpha \circ i$.
2. $f'' \otimes \mathrm{id} : R \otimes G \longrightarrow \tilde{R} \otimes G$ maps $\ker(i \otimes \mathrm{id})$ to $\ker(\tilde{i} \otimes \mathrm{id})$. The restriction $\Phi(f, \mathcal{S}, \widetilde{\mathcal{S}})$ to this kernel is independent of choices.
3. If $g : \tilde{A} \longrightarrow \tilde{\tilde{A}}$ and $\widetilde{\widetilde{\mathcal{S}}}$ is a free resolution of $\tilde{\tilde{A}}$, then

$$\Phi(g \circ f, \mathcal{S}, \widetilde{\widetilde{\mathcal{S}}}) = \Phi(g, \widetilde{\mathcal{S}}, \widetilde{\widetilde{\mathcal{S}}}) \circ \Phi(f, \mathcal{S}, \widetilde{\mathcal{S}}).$$

4. $\Phi(\mathrm{id}_A, \mathcal{S}, \mathcal{S}) = \mathrm{id}$.

- **Proof:**
  1. Existence of f' follows from the fact that F is free. Once f' is defined, $f'' = \tilde{i}^{-1} \circ f' \circ i$. In particular, f'' is completely determined by f'.
     Let f_1' be another map as f'. Then $(\tilde{p} \circ (f' - f_1'))(x) = 0$, so there is a (unique) $\alpha(x) \in \tilde{R}$ such that $\tilde{i}(\alpha(x)) = (f' - f_1')(x)$. The equality $f'' - f_1'' = \alpha \circ i$ follows.
  2. After tensoring, the above diagram still commutes. If $i \otimes \mathrm{id}(\rho \in R \otimes G) = 0$, then

$$(\tilde{i} \otimes \mathrm{id}) \circ (f'' \otimes \mathrm{id})(\rho) = f' \otimes \mathrm{id}(0) = 0.$$

     Moreover, $(f'' - f_1'') \otimes \mathrm{id} = (\alpha \circ i) \otimes \mathrm{id} = (\alpha \otimes \mathrm{id}) \circ (i \otimes \mathrm{id})$.
  3. Follows since one can choose $g'' \circ f''$ for $(g \circ f)''$.
  4. Choose $f'' = \mathrm{id}$.
- If f is an isomorphism, then $\Phi(f, \mathcal{S}, \widetilde{\mathcal{S}})$ has an inverse: $\Phi(f^{-1}, \widetilde{\mathcal{S}}, \mathcal{S})$.
- **Theorem:** For every free resolution $\mathcal{S}$ there is a canonical isomorphism

$$\Phi(S) : \ker(i \otimes \mathrm{id}) \longrightarrow Tor(A, G).$$

- **Examples:**
  - If A is free, then $Tor(A, G) = 0$ for all G.
  - For $n \geq 1$, $Tor(\mathbb{Z}_n, G) \simeq \{g \in G \,|\, ng = 0\}$. For this use the free resolution $0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}_n \longrightarrow 0$ (the second map is multiplication by n).
  - $Tor(\mathbb{Z}_n, \mathbb{Z}_m) \simeq \mathbb{Z}_{gcd(m,n)} = \ker(n \cdot \otimes \mathrm{id} : \mathbb{Z} \otimes \mathbb{Z}_m \longrightarrow \mathbb{Z} \otimes \mathbb{Z}_m = \mathbb{Z}_m)$.
  - $Tor(A_1 \oplus A_2, G) \simeq Tor(A_1, G) \oplus Tor(A_2, G)$.
- We now study the effect on H_* of tensoring a chain complex (C, ∂) of free Abelian groups with a group G. Note that the singular chain complex is free.
- Let $Z_n = \ker(\partial_n) \subset C_n, B_n = \mathrm{im}(\partial_{n+1}) \subset C_n$. Then there is a short exact sequence of chain complexes

$$\begin{tikzcd}
0 \arrow[r] & Z_n \arrow[r] \arrow[d, "\partial_n"] & C_n \arrow[r, "\partial_n"] \arrow[d, "\partial_n"] & B_{n-1} \arrow[r] \arrow[d, "\partial_{n-1}"] & 0 \\
0 \arrow[r] & Z_{n-1} \arrow[r] & C_{n-1} \arrow[r, "\partial_n"] & B_{n-2} \arrow[r] & 0
\end{tikzcd}$$

The differentials of the outer complex are actually trivial. Because all groups are free, the sequences split and remain exact after $\otimes G$.

$$\begin{tikzcd}
0 \arrow[r] & Z_n \otimes G \arrow[r] \arrow[d, "\partial_n\otimes\mathrm{id}"] & C_n \otimes G \arrow[r, "\partial_n\otimes\mathrm{id}"] \arrow[d, "\partial_n\otimes\mathrm{id}"] & B_{n-1} \otimes G \arrow[r] \arrow[d, "\partial_{n-1}\otimes\mathrm{id}"] & 0 \\
0 \arrow[r] & Z_{n-1} \otimes G \arrow[r] & C_{n-1} \otimes G \arrow[r, "\partial_n\otimes\mathrm{id}"] & B_{n-2} \otimes G \arrow[r] & 0
\end{tikzcd}$$

This gives rise to a long exact sequences of the form

$$\ldots \longrightarrow B_n \otimes G \longrightarrow Z_n \otimes G \longrightarrow H_n(C;G) \longrightarrow B_{n-1} \otimes G \longrightarrow Z_{n-1} \otimes G \longrightarrow \ldots$$

The connecting homomorphisms in this sequence are

$$(i_n \otimes \mathrm{id}) : B_n \otimes G \longrightarrow Z_n \otimes G$$

where $i_n : B_n \longrightarrow Z_n$ is the inclusion. From the long exact sequence we get short exact sequences

$$0 \longrightarrow \mathrm{coker}(i_n \otimes \mathrm{id}) = \frac{Z_n \otimes G}{\mathrm{im}(i_n \otimes \mathrm{id})} \longrightarrow H_n(C;G) \longrightarrow \ker(i_{n-1} \otimes G) \longrightarrow 0.$$

Interpretation of $\ker(i_{n-1} \otimes \mathrm{id})$: This fits into an exact sequence

$$0 \longrightarrow B_{n-1} \otimes G \xrightarrow{i_{n-1}\otimes\mathrm{id}} Z_{n-1} \otimes G \longrightarrow H_{n-1}(C;\mathbb{Z}) \otimes G \longrightarrow 0$$

This is a free resolution of $H_{n-1}(C;\mathbb{Z})$ after tensoring with G. Thus, $\ker(i_{n-1} \otimes \mathrm{id})$ is canonically isomorphic to $Tor(H_{n-1}(C;\mathbb{Z}), G)$.

Interpretation of $\mathrm{coker}(i \otimes G)$: The map defined in (5) applied to the present setting $B_n \longrightarrow Z_n \longrightarrow H_n(C;\mathbb{Z}) \longrightarrow 0$ defines a natural isomorphism

$$H_n(C;\mathbb{Z}) \otimes G \longrightarrow \mathrm{coker}(i_n \otimes \mathrm{id}) = \frac{Z_n \otimes G}{\mathrm{im}(i_n \otimes \mathrm{id})} \tag{8}$$

- **Theorem (universal coefficient theorem for homology):** There is a natural short exact sequence

$$0 \longrightarrow H_n(C;\mathbb{Z}) \otimes G \longrightarrow H_n(C;G) \longrightarrow Tor(H_{n-1}(C;\mathbb{Z}), G) \longrightarrow 0.$$

  The sequence splits (but not naturally).
- **Proof:** We still have to show that the sequence splits. Recall that this is true for $0 \longrightarrow Z_n \xrightarrow{i_n} C_n \xrightarrow{\partial_n} B_{n-1} \longrightarrow 0$ since B_{n-1} is free. Thus, we can *choose* a left inverse $p_n : C_n \longrightarrow Z_n$ for i_n, and we get maps

$$P_n = \text{quotient map} \circ p_n : C_n \longrightarrow H_n(C;\mathbb{Z})$$

  for all n. When one views $H_n(C;\mathbb{Z})$ as a chain complex with trivial differential this collection of maps is a chain map. (The homology of the chain complex $(H_*(C;\mathbb{Z}), 0)$ is $H_*(C;\mathbb{Z})$.) After tensoring with G, we get maps

$$H_n(C;G) \longrightarrow H_n(C;\mathbb{Z}) \otimes G$$
$$\left[\sum_\sigma g_\sigma \cdot \sigma\right] \longmapsto \sum_\sigma \underbrace{P_*(\sigma)}_{\in H_n(C;\mathbb{Z})} \otimes g_\sigma.$$

Precomposing this with the map

$$H_n(C;\mathbb{Z}) \otimes G \longrightarrow H_n(C;G)$$
$$\sum_i [\underbrace{\sigma_i}_{\in Z_n}] \otimes g_i \longmapsto \sum_i g_i \cdot \sigma_i.$$

we get the identity of $H_n(C;\mathbb{Z}) \otimes G$ since p_* is a left inverse of the inclusion $i_* : Z_n \longrightarrow C_n$.

- C was a chain complex, for example the singular chain complex of a pair (X, A) of spaces.
- **Reality check:** Recall that for even n

$$H_k(\mathbb{RP}^n;\mathbb{Z}) \simeq \begin{cases} \mathbb{Z} & k = 0 \\ \mathbb{Z}_2 & 0 < k < n \text{ odd} \\ 0 & \text{otherwise.} \end{cases}$$

Using the universal coefficient theorem we get

$$H_k(\mathbb{RP}^n;\mathbb{Z}_2) \simeq \begin{cases} \mathbb{Z}_2 & 0 \leq k \leq n \\ 0 & \text{otherwise.} \end{cases}$$

- **Remark:** The universal coefficient sequence does not split naturally, i.e. one can not choose the isomorphism $H_*(C;G) \simeq (H_*(C;\mathbb{Z})\otimes G) \oplus Tor(H_{*-1}(C;\mathbb{Z}), G)$ for all chain complexes such that for every chain map $f : C \longrightarrow D$ the following diagram commutes:

$$\begin{tikzcd} H_*(C;G) \arrow[r] \arrow[d, "f_*"'] & (H_*(C;\mathbb{Z}) \otimes G) \oplus Tor(H_{*-1}(C;\mathbb{Z}), G) \arrow[d, "f_* \oplus (Tor(f_{*-1}) \otimes \mathrm{id})"] \\ H_*(D;G) \arrow[r] & (H_*(D;\mathbb{Z}) \otimes G) \oplus Tor(H_{*-1}(D;\mathbb{Z}), G) \end{tikzcd}$$

The exercises provide an example illustrating this by a map of a Moore space $M(\mathbb{Z}_m, n) \longrightarrow S^{n+1}$ ([Ha], Example 2.51) in singular/cellular homology.

5. Lecture on April, 26. More on Tor

- For computations of the torsion product $Tor(\cdot, \cdot)$ the following facts are useful.
- **Fact:** $Tor(A, F) = 0$ is F is free (since tensoring with a free group preserves exactness).
- **Proposition:** For a short exact sequence $0 \longrightarrow B \longrightarrow C \longrightarrow D \longrightarrow 0$ and A Abelian there is a natural exact sequence

$$0 \longrightarrow Tor(A, B) \longrightarrow Tor(A, C) \longrightarrow Tor(A, D) \ldots$$
$$\ldots \longrightarrow A \otimes B \longrightarrow A \otimes C \longrightarrow A \otimes D \longrightarrow 0.$$

- **Proof:** Let $0 \longrightarrow R \longrightarrow F_0 \longrightarrow A \longrightarrow 0$ be a free resolution of A. Consider the following exact sequence of chain complexes (the chain complexes are rows and extended by zero). Tensoring with free groups does not affect exactness.

$$\begin{tikzcd} 0 \arrow[r] & R \otimes B \arrow[r] \arrow[d] & R \otimes C \arrow[r] \arrow[d] & R \otimes D \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & F \otimes B \arrow[r] & F \otimes C \arrow[r] & F \otimes D \arrow[r] & 0 \end{tikzcd}$$

The long exact sequence induced by this is what we want to prove (since $F/R = A$ and R, F are free).

- **Proposition:** $Tor(A,B) \simeq Tor(B,A)$.
- **Proof:** Pick a free resolution $0 \longrightarrow R \longrightarrow F \longrightarrow B \longrightarrow 0$ of B and apply the previous proposition to obtain

$$\begin{tikzcd} 0 \arrow[r] & Tor(A,B) \arrow[r] & A\otimes R \arrow[r] \arrow[d, "\simeq"] & A\otimes F \arrow[r] \arrow[d, "\simeq"] & A\otimes B \arrow[r] \arrow[d, "\simeq"] & 0 \\ 0 \arrow[r] & Tor(B,A) \arrow[r] & R\otimes A \arrow[r] & F\otimes A \arrow[r] & B\otimes A \arrow[r] & 0 \end{tikzcd}$$

The lower row is the definition of $Tor(B,A)$, the vertical arrows are all isomorphisms coming from interchanging factors. Therefore there is a natural iso. $Tor(A,B) \longrightarrow Tor(B,A)$.
- **Proposition:** If B has no torsion, then $Tor(A,B) = 0$.
- This is of interest for $B = \mathbb{Q}, \mathbb{R}$.
- **Proof:** Let $0 \longrightarrow R \longrightarrow F \longrightarrow A \longrightarrow 0$ be a free resolution of A. If B is finitely generated and torsionfree, then B is free and the statement follows. The general case reduces to the finitely generated case as follows:

  Assume $\sum_i r_i \otimes b_i$ is trivial in $F\otimes B$. Then this sum can be reduced in finitely many steps using the defining relations of tensor products (like $r\otimes(b+b') = r\otimes b + r\otimes b'$). Only finitely many elements of B appear in this process and all these elements lie in a finitely generated subgroup of B.
- **Corollary:** $H_n(X;\mathbb{Q}) = H_n(X;\mathbb{Z})\otimes\mathbb{Q}$ and $\operatorname{rank}(H_n(X;Z)) = \dim_{\mathbb{Q}}(X;\mathbb{Q})$.
- **Corollary:** $\widetilde{H}_*(X;Z) = 0$ if and only if $\widetilde{H}_*(X;\mathbb{Z}_p) = 0$ for all primes and $\widetilde{H}_*(X;Q)$.
- **Reminder:** Reduced homology is obtained from the singular chain complex by replacing $\partial_0 : C_0(X) \longrightarrow 0$ with $\varepsilon : C_0(X) \longrightarrow \mathbb{Z}, \sum n_i\sigma_i \longmapsto \sum_i n_i$.
- **Proof:** One direction is clear. For the other: Assume $A = \widetilde{H}_*(X,\mathbb{Z})$ such that the homology groups with $\mathbb{Z}_p$ and $\mathbb{Q}$-coefficients vanishes. We want to show $A = 0$.

  From the six-term exact sequence above applied to $0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}_p \longrightarrow 0$ (the second arrow is multiplication with p) we get

$$0 \longrightarrow Tor(A;Z_p) \xrightarrow{p\cdot} A \longrightarrow A\otimes\mathbb{Z} \simeq A \longrightarrow A\otimes\mathbb{Z}_p \longrightarrow 0$$

  The assumptions imply $Tor(A,\mathbb{Z}_p) = 0$, so A is torsion free since multiplication by p is injective. The assumption also implies, that $A\otimes\mathbb{Q} = 0$.

  Now consider $0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0$. Since A is torsion free, $Tor(A,\cdot) = 0$. Hence the six term sequence is

$$0 \longrightarrow Tor(A,\mathbb{Q}/Z) = 0 \longrightarrow A \longrightarrow A\otimes\mathbb{Q} \longrightarrow A\otimes\mathbb{Q}/\mathbb{Z} \longrightarrow 0.$$

  Therefore $A \longrightarrow A\otimes\mathbb{Q} = 0$ is injective. Hence $A = 0$.
- **Corollary:** Let $f : X \longrightarrow Y$ be continuous. Then f is an isomorphism in integral homology iff the same is true for rational and $\mathbb{Z}_p$-homology (for all primes p).
- **Proof:** Let $C(f) = X\times[0,1]\cup Y/\sim$ with $(x,0)\sim f(x)$ and $(x,1)$ $(x',1)$ for all $x,x'\in X$ be the mapping cone of f. From the Mayer-Vietoris sequence (applied to $A = C(f)\setminus[X\times 1]$ and B the image of $X\times(0,1]$ in $C(f)$) one gets a long exact sequence

$$H_n(X) \xrightarrow{f_*} H_n(Y) \longrightarrow H_n(C(f)) \longrightarrow \dots$$

  The reduced homology of $C(f)$ vanishes if and only if f is an isomorphism.

- **Reminder:** Assume $H_*(X,\mathbb{Z})$ is finitely generated. Then we have defined the Euler characteristic
$$\chi(X)=\sum_i(-1)^i\mathrm{rank}(H_i(X;\mathbb{Z})).$$
Using the universal coefficient theorem it follows that
$$\chi(X)=\sum_i(-1)^i\dim_F(H_i(X;F))$$
for all fields $F=\mathbb{Z}_p$ with p prime or $F=\mathbb{Q},\mathbb{R},\mathbb{C}$.
- **Break for an important application of homology:** One of the most important applications of homology and invariants in the spirit of the Euler characteristic is the Lefschetz fixed point theorem. We discuss it for simplicial complexes in $\mathbb{R}^n$ which we first review. We omit proofs and explain only the vocabulary. For details see Chapter 3 of [StZ].
- **Simplices in $\mathbb{R}^n$:** Let $x_0,\dots,x_q\in\mathbb{R}^n$. Then
$$T=\left\{x\in\mathbb{R}^n\,|\,x=\sum_i\lambda_ix_i,\lambda_i\in\mathbb{R}\text{ and }\sum_i\lambda_i=1\right\}$$
is the affine space spanned of $x_0,\dots,x_q$. The following are equivalent:
  1. $\dim(T)=q$
  2. $x_1-x_0,\dots,x_q-x_0$ are linearly independent.
  3. There is no affine subspace S containing $x_0,\dots,x_q$ and $S\neq T$.
  4. If $\sum_i\lambda_ix_i=\sum_i\lambda_i'x_i$, then $\lambda_i=\lambda_i'$. Then, $\lambda_0,\dots,\lambda_q$ are the *barycentric coordinates* of x in $\overline{\sigma}$.

  If any of these conditions is satisfied, then $x_0,\dots,x_q$ are in *general position* and
$$\sigma=\left\{x\in T\,\middle|\,x=\sum_i\lambda_ix_i\text{ with }\lambda_i>0\right\}$$
is the *open simplex* with vertices $x_0,\dots,x_q$ (not open as subset of $\mathbb{R}^n$ when $q\neq n$). The boundary of σ is
$$\dot{\sigma}=\overline{\sigma}\setminus\sigma.$$
Then $(\overline{\sigma},\dot{\sigma})$ is homeomorphic to (D^q,S^{q-1}). Let σ be a simplex and τ the simplex spanned by a subset of $\{x_0,\dots,x_q\}$. Then $\tau\subset\dot{\sigma}$ is a face of σ, we write $\tau\leq\sigma$.
- **Definition:** A *simplicial complex* in $\mathbb{R}^n$ is a finite collection K of (open) simplices in $\mathbb{R}^n$ such that
  1. If $\sigma\in K$ and $\tau\leq\sigma$, then $\tau\in K$.
  2. If $\sigma,\tau\in K$ and $\sigma\neq\tau$, then $\sigma\cap\tau=\emptyset$.

  The simplices with dimension 0 are the *vertices* of the complex. The space
$$|K|=\bigcup_{\sigma\in K}\sigma$$
with the subset topology is the space underlying the simplicial complex. As a subset of $\mathbb{R}^n$ it is metric and it is compact. A subset $L\subset K$ is a subcomplex if it is a simplicial complex in $\mathbb{R}^n$. For each $x\in|K|$ there is a unique (open) simplex $\sigma(x)\in K$ containing x.
- **Definition:** Let L,K be two simplicial complexes in (maybe different) Euclidean spaces. A *simplicial map* is a map $f:K\longrightarrow L$ such that

1. f maps vertices to vertices.
2. f maps the simplex σ onto the simplex in L whose vertices are images of the vertices of σ. The restriction of f to σ is affine.

Given a simplicial map $f : K \longrightarrow L$ there is a unique map $|f| : |K| \longrightarrow |L|$ which coincides with f on vertices and is affine on each simplex of K. The map $|f|$ is continuous. Finally, two simplicial complexes K, L are *isomorphic* if there is simplicial map $f : K \longrightarrow L$ which is bijective on vertices. One has to show that if f is a simplicial isomorphism, then

- the map induced by the restriction of f^{-1} to vertices induces a simplicial map f^{-1} such that $|f^{-1}| = |f|^{-1}$, and
- $|f|$ is a homeomorphism.

- **Remark:** One may think of a simplicial complex as some special CW-complex. However, simplicial maps are much more rigid than cellular maps: For two simplicial complexes K, L there are only finitely many simplicial maps $f : K \longrightarrow L$. If for example $|K| = |L| (\simeq S^k)$, then only finitely many maps are homotopic to simplicial maps. Since there are infinitely many homotopy classes of maps $S^k \longrightarrow S^k$, not every map $f : |K| \longrightarrow |L|$ is homotopic to map which can be realized by a simplicial map $g : K \longrightarrow L$.
- **Reminder:** Barycentric subdivision of a simplex σ, this was discussed last semester for the standard simplex $\Delta^k \subset \mathbb{R}^{k+1}$ but works analogously for all simplices in Euclidean space.
- **Theorem:** Let K be a simplicial complex (in Euclidean space). Then there is a simplicial complex $K^{(1)}$ such that
  - the vertices of $K^{(1)}$ are the barycenters of simplices of K (this means in particular that for each vertex y of $K^{(1)}$ there is a unique open simplex $\sigma(y)$ of K containing it),
  - $|K| = |K^{(1)}|$,
  - vertices $y_0, \ldots, y_q$ span a simplex of $K^{(1)}$ if and only if $\sigma(y_0) < \sigma(y_1) \ldots < \sigma(y_q)$ (after renumbering).

  $K^{(1)}$ is the *barycentric subdivision* of K. This can be iterated, $K^{(n)} := \left(K^{(n-1)}\right)^{(1)}$. If m is the maximal dimension of simplices in K (a finite set). Then the diameter of a simplex $\sigma^1$of $K^{(1)}$ is at most

$$\frac{m}{m+1} \cdot (\text{ diameter of the simplex of } K \text{ containing } \sigma^1).$$

- **Theorem (simplicial approximation theorem):** Let $f : |K| \longrightarrow |L|$ be a continuous map. f is homotopic to a simplicial map after sufficiently many barycentric subdivisions of K.
- **Remark:** the proof of this can be found in [StZ], Section 3.2–3.3. One part is similar to the cellular approximation theorem (which does not require subdivision): One homotopes the map pushing images of low dimensional simplices out of higher dimensional ones. The other (first) part uses the following definition:
- **Definition:** Let K be a simplicial complex and p a vertex (i.e. a 0-dimensional simplex in K). The *star* of p in K is

$$St(p) = \{x \in \tau \,|\, p \leq \sigma(x)\} \subset |K|.$$

- **Lemma:** $(St(p))_{p \text{ a vertex of } K}$ is an open cover of $|K|$. If $\mathfrak{U}$ is any open cover, then after finitely many barycentric subdivisions, the open cover by stars is subordinate to $\mathfrak{U}$.

- **Startegy for Proof of simplicial approximation:** Let $f : |K| \longrightarrow |L|$ be continuous. After sufficiently many subdivisions of K the open cover of $|K|$ by stars of vertices of $K^{(n)}$ is subordinate to the open cover
$$\left(f^{-1}(St(y))\right), y \text{ a vertex of } L.$$
Thus, for each vertex x of $K^{(n)}$ we can choose a vertex y_x of L such that $f(St(x)) \subset St(y_x)$.
- **Lemma:** The assignment $x \longmapsto y_x$ defines a simplicial map $\varphi : K^{(n)} \longrightarrow L$. φ is homotopic to f.
- **Fact:** Throughout the homotopy $St(x)$ is mapped to $St(y_x)$. When one subdivides L sufficiently often, then one can arrange that the simplicial approximation φ of f is C^0-close to f. This is essential in the proof of the Lefschetz fixed point theorem.
- **Definition:** Let $f : |K| \longrightarrow |K|$ be a continuous map of a simplicial complex. The Lefschetz-number of f is
$$\lambda(f) = \sum_i (-1)^i \mathrm{trace}(f_i : H_i(|K|;\mathbb{R}) \longrightarrow H_i(|K|;\mathbb{R})).$$
The cellular chain complex is finitely generated, so the same is true for H_* and the sum above is defined. Note that $\chi(|K|) = \lambda(\mathrm{id}_{|K|})$.
- **Theorem (Lefschetz fixed point theorem):** If $\lambda(f) \neq 0$, then f has a fixed point.
- **Proof:** Let $f : |K| \longrightarrow |K|$ be continuous without fixed points. Since $|K|$ is metric and compact, there is $\varepsilon > 0$ such that $d(x, f(x)) > \varepsilon$. After subdivision of K we can assume that the diameter of stars of vertices in $K^{(n)}$ is smaller than $\varepsilon/2$. After subdivision of $K^{(n)}$ and homotopy we find a map $\varphi : |K| \longrightarrow |K|$ which is simplicial with respect to $K^{(m)}$ and $\varepsilon/2$-close to f. Therefore, f does not map a simplex to itself. That implies that the trace of
$$f_i : C_i^{CW}(|K|;\mathbb{R}) \longrightarrow C_i^{CW}(|K|;\mathbb{R})$$
vanishes for all i (we use the CW-decomposition induced by $K^{(m)}$ on $|K|$). To conclude we use the Hopf trace formula:
- **Lemma:**
$$\begin{aligned}&\sum_i (-1)^i \cdot \mathrm{trace}\left(f_i : C_i^{CW}(|K|;\mathbb{R}) \longrightarrow C_i^{CW}(|K|;\mathbb{R})\right)\\ &\qquad = \sum_i (-1)^i \cdot (\mathrm{trace}\,(f_i : H_i(|K|;\mathbb{R}) \longrightarrow H_i(|K|;\mathbb{R}))\end{aligned} \tag{9}$$
- **Proof:** exercise.
- **Applications:**
  - The closed disc is homeomorphic to the standard simplex, all maps are homotopic and $\chi(\Delta^k) = 1$. Thus the Lefschetz theorem generalizes the Brouwer fixed point theorem.
  - If $f : S^k \longrightarrow S^k$ for k even has degree $\neq -1$, then f has a fixed point. The antipodal map has degree -1 and no fixed point.
  - Assume $f : |K| \longrightarrow |K| \neq \emptyset$ is null homotopic. Then $\lambda(f) > 0$ and f has a fixed point.
  - Every continuous map $\mathbb{RP}^n \longrightarrow \mathbb{RP}^n$ for n even has a fixed point. (Compute the homology of $\mathbb{RP}^n$ with real/rational coefficients). If $n = 2k - 1$

is odd, then

$$\begin{aligned}\mathbb{RP}^n &\longrightarrow \mathbb{RP}^n\\ [x_1 : x_2 : \ldots : x_{2k-1} : x_{2k}] &\longmapsto [x_2 : -x_1 : \ldots : x_{2k} : -x_{2k-1}]\end{aligned}$$

has no fixed point.

– The same is true for $\mathbb{CP}^n$ with even n, but we do not yet have the technology to prove that. For odd n

$$\begin{aligned}\mathbb{CP}^n &\longrightarrow \mathbb{CP}^n\\ [z_1 : z_2 : \ldots : z_{2k-1} : z_{2k}] &\longmapsto [\overline{z}_2 : -\overline{z}_1 : \ldots : \overline{z}_{2k} : -\overline{z}_{2k-1}]\end{aligned}$$

has no fixed point.

– If $f : |K| \longrightarrow |K|$ is homotopic to the identity and $\chi(|K|) \neq 0$, then f has a fixed point.

6. Lecture on May 3 – Eilenberg-Zilber equivalences

- **Reference:** Mostly [StZ], Chapter 12
- **Goal:** Describe the homology of a product space $X \times Y$. For products of CW-complexes this would be easy using the cellular chain complex provided that $X \times Y$ is actually a CW-complex. However, this is not true in general and there is a different method using singular chains and a technology that is used elsewhere.
- **Definition:** Let $A = (A_n), B = (B_n)$ be chain complexes. Then $A \otimes B$ is defined via

$$\begin{aligned}(A \otimes B)_n &= \bigoplus_i A_i \otimes B_{n-i} \text{ and}\\ \partial(a \otimes b) &= (\partial a) \otimes b + (-1)^p a \otimes (\partial b) \text{ when } a \in A_p.\end{aligned} \tag{10}$$

This is a chain complex and the construction is functorial with respect to chain maps of A, B.

- **Remark:** If A_n, B_k are all free Abelian, then the same is true for all $(A \otimes B)_l$.
- We will first state what we would like to do. The proofs will be more indirect/less explicit.

  Let Δ^k be the standard k-simplex.
- **Definition:** A singular $p+q$-chain $m_{p,q}$ on $\Delta^p \times \Delta^q$ is a model product chain if $m_{p,q}$ is a cycle relative to $\partial(\Delta^p \times \Delta^q)$ which generates

$$H_{p+q}(\Delta^p \times \Delta^q, \partial(\Delta^p \times \Delta^q); \mathbb{Z}) \simeq \mathbb{Z}.$$

- **Example:** If $p = q = 0$, then Δ^0 is a single point and there is only one possible choice for

$$m_{0,0} : \Delta^0 \longrightarrow \Delta^0 \times \Delta^0.$$

- **Definition:** Assume that $m_{p,q}$ were chosen. Let $\sigma : \Delta^p \longrightarrow X$ and $\tau : \Delta^q \longrightarrow Y$ be singular simplices. The *product chain* $\sigma \times \tau : \Delta^p \times \Delta^q \longrightarrow X \times Y$ is defined as

$$\sigma \times \tau = (\sigma \times \tau)_*(m_{p,q}) \in C_{p+q}(X \times Y). \tag{11}$$

On the right hand side, $\sigma \times \tau$ is viewed as a map, on the left hand side it is a singular chain.

For singular chains $c = \sum_i c_i\sigma_i \in C_*(X), c_i \in \mathbb{Z}$, and $d = \sum_j d_j\tau_i \in C_*(Y)$ set

$$c \times d = \sum_{i,j} c_i d_j \sigma_i \times \tau_j \in C_*(X \times Y).$$

- **Remark:** Here we use the ring structure on $\mathbb{Z}$. Any other commutative coefficient ring would be fine.
- **Lemma (product boundary operator):** There is a choice of model product chains such that for $c \in C_p(X)$

$$\partial(c \times d) = (\partial c) \times d + (-1)^p c \times (\partial d). \tag{12}$$

- **Theorem:** (11) defines a chain map

$$\begin{aligned} P : C_*(X) \otimes C_*(Y) &\longrightarrow C_*(X \times Y) \\ c \otimes d &\longmapsto c \times d. \end{aligned}$$

For continuous maps $f : X \longrightarrow X', g : Y \longrightarrow Y'$ the diagram

$$\begin{tikzcd} C_*(X) \otimes C_*(Y) \arrow[r, "P"] \arrow[d, "f_* \otimes g_*"'] & C_*(X \times Y) \arrow[d, "(f \times g)_*"] \\ C_*(X') \otimes C_*(Y') \arrow[r, "P'"] & C_*(X' \times Y') \end{tikzcd}$$

commutes and P is normalized, i.e. P is defined on $C_0(X) \otimes C_0(Y) = (C_*(X) \otimes C_*(Y))_0$ via $P(x \otimes y) = (x, y)$.
- **Remarks:** (10) and (12) together imply that P is a chain map. That the diagram commutes is immediate from the definition: For σ a singular p-simplex in X and τ a singular q-simplex in Y we have

$$\begin{aligned} (f \times g)_* P(\sigma \otimes \tau) &= (f \times g)_*(\sigma \times \tau)_*(m_{p,q}) = \big((f \circ \sigma) \times (g \circ \tau)\big)_*(m_{p,q}) \\ &= P\big((f_* \otimes g_*)(\sigma \otimes \tau)\big) \end{aligned}$$

The normalization is obvious. What is still missing is the proof that there is a choice $m_{p,q}$.
- **Theorem:** $P : C_*(X) \otimes C_*(Y) \longrightarrow C_*(X \times Y)$ is a chain homotopy equivalence.
- This is what we want. For this one has to find a chain homotopy inverse of P and proof all auxiliary statements. This is the content of the Eilenberg-Zilber Lemma/Theorem.
- Moreover, we need to understand how to compute the homology of a tensor product of chain complexes whose chain groups are free Abelian (and vanish for negative degrees). That yields the Künneth formula. One instance of this formula goes into the proof of the Eilenberg-Zilber Lemma.
- **Lemma:** Let C, C' be chain complexes with C_n free Abelian for all n such that
  (i) $C_n = C'_n = 0$ for $n < 0$ and
  (ii) $H_n(C') = 0$ for $n > 0$.

  Then
  1. Any two chain maps $f, g : C \longrightarrow C'$ with $f|_{C_0} = g|_{C_0}$ are chain homotopic.
  2. For every homomorphism $\varphi : C_0 \longrightarrow C'_0$ which maps boundaries to boundaries there is a chain map $f : C \longrightarrow C'$ such that $f|_{C_0} = \varphi$.

- **Proof of (1):** We seek $D_n : C_n \longrightarrow C'_{n+1}$ such that $f_n - g_n = \partial'_{n+1} \circ D_n + D_{n-1} \circ \partial_n$. Set $D_n = 0$ for $n \leq 0$ and assume D_i is defined for $i \leq n-1$. Pick a basis of C_n and x an element of the basis. Then for $z(x) = f_n x - g_n x - D_{n-1}\partial_n x$

$$\begin{aligned} \partial'_n z(x) &= f_{n-1}\partial_n x - g_{n-1}\partial_n x - \partial'_n D_{n-1}\partial_n x \\ &= f_{n-1}\partial_n x - g_{n-1}\partial_n x - (-D_{n-2}\partial_{n-1} + f_{n-1} - g_{n-1})\partial_n x \\ &= 0, \end{aligned}$$

so $z(x) \in C'_n, n \geq 1$ is a cycle. Because of $H_n(C') = 0$ for $n \geq 1$ we can choose $b(x) \in C'_{n+1}$ such that $\partial b(x) = z(x)$ and define $D_n x = b(x)$. This defines D_n.
- **Proof of (2):** Set $f_0 = \varphi_0$. Assume that f_i is defined for $i \leq n-1$ such that $f_{i-1} \circ \partial_i = \partial'_i \circ f_i$. Pick a basis for C_n and let x be a basis element. Then $f_{n-1}\partial_n x$ is a cycle in C'_{n-1}. Then by $H_{n-1}(C') = 0$ if $n > 1$ and by assumption if $n = 1$ there is $b(x) \in C'_n$ such that $\partial'_n b(x) = f_{n-1}\partial x$. Thus one finds the desired f.
- **Theorem:** Any two normalized chain maps $P, P' : C_*(X) \otimes C_*(Y) \longrightarrow C_*(X \times Y)$ which are natural in X, Y are chain homotopic.
- **Proof:** The model case: Let $X = \Delta^p, Y = \Delta^q$ for fixed p, q. Since $\Delta^p \times \Delta^q$ is contractible, $C_*(\Delta^p) \otimes C_*(\Delta^q)$ has free Abelian chain groups, and $P = P'$ in degree 0 (and $C_k = 0$ for $k < 0$) we can apply part 1 of the Lemma to obtain the chain homotopy $D_n^\Delta : [C_*(\Delta^p) \otimes C_*(\Delta^q)]_n \longrightarrow C_{n+1}(\Delta^p \times \Delta^q)$.

  The general case: Define D via

$$\begin{aligned} D : [C_*(X) \otimes C_*(Y)]_n &\longrightarrow C_{n+1}(X \times Y) \\ \sigma \otimes \tau &\longmapsto (\sigma \times \tau)_* D^\Delta(\mathrm{id}_p \times \mathrm{id}_q) \end{aligned}$$

  for $\sigma : \Delta^p \longrightarrow X, \tau : \Delta^q \longrightarrow Y$ with $p + q = n$. First, we show that this is natural:

$$\begin{aligned} (f \times g)_* D(\sigma \otimes \tau) &= (f \times g)_*(\sigma \times \tau)_* D^\Delta(\mathrm{id}_p \times \mathrm{id}_q) \\ &= (f \circ \sigma \times g \circ \tau)_* D^\Delta(\mathrm{id}_p \times \mathrm{id}_q) \\ &= D(f \circ \sigma \otimes g \circ \tau) \\ &= D(f_* \otimes g_*)(\sigma \otimes \tau). \end{aligned} \tag{13}$$

  This is used in the following computation:

$$\begin{aligned} \partial D(\sigma \otimes \tau) &= \partial(\sigma \times \tau)_* D^\Delta(\mathrm{id}_p \times \mathrm{id}_q) \\ &= (\sigma \times \tau)_* \partial D^\Delta(\mathrm{id}_p \times \mathrm{id}_q) \\ &= (\sigma \times \tau)_* \big((P - P' - D^\Delta\partial)(\mathrm{id}_p \times \mathrm{id}_q)\big) \\ &= (P - P' - D\partial)\,(\sigma_* \otimes \tau_*)(\mathrm{id}_p \otimes \mathrm{id}_q) \\ &= (P - P' - D\partial)(\sigma \otimes \tau). \end{aligned}$$

  This shows that D is the desired chain homotopy.

7. Lecture on May, 7 – Acyclic model method, again

- **Theorem:** There are natural normalized chain maps $P : C_*(X) \otimes C_*(Y) \longrightarrow C_*(X \times Y)$.

- **Proof:** model case: $X = \Delta^p, Y = \Delta^q$. Set
$$\begin{aligned}\varphi_0 : [C_*(\Delta^p) \otimes C_*(\Delta^q)]_0 &\longrightarrow C_0(X \times Y)\\ (x \otimes y) &\longmapsto (x, y)\end{aligned}$$
for all $x \in C_0(\Delta^p), y \in C_0(\Delta^q)$. By part (2) of the Lemma there is a chain map
$$P^\Delta : C_*(\Delta^p) \otimes C_*(\Delta^q) \longrightarrow C_*(\Delta^p \times \Delta^q)$$
such that $P^\Delta|_{C_0(\Delta^p)\otimes C_0(\Delta^q)} = \varphi_0$.
General spaces: Let $\sigma : \Delta^p \longrightarrow X$ and $\tau : \Delta^q \longrightarrow Y$ be singular simplices. Set
$$P(\sigma \otimes \tau) = (\sigma \times \tau)_* \underbrace{P^\Delta(\mathrm{id}_p \otimes \mathrm{id}_q)}_{\text{model product chain}} .$$
P is normalized and natural (computation similar to (13)).
- We want to show that P defines a chain homotopy equivalence, i.e. we want to use a similar approach to find a natural chain homotopy inverse to P. For this we need know something about $H_*(C_*(\Delta^p) \otimes C_*(\Delta^q))$ for all p, q.
- **Fact:** $H_n(C_*(\Delta^p) \otimes C_*(\Delta^q)) = 0$ for $n > 0$.
- This is the content of the corollary of the Künneth Formula on p. 22
- **Theorem:** There are chain maps
$$Q : C_*(X \times Y) \longrightarrow C_*(X) \otimes C_*(Y)$$
with the following properties
  1. Q is natural in X, Y.
  2. Q is normalized, i.e. $Q(x, y) = x \otimes y$ for $x \in X$ and $y \in Y$.
  3. Q is a chain homotopy inverse of P.
  4. Any two maps Q', Q with these properties are chain homotopic.
- **Proof:** Again, we first consider the case $X = \Delta^p, Y = \Delta^q$. Define
$$\begin{aligned}\varphi_0 : C_0(\Delta^p \times \Delta^q) &\longrightarrow C_0(\Delta^p) \otimes C_0(\Delta^q) = [C_*(\Delta^p) \otimes C_*(\Delta^q)]_0\\ (x, y) &\longrightarrow x \otimes y.\end{aligned}$$
The following computation shows by linearity that φ_0 maps boundaries to boundaries. Let (γ^p, γ^q) be a 1-simplex $\Delta^1 \subset \mathbb{R}^2$ to $\Delta^p \times \Delta^q$. Then by the sign convention for the boundary operator in a tensor product of chain complexes
$$\begin{aligned}\gamma^p((0,1)) \otimes \gamma^q((0,1)) - \gamma^p((1,0)) \otimes \gamma^q((1,0)) &= \gamma^p((0,1)) \otimes \gamma^q((0,1)) - \gamma^p((1,0)) \otimes \gamma^q((0,1))\\ &\quad + \gamma^p((1,0)) \otimes \gamma^q((0,1)) - \gamma^p((1,0)) \otimes \gamma^q((1,0))\\ &= (\partial\gamma^p) \otimes \gamma^q((0,1)) + \gamma^p((1,0)) \otimes (\partial\gamma^q)\\ &= \partial(\gamma^p \otimes \gamma^q((0,1))) + \partial(\gamma^p((1,0)) \otimes \gamma^q).\end{aligned}$$
By the Lemma there is a chain map $Q^\Delta : C_*(\Delta^p \times \Delta^q) \longrightarrow C_*(\Delta^p) \otimes C_*(\Delta^q)$ extending φ_0.
For the general case set
$$\begin{aligned}Q : C_*(X \times Y) &\longrightarrow C_*(X) \otimes C_*(Y)\\ \sigma &\longmapsto ((\mathrm{pr}_X \circ \sigma)_* \otimes (\mathrm{pr}_Y \circ \sigma)_*) \left(Q^\Delta(d_n)\right)\end{aligned}$$
where $\sigma : \Delta^n \longrightarrow X \times Y$ is a n-simplex and $d_n : \Delta^n \longrightarrow \Delta^n \times \Delta^n$ is the diagonal map $u \longmapsto (u, u)$.
One can check that Q is a natural (and normalized) chain map.
- **Theorem:** P, Q are mutually inverse chain equivalences.

- **Proof:** By definition $P \circ Q$ and $Q \circ P$ are the respective identities in degree zero. By the first part of the lemma, both these maps are chain homotopic to the identity in the model case. The general case follows by naturality.
- This concludes the proofs of the Lemmas/Theorems from the beginning. It also shows that all choices in the construction lead to naturally chain equivalent results.
- One can describe a map $Q : C_*(X \times Y) \longrightarrow C_*(X) \otimes C_*(Y)$ explicitly. For this one does not have to choose model product chains.
- **Definition:** For $0 \le q \le n$ consider the following maps between standard simplices

$$\begin{aligned}[v_0, \dots, v_q] &: \Delta^q \longrightarrow \Delta^n \\ [v_q, \dots, v_n] &: \Delta^{n-q} \longrightarrow \Delta^n\end{aligned}$$

For a singular simplex $\sigma : \Delta^n \longrightarrow Z$ the compositions

$$\begin{aligned}\sigma \circ [v_0, \dots, v_q] &: \Delta^q \longrightarrow Z \\ \sigma \circ [v_q, \dots, v_n] &: \Delta^{n-q} \longrightarrow Z\end{aligned}$$

are the front/back side of the simplex.
- **Theorem:** Let σ be a n-simplex in $X \times Y$ and

$$Q(\sigma) = \sum_{q=0}^{n} (\mathrm{pr}_1 \circ \sigma \circ [v_0, \dots, v_q]) \otimes (\mathrm{pr}_2 \circ \sigma \circ [v_q, \dots, v_n]) \in [C_*(X) \otimes C_*(Y)]_n. \tag{14}$$

This defines a natural and normalized equivalence of chain complexes

$$C_*(X \times Y) \longrightarrow C_*(X) \otimes C_*(Y).$$

8. Lecture on May, 14 – Künneth formula, Homology cross product

- We have shown that $C_*(X) \otimes C_*(Y)$ and $C_*(X \times Y)$ are naturally chain homotopy equivalent. In order to compute the homology of a product in terms of the homologies of the factors we want to understand the homology of a tensor product of chain complexes.
- In the last section, every coefficient ring instead of $\mathbb{Z}$ would have been fine. In this section, one has to require that R is a principal ideal domain. Then $C_*(\cdot; R)$ is a free R-module. The assumption that R is a principal ideal domain ensures that submodules of free modules are again free. Recall that a principal ideal domain is a ring without zero divisors such that every ideal is generated by one element.

  An example is $\mathbb{Z}$, of course. The submodule $\{0, 2\} \subset \mathbb{Z}_4$ of the free $\mathbb{Z}_4$-module $\mathbb{Z}_4$ is not free.
- One first proves an auxiliary result:
- **Lemma:** Let C, C' be free chain complexes such that $\partial \equiv 0$, i.e. $H_p(C) = C_p$, then

$$\begin{aligned}\lambda : [C \otimes H(C')]_n &\longrightarrow H_n(C \otimes C') \\ c \otimes [d] &\longmapsto [c \otimes d]\end{aligned} \tag{15}$$

is an isomorphism.
- **Note:** This map is well defined since cycle$\otimes$boundary is a boundary, etc.

- **Proof:** By definition
$$[C \otimes C']_n = \bigoplus_{p+q=n} C_p \otimes C_p$$
and
$$\begin{aligned}\partial^{\otimes} : C_p \otimes C_q &\longrightarrow C_p \otimes C_{q-1}\\ c \otimes c' &\longmapsto (-1)^p c \otimes \partial' c'.\end{aligned}$$
The sign can be ignored when computing homology. Hence $H_n(C \otimes C') = \bigoplus_{p+q=n} H_q(K_p)$ where K_p is the chain complex (with boundary map $(-1)^p \mathrm{id}_{C_p} \otimes \partial'$)
$$\ldots \longrightarrow C_p \otimes C'_{n-p} \longrightarrow C_p \otimes C'_{n-p-1} \longrightarrow \ldots$$
This is the chain complex C' after tensoring with C_p and a degree shift. By the universal coefficient theorem (and since C_p is free Abelian)
$$\begin{aligned}\lambda : C_p \otimes H_q(C') &\longrightarrow H_q(K_p)\\ c \otimes [c'] &\longmapsto [c \otimes c']\end{aligned}$$
is an isomorphism.
- **Theorem (Künneth Formula):** Let C, C' be free chain complexes. Then for all n there is natural exact sequence
$$0 \longrightarrow [H_*(C) \otimes H_*(C')]_n \xrightarrow{\lambda} H_n(C \otimes C') \xrightarrow{\mu} \bigoplus_{p+q=n} Tor(H_{p-1}(C), H_q(C')) \longrightarrow 0$$
which splits (not naturally).
- **Proof:** The proof strategy is similar to the proof of the univ. coefficient theorem. Consider the exact sequence $0 \longrightarrow Z \longrightarrow C \longrightarrow B^- \longrightarrow 0$ of chain complexes
$$\begin{tikzcd} 0 \arrow[r] & Z_p \arrow[r] \arrow[d, "\partial"] & C_p \arrow[r, "\partial"] \arrow[d, "\partial"] & B_{p-1} \arrow[r] \arrow[d, "\partial"] & 0 \\ 0 \arrow[r] & Z_{p-1} \arrow[r] & C_{p-1} \arrow[r, "\partial"] & B_{p-2} \arrow[r] & 0. \end{tikzcd}$$
Here $B^-_p = B_{p-1}$ and all horizontal maps are chain maps. Because all groups involved are free the sequence splits, and tensoring with C' (i.e. tensoring with C'_q, direct summing over $p + q = n$) does not affect exactness. Thus we obtain an exact sequence
$$0 \longrightarrow Z \otimes C' \longrightarrow C \otimes C' \longrightarrow B^- \otimes C' \longrightarrow 0.$$
Hence, there is a long exact sequence
$$\ldots H_n(Z \otimes C') \xrightarrow{j \otimes \mathrm{id}'} H_n(C \otimes C') \xrightarrow{\partial \otimes \mathrm{id}'} H_n(B^- \otimes C') \xrightarrow{\partial_*} H_{n-1}(Z \otimes C') \ldots \tag{16}$$
The connecting homomorphism is induced by the inclusion $B^- \hookrightarrow Z$, i.e. it fits into a commutative diagram
$$\begin{tikzcd} {[B^- \otimes H_*(C')]_n} \arrow[r, "i \otimes \mathrm{id}"] \arrow[d, "\lambda"] & {[Z \otimes H_*(C')]_n} \arrow[d, "\lambda"] \\ H_n(B^- \otimes C') \arrow[r, "\partial_*"] & H_n(Z \otimes C') \end{tikzcd}$$

whose vertical maps are isomorphisms by the previous Lemma. Hence, $\ker(\partial_*) \simeq \lambda^{-1}(\ker(i \otimes \mathrm{id}))$ where i is part of a free resolution of $H_{p-1}(C)$:

$$B_p^- \xrightarrow{i} Z_{p-1} \longrightarrow H_{p-1}(C) \longrightarrow 0.$$

(again tensor with C'_q and sum over $p+q=n$). Therefore, there is a natural isomorphism

$$\phi : \ker(i \otimes id) \longrightarrow \bigoplus_{p+q=n} Tor(H_{p-1}(C), H_q(C'))$$

Thus from (16) we get an exact sequence

$$\begin{tikzcd}
H_n(Z \otimes C') \arrow[r, "j\otimes\mathrm{id}'"] \arrow[d] & H_n(C \otimes C') \arrow[r, "(\partial\otimes\mathrm{id}')\circ\phi^{-1}"] & \bigoplus_{p+q} Tor(H_{p-1}(C), H_q(C')) \arrow[r] & 0 \\
\bigoplus_{p+q=n} H_p(C) \otimes H_q(C') \arrow[ur, "\lambda"'] & & &
\end{tikzcd}$$

The image of λ coincides with the image of $j \otimes \mathrm{id}'$. We construct a left inverse for λ. This will conclude the proof. Recall that

$$0 \longrightarrow Z_p \xrightarrow{r} C_p \longrightarrow B_{p-1} \longrightarrow 0$$

is free and exact, i.e. it splits. Hence there is $l : C_p \longrightarrow Z_p$ such that $l \circ j = \mathrm{id}$. Moreover, this also induces a map into $H_*(C)$, and the map $l \otimes l'$ sends boundaries in $C \otimes C'$ to zero. Thus there is a well-defined map

$$H_n(C \otimes C') \longrightarrow \bigoplus_{p+q=n} H_p(C) \otimes H_q(C')$$
$$[c \otimes c'] \longrightarrow [l(c)] \otimes [l'(c')]$$

which is a left-inverse to λ. In particular, λ is injective.

- This can be used to prove a statement we used to show that P has a chain homotopy inverse:
- **Corollary:** If C, C' satisfy $H_i(C) \neq 0 \neq H_i(C')$ only if $i = 0$, then the same is true for $C \otimes C'$.
- Let X, Y be topological spaces and $P : C_*(X) \otimes C_*(Y) \longrightarrow C_*(X \times Y)$ an Eilenberg-Zilber equivalence, i.e. a natural, normalized chain equivalence. The particular choice of P is irrelevant as any two such maps are naturally chain equivalent.
- **Definition:** The homology cross product is

$$H_p(X) \times H_q(Y) \longrightarrow H_{p+q}(X \times Y)$$
$$([a],[b]) \longmapsto [P(a \otimes b)] =: [a] \times [b].$$

- This is well defined since P is a chain map.
- **Theorem:** The homology cross product has the following properties:
  1. Naturality: $(f \times g)_*(a \times b) = (f_* a) \times (g_* b)$
  2. Bilinear: $(a + a') \times b = a \times b + a' \times b$, $a \times (b + b') = a \times b + a \times b'$.
  3. (Skew)Commutativity: Let $t : X \times Y \longrightarrow Y \times X$ be the map interchanging the factors. Then $b \times a = (-1)^{pq} t_*(b \times a)$.
  4. Associativity: $(a \times b) \times c = a \times (b \times c)$ where $c \in H_*(Z)$.
  5. Unit: Let $x \in X$. Then $[x] \in H_0(X)$ and $[x] \times b = j_{x*}(b)$ where $j_x : Y \longrightarrow X \times Y$ is the inclusion $j_x(y) = (x, y)$.
- **Proof:**

1. Follows immediately from the naturality of P.
2. by definition of P and $\otimes$.
3. Let $\tau : C_*(X) \times C_*(\tau) \longrightarrow C_*(Y) \otimes C_*(X)$ be the map defined by $\alpha \otimes \beta = (-1)^{pq} \beta \otimes \alpha$ where $\alpha \in C_p(X)$ and $\beta \in C_*(Y)$. Checking that this is a natural chain map is an exercise.
   Consider the commutative diagram
$$\begin{tikzcd} C_*(X) \otimes C_*(Y) \arrow[r, "P"] \arrow[d, "\tau"] & C_*(X \times Y) \arrow[d, "t_*"] \\ C_*(Y) \otimes C_*(X) \arrow[r, "P"] & C_*(Y \times X) \end{tikzcd}$$
   We will conclude that this diagram commutes up to chain homotopy. Both maps $C_*(X) \otimes C_*(Y) \longrightarrow C_*(Y \times X)$ are chain maps and they coincide in degree zero. When $X = \Delta^p$ and $Y = \Delta^q$, the resulting two maps are chain thus homotopic. By naturality, they are chain homotopic in general.
4. This is done using the same type of argument as for commutativity using the diagram (and the associativity of $\otimes$ up to natural isomorphism)
$$\begin{tikzcd} C_*(X) \otimes C_*(Y) \otimes C_*(Z) \arrow[r, "P\otimes\mathrm{id}"] \arrow[d, "\mathrm{id}\otimes P"] & C_*(X \times Y) \otimes C_*(Z) \arrow[d, "P"] \\ C_*(X) \otimes C_*(Y \times Z) \arrow[r, "P"] & C_*(X \times Y \times Z). \end{tikzcd}$$
5. Assume first that $X = \Delta^0$ is an one-point space. Let $\chi : C_*(Y) \longrightarrow C_*(X) \otimes C_*(Y)$ be the chain map (natural in Y) $\tau \longmapsto x \otimes \tau$ with $x \in C_0(X)$. Then $P \circ \chi = j_{x*}$ up to chain homotopy (do it first for $Y = \Delta^p$, then the general case by naturality).

9. Lecture on May, 17 – Künneth formula for spaces, with field coefficients

- Together, the Eilenberg-Zilber Theorem and the Künneth theorem yield the following:
- **Künneth formula for spaces:** The homology cross product defines an injective map
$$\bigoplus_{p+q=n} H_p(X) \otimes H_q(Y) \longrightarrow H_n(X \times Y).$$
  The cokernel is isomorphic to $\bigoplus_{p+q=n} Tor(H_{p-1}(X), H_q(Y))$.
- **Example:** Let X, Y be spaces such that $H_p(X) \simeq \mathbb{Z}_m$ (let $[\sigma]$ be a generator) and $H_q(X) \simeq \mathbb{Z}_n$ (let $[\tau]$ be a generator). Let $r = \gcd(m, n)$ and pick chains $x \in C_{p+1}(X), y \in C_{q+1}(Y)$ such that $\partial x = m\sigma$ and $\partial y = n\tau$. Then
$$c = \frac{n}{r}(x \otimes \tau) - (-1)^p \frac{m}{r}(\sigma \otimes y)$$
  is a cycle. The chain $P(c) \in H_{p+q+1}(X \times Y)$ is a cycle such that $rP(c)$ is a boundary:
$$rc = x \otimes (\partial y) - (-1)^p (\partial x) \otimes y = -(-1)^p \partial(x \otimes y).$$

- There are two instances which are particularly useful for computations: One can use coefficients in a field, and one can use the cellular chain complex for a product of CW-complexes (provided that the product cellular structure is the CW-structure).
- **Definition:** Let V, W be vector spaces of a field K, $V \otimes W$ the tensor product of the underlying Abelian groups, and U the subgroup generated by

$$(kv) \otimes w - v \otimes (kw) \text{ for all } k \in K, v \in V, w \in W.$$

  The factor group $V \otimes W/U$ is $V \otimes_K W$, it is naturally a K-vector space. The following universal property characterizes $\otimes_K$.
- **Theorem:** Let V, W be K-vector spaces. There is a vector space $V \otimes_K W$ and a bilinear map $i : V \times W \longrightarrow (V \otimes_K W)$ such that for all K-bilinear maps $f : V \times W \longrightarrow X$ there is a unique K-linear map $\psi : V \otimes W \longrightarrow X$ with $\psi \circ i = f$. $V \otimes_K W$ is unique up to isomorphism.
- The analogous discussion works for R-modules (R a commutative ring).
- In the above discussions one can use chain complexes whose chain groups are K-vector spaces, like $\mathbb{Q} \otimes C_*(X, \mathbb{Z})$ with $K = \mathbb{Q}$. The Eilenberg-Zilber discussion remains unchanged, the only assumption there was that the coefficients form a *commutative ring*

  For the remainder, one assumes that R is a principal ideal domain, in particular there are no zero-divisors. This ensures that submodules of free modules are free.

  One defines the torsion modules $Tor(A, B)$ for R-modules as before as kernel of $i \otimes_R \mathrm{id}_B$ using a map i associated to a resolution of A through free R-modules. In the case when K is a field, the homology groups will be again vector spaces.
- **Theorem (Künneth theorem with coefficients in a field):** Let C, C' be chain complexes whose chain groups are K-vector spaces and the boundary operator is K linear. Then the product map

$$\lambda : H_*(C) \otimes_K H_*(C') \longrightarrow H_*(C \otimes_K C')$$
$$[c] \otimes_K [c'] \longmapsto [c \otimes_K c']$$

  is a well defined isomorphism.
- **Proof:** The proof consists of four steps.
  1. **Observation:** If $V \longrightarrow V'$ and $g : W \longrightarrow W'$ are injective, linear maps of K-vector spaces, then $f \otimes g : V \otimes_K V \longrightarrow V' \otimes_K W'$ is injective (construct a left inverse of f, g to get a left inverse of $f \otimes g$).
  2. **Universal coefficient theorem in the present situation:** Let (C_n, ∂_n) be a chain complex whose chain groups are K-vector spaces, ∂_n is K-linear, and V a K-vector space. Then $(C_n \otimes_K V, \partial_n \otimes \mathrm{id}_V)$ is again a chain complex whose chain groups are vector spaces, etc. The map $[c] \otimes v \longmapsto [c \otimes v]$ defines a map

$$H_*(C) \otimes_K V \longrightarrow H_*(C \otimes_K V).$$

     The proof is analogous to the proof of the universal coefficient theorem, but easier since $i_q \otimes \mathrm{id}_V$ is injective by the observation above.
  3. The auxiliary lemma (p. 20) used in the proof of remains the same, one replaces the univerlas coefficient theorem with the version discussed in the last step.

4. The proof of the Künneth theorem works again, $i_q \otimes \mathrm{id}$ is still injective. Therefore, no torsion products appear.

- Combining this with the Eilenberg-Zilber theorem one obtains the following theorem.
- **Theorem:** Let X, Y be topological spaces and $a = [\sum \sigma \otimes k_\sigma] \in H_*(X;K), b = [\sum \tau \otimes k_\tau]$. The homology cross product defined via

$$\begin{aligned} H_*(X;K) \otimes_K H_*(Y;K) &\longrightarrow H_*(X \times Y;K) \\ a \otimes_K b &\longmapsto \left[\sum P(\sigma \otimes \tau) \otimes k_\sigma k_\tau\right] \end{aligned} \tag{17}$$

is an isomorphism.
- Note that the chain complex computing $H_*(X \times Y;K)$ is $C_*(X \times Y) \otimes K$. Then there are obvious maps

$$(C_*(X) \otimes K) \otimes_K (C_*(Y) \otimes K) \xrightarrow{\varphi} (C_*(X) \otimes C_*(Y)) \otimes K \overset{P\otimes \mathrm{id}}{\longrightarrow} C_*(X \times Y) \otimes K$$

re-explaining the definition (17), the first map is an isomorphism.
- In the following we assume that X, Y are CW-complexes such that the product topology on $X \times Y$ is the weak topology of the induced cellular structures. Like we did for the cellular chain complex we use information about homology groups to express the Künneth theorem for spaces in terms of the cellular chain complexes.
- **Fact:** If X, Y are CW-complexes and one of these to spaces is locally compact (i.e. locally finite), then $X \times Y$ is a CW-complex ([StZ], 4.2.9).
- **Fact (established in the exercises):** Let $a = [\alpha] \in H_p(X,A)$ and $b = [\beta] \in H_q(Y,B)$. One can form the relative cross product

$$a \times b = [P(\alpha \otimes \beta)] \in H_{p+q}(X \times Y, X \times B \cup A \times Y)$$

where $P : C_*(X) \otimes C_*(X) \longrightarrow C_*(X \times Y)$. By naturality of the chain map P one gets a well defined natural chain map

$$C_*(A) \otimes C_*(Y) \oplus C_*(X) \otimes C_*(B) \longrightarrow C_*(A \times Y \cup X \otimes B).$$

- **Lemma:** $[D^k] \in H_k(D^k, \partial D^k), [D^l] \in H_l(D^l, \partial D^l)$ are generators, then

$$[D^k] \times [D^l] \in H_{k+l}(D^k \times D^l, \partial D^k \times D^l \cup D^k \times \partial D^l) \tag{18}$$

is a generator.
- **Proof:** For $k = 0$ or $l = 0$ the claim follows from the unit-property of the cross product. Recall that a pair (X, A) of spaces is good if $A \subset X$ is closed and there is a neighborhood U of A in X such that $A \longrightarrow U$ is a deformation equivalence. For good pairs, the quotient map induces an isomorphism (see Jan. 22 of Topology 1).

$$H_*(X,A) \longrightarrow \widetilde{H}_*(X/A)$$

Let $[D^p] \in H_p(D^p, S^{p-1})$ and $[D^q] \in H_q(D^q, S^{q-1})$ be generators. Fix relative homeomorphisms $f_k : (D^k, S^{k-1}) \longrightarrow S^k$ and let $f_k([D^k]) = [S^k] \in H_k(S^k)$.

Now consider the diagram

$$\begin{array}{ccccc}
H_p(D^p,S^{p-1})\times H_q(D^q,S^{q-1}) & & \xrightarrow{\quad\times\quad} & & H_{p+q}(D^{p+q},D^p\times S^{q-1}\cup S^{p-1}\cup D^q) \\
\simeq\Big\downarrow f_{p*}\times f_{q*} & & & & \simeq\Big\downarrow f_{p+q*} \\
\widetilde{H}_p(S^p)\times\widetilde{H}_q(S^q) & \xrightarrow{\times} & \widetilde{H}_{p+q}(S^q\times S^q) & \xrightarrow{q_*} & \widetilde{H}_{p+q}(S^p\times S^q/S^p\vee S^q\simeq S^{p+q}) \\
 & & & & \\
[S^p],[S^q] & \longmapsto & [S^p]\times[S^q] & \longmapsto & [S^p\times S^q]
\end{array}$$

The first lower horizontal map maps generators to generators by the Künneth formula, the same is true for the second map (induced by the quotient map) by cellular homology. The diagram commutes up to sign. Thus $[D^p]\times[D^q]$ is a generator of $H_{p+q}(D^{p+q},D^p\times S^{q-1}\cup S^{p-1}\cup D^q)$.

10. Lecture on May, 24 – Künneth formula for CW-complexes, Cohomology

- **Definition:** Let $\alpha\in C_p^{CW}(X)=H_p(X^p,X^{p-1})$ and $\beta\in C_q^{CW}(Y)=H_p(Y^p,Y^{p-1})$. Then $\alpha\times\beta$ is defined as element of $H_{p+q}(X^p\times Y^q,X^{p-1}\times Y^q\cup X^p\times Y^{q-1})$. The image of this element under the inclusion

  $$X^p\times Y^q,X^{p-1}\times Y^q\cup X^p\times Y^{q-1}\longrightarrow H_{p+1}((X\times Y)^{p+q},(X\times Y)^{p+q-1})$$

  is still denoted by $\alpha\times\beta$. It defines the cellular product chain of α,β.

  Recall also, that the cellular boundary operator $\partial_k^{CW}:C_k^{CW}(X)\longrightarrow C_{k-1}^{CW}(X)$ is the composition $j_k\circ\partial_*$ where

  $\partial_*:H_k(X^k,X^{k-1})\longrightarrow H_{k-1}(X^{k-1})$ is the connecting homomorphism of the long exact sequence of the pair (X^k,X^{k-1})

  $j_{k-1}:(X^{k-1},\emptyset)\longrightarrow(X^{k-1},X^{k-2})$ is the inclusion.

  Finally, since $C_k^{CW}(X)=H_k(X^k,X^{k-1})$ is a free Abelian group generated by so-called fundamental classes of oriented cells: Recall that an orientation of a cell e in X is $F_*([D^p])$ where is a choice of a characteristic map F and $[D^p]\in H_p(D^p,S^{p-1})$ is a chosen generator.
- **Lemma:** $\partial^{CW,\times}(\alpha\times\beta)=(\partial^{CW,X}\alpha)\times\beta+(-1)^p\alpha\times(\partial^{CW,Y}\beta)$.
- **Proof:** One translates into singular homology. Let $x\in C_p(X^p)$ respectively y be singular chains representing α respectively β. Then

$$\begin{aligned}
\alpha&=[x]\text{ in }H_p(X^p,X^{p-1})\\
\beta&=[y]\text{ in }H_p(Y^q,Y^{q-1})\\
\partial^{CW,X}\alpha&=[\partial x]\text{ in }H_{p-1}(X^{p-1},X^{p-2})\\
\partial^{CW,Y}\beta&=[\partial y]\text{ in }H_{q-1}(X^{q-1},X^{q-2})\\
(\partial^{CW,X}\alpha)\times\beta&=[P(\partial x\otimes y)]\text{ in }H_{p+q-1}((X\times Y)^{p+q-1},(X\times Y)^{p+q-2})\\
\alpha\times\partial^{CW,Y}\beta&=[P(x\otimes\partial y)]\text{ in }H_{p+q-1}((X\times Y)^{p+q-1},(X\times Y)^{p+q-2})\\
\alpha\times\beta&=[P(x\otimes y)]\text{ in }H_{p+q}((X\times Y)^{p+q},(X\times Y)^{p+q-1})\\
\partial^{CW,\times}(\alpha\times\beta)&=[\partial P(x\otimes y)]\text{ in }H_{p+q-1}((X\times Y)^{p+q-1},(X\times Y)^{p+q-2}).
\end{aligned}$$

The claim follows since P is a chain map (and by the definition of the boundary operator on tensor products). The boundary operator in question here is the one from singular homology (not ∂^{CW}).

- **Lemma:** If $\alpha \in C_p(X), \beta \in C_q(Y)$ are oriented cells e, d in X, Y (i.e. represent a generator of $H_p(X^p, X^{p-1} \cup (X^p \setminus e))$ etc.) then $\alpha \times \beta$ is an orientation of the cell $e \times d$ in $X \times Y$.
- **Proof:** There are characteristic maps

$$F : (D^p, S^{p-1}) \longrightarrow (X^p, X^{p-1})$$
$$G : (D^q, S^{q-1}) \longrightarrow (Y^q, Y^{q-1})$$

such that $F_*([D^p]) = \alpha$ and $G_*([D^q]) = \beta$. For a chosen homeomorphism h consider

$$\begin{tikzcd}[column sep=small] & (D^{p+q}, S^{p+q-1}) \arrow[r, "h"] & (D^p \times D^q, S^{p-1} \times D^q \cup D^p \times S^{q-1}) \arrow[dl, "F\times G"'] & \\ (X^p \times Y^q, X^p \cup Y^{q-1} \cup X^{p-1} \cup Y^q) \arrow[rr, "i"] & & ((X \times Y)^{p+q}, (X \times Y)^{p+q-1}). \end{tikzcd}$$

The composition H of these maps is a characteristic map for $e \times d$. Let $[D^{p+q}] = h^{-1}([D^p] \times [D^q])$. This is a generator of $H_{p+q}(D^{p+q}, S^{p+q-1})$, see (18). Then

$$H_*([D^{p+q}]) = i_*\left((F \times G)_*([D^p \times D^q]\right) = i_*(\alpha \times \beta)$$

defines an orientation of $e \times d$.

- **Summary/Theorem:** A basis of $C_n^{CW}(X \times Y)$ is formed by $e \times d$ with the product orientation where e, d denotes p, q cells of X, Y such that $p + q = n$. The boundary operator $\partial^{CW,\times} : C_n(X \times Y) \longrightarrow C_{n-1}(X \times Y)$ is given by

$$\partial^{CW,\times}(e \times d) = \left(\partial^{CW,X} e\right) \times d + (-1)^p e \times \left(\partial^{CW,Y} d\right).$$

The map

$$C_*^{CW}(X) \otimes C_*^{CW}(Y) \longrightarrow C_*^{CW}(X \times Y)$$
$$e \otimes d \longmapsto e \times d (= \text{ oriented cell, i.e. at the same time a relative homology class})$$

is an isomorphism of chain complexes.

- **Example:** Consider $\mathbb{RP}^2$ with the standard CW-decomposition, $\mathbb{RP}^2 = e^0 \cup e^1 \cup e^2$. For a choice of orientations

$$\partial e^0 = 0 \qquad \partial e^1 = 0 \qquad \partial e^2 = 2e^1.$$

Then the cellular complex of $\mathbb{RP}^2 \times \mathbb{RP}^2$ is generated by

$$\begin{aligned} e^0 \times e^0 &\text{ in degree } 0 \\ e^1 \times e^0, e^0 \times e^1 &\text{ in degree } 1 \\ e^2 \times e^0, e^1 \times e^1, e^0 \times e^2 &\text{ in degree } 2 \\ e^2 \times e^1, e^1 \times e^2 &\text{ in degree } 3 \\ e^2 \times e^2 &\text{ in degree } 4. \end{aligned}$$

The boundary operator is given by

$$\partial(e^0 \times e^0) = 0$$
$$\partial(e^1 \times e^0) = \partial(e^0 \times e^1) = 0$$
$$\partial(e^2 \times e^0) = 2e^1 \times e^0, \partial(e^0 \times e^2) = 2e^0 \times e^1, \partial(e^1 \times e^1) = 0$$
$$\partial(e^2 \times e^1) = 2(e^1 \times e^1) = -\partial(e^1 \times e^2)$$
$$\partial(e^2 \times e^2) = 2e^1 \times e^2 + 2e^2 \times e^1$$

The homology of $\mathbb{RP}^2 \times \mathbb{RP}^2$ with coefficients in $\mathbb{Z}$ is

$$H_0(\mathbb{RP}^2 \times \mathbb{RP}^2) = \mathbb{Z} \text{ generated by } e^0 \times e^0$$
$$H_1(\mathbb{RP}^2 \times \mathbb{RP}^2) = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \text{ generated by } e^1 \times e^0, e^0 \times e^1$$
$$H_2(\mathbb{RP}^2 \times \mathbb{RP}^2) = \mathbb{Z}_2 \text{ generated by } e^1 \times e^1$$
$$H_3(\mathbb{RP}^2 \times \mathbb{RP}^2) = \mathbb{Z}_2 \text{ generated by } e^2 \times e^1 + e^1 \times e^2$$
$$H_4(\mathbb{RP}^2 \times \mathbb{RP}^2) = 0$$

- **Reference**: [Ha], early parts of Chapter 3.
- For Abelian groups A, G we denote the Abelian group of homomorphisms $A \longrightarrow G$ by $\mathrm{Hom}(A, G)$. Let $A_i, i \in I$ be a family of Abelian groups. Then
$$Hom\,(\oplus_i A_i, G) = \prod_i \mathrm{Hom}(A_i, G).$$
If $f : A \longrightarrow B$ is a homomorphism, then $f^* : \mathrm{Hom}(B, G) \longrightarrow \mathrm{Hom}(A, G)$ is defined via $f^*\varphi(a) = \varphi(f(a))$. Note $(f \circ g)^* = g^* \circ f^*$.
- **Example:** When A is free of rank k and $a_1, \ldots, a_k$ i a basis, then $\mathrm{Hom}(A, \mathbb{Z})$ is free of rank k with the dual basis $\widetilde{a}_i(a_j) = \delta_{i,j}$.
- **Definition:** Let $(C_n, \partial_n), n \in \mathbb{Z}$, be a chain complex and G an Abelian group. Then
$$C^n := \mathrm{Hom}(C_n, G)$$
$$\delta_n : C^n \longrightarrow C^{n+1} = \mathrm{Hom}(C_{n+1}, G)$$
$$\varphi \longmapsto \big(c \longmapsto \varphi(\partial_{n+1}c)\big).$$
This defines the dual chain complex C^*
$$\ldots C^{n-1} \xrightarrow{\delta_{n-1}} C^n \xrightarrow{\delta_n} C^{n+1} \xrightarrow{\delta_{n+1}} C^{n+2} \ldots$$
and $\delta_n \circ \delta_{n-1} = 0$ for all n, since $\delta = \partial^*$. The maps δ_n are *coboundary operators*, elements of C^* are *cochains*, elements of $\ker(\delta)$ are *cocycles* (who vanish on boundaries), elements of $\mathrm{image}(\delta)$ are *coboundaries* (who vanish on cycles), and
$$H^n(C; G) := \ker(\delta_n : C^n \longrightarrow C^{n+1})/\mathrm{image}(\delta_{n-1} : C^{n-1} \longrightarrow C^n)$$
are the *cohomology groups*.
- One can define cochain complexes without referring to chain complexes as a chain complex where the boundary operator increases the degree. However, if (C^n, δ) is a cochain complex where the boundary operator increases the degree, then $(D^n := C^{-n}, \delta)$ is a standard chain complex. In particular, as in the case of chain complexes, a short exact sequence of cochain complexes induces a long exact sequences of cohomology groups.

- **Definition:** A chain map $f : C^* \longrightarrow D^*$ between two cochain complexes is a map f such that $\delta^D \circ f = f \circ \delta^C$. If $f : C \longrightarrow C'$ is a chain map between chain complexes, then $f^* : C'^* \longrightarrow C^*$ is a chain map between the dual chain complexes. In particular, $(f \circ g)^* = g^* \circ f^*$.
- **Notation:** There is a bilinear pairing
$$\begin{aligned} \langle \cdot, \cdot \rangle : C^q(C; G) \times C_q(C) &\longrightarrow G \\ (\varphi, x) &\longmapsto \varphi(x). \end{aligned}$$
It has the following property for φ a q-cochain and x a $q+1$-chain.
$$\langle \delta\varphi, x \rangle = \langle \varphi, \partial x \rangle.$$
If f_* is a chain map with dual f^*, then $\langle f^*\varphi, x \rangle = \langle \varphi, f_* x \rangle$ by definition.
- This induces a bilinear pairing on $H^n(C; G) \times H_n(C) \longrightarrow G$. Moreover, a chain map induces a homomomorphism between cohomology groups (opposite direction).
- When $\alpha : G \longrightarrow G'$ is a group homomorphism and C a chain complex, then α induces a chain map $\mathrm{Hom}(C; G) \longrightarrow \mathrm{Hom}(C, G')$.
- If $0 \longrightarrow G' \longrightarrow G' \longrightarrow G'' \longrightarrow 0$ is exact, then the same is true for the induced sequence of cochain complexes.
- We want to relate $H^n(C)$ and $H_n(C; G)$ via the a map
$$\begin{aligned} h : H^n(C; G) &\longrightarrow \mathrm{Hom}(H_n(C), G) \\ [\varphi] &\longmapsto ([c] \longmapsto \varphi(c)) \end{aligned} \tag{19}$$
This is well defined and linear.
- **Lemma:** h is surjective and the exact sequence
$$0 \longrightarrow \ker(h) \hookrightarrow H^n(C; G) \xrightarrow{h} \mathrm{Hom}(H_n(C), G) \longrightarrow 0$$
splits.
- **Proof:** Consider the split-exact sequence (B_{n-1} is free)
$$0 \longrightarrow Z_n \longrightarrow C_n \longrightarrow B_{n-1} = \partial C_n \longrightarrow 0$$
and fix splitting $p : C_n \longrightarrow Z_n$ such that $p(z) = z$ for $z \in Z_n \subset C_n$. Let $\alpha : H_n(C) \longrightarrow G$ be a homomorphism. Then $\alpha \circ p : C_n \longrightarrow G$ is a cocycle since
$$(\delta_n(\alpha \circ p)))\,(c_{n+1}) = \alpha \circ p(\partial_{n+1} c_{n+1}) = \alpha(\partial_{n+1} c_{n+1}) = 0.$$
It represents a cohomology class φ such that $h(\varphi) = \alpha$. Moreover, if α is of the form $h(\varphi)$, then $(h(\varphi)) \circ p = \varphi$ on cycles.
- **Goal:** We want to analyze $\ker(h)$.
- **Example:** Consider the exact sequence $0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}_2 \longrightarrow 0$. Then the dual sequence ($G = \mathbb{Z}$)
$$0 \longrightarrow \mathrm{Hom}(\mathbb{Z}_2, \mathbb{Z}) = 0 \longrightarrow \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) \xrightarrow{(\cdot 2)^*} \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) \longrightarrow 0.$$
is not exact (since $(\cdot 2)^*$ is not surjective).
- **Theorem:** When $A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ is exact, then the same is true for
$$0 \longrightarrow \mathrm{Hom}(C, G) \xrightarrow{g^*} \mathrm{Hom}(B, G) \xrightarrow{f^*} \mathrm{Hom}(A, G).$$
- When $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ is exact and split, then the same is true for
$$0 \longrightarrow \mathrm{Hom}(C, G) \xrightarrow{g^*} \mathrm{Hom}(B, G) \xrightarrow{f^*} \mathrm{Hom}(A, G) \longrightarrow 0.$$

- **Proof:** Assume $g^*(\varphi) = 0$, then $\varphi \equiv 0$ on the image of $g(B) = C$. Hence $\varphi = 0$. Since $0 = g \circ f$, the composition $f^* \circ g^*$ is trivial. Now assume $f^*\psi = 0$. Then $\psi \equiv 0$ on the image of f, i.e. on the kernel of g. Thus $\phi(c) = \psi(g^{-1}(c))$ is well defined and $g^*\phi = \psi$.
- Let $0 \longrightarrow R \longrightarrow F \longrightarrow A \longrightarrow 0$ be a free resolution of A and G Abelian. Then $0 \longrightarrow \mathrm{Hom}(A, G) \longrightarrow \mathrm{Hom}(F, G) \longrightarrow \mathrm{Hom}(R, A)$ is exact, and the last map is not surjective in general.

11. Lecture on May, 28 – Ext, Universal coefficient theorem for cohomology

- **Definition:** Let $0 \longrightarrow R(A) \longrightarrow F(A) \longrightarrow A \longrightarrow 0$ be the standard resolution of A. Then let $i^* : \mathrm{Hom}(F(A), G) \longrightarrow \mathrm{Hom}(R(A), G)$ be the dual of the inclusion $R(A) \longrightarrow F(A)$. The group
$$Ext(A, G) = \mathrm{Hom}(R(A), G)/\mathrm{image}(i^*)$$
is the Ext-product of A, G.
- Let $A, \tilde{A}$ be Abelian groups, $f : A \longrightarrow \tilde{A}$ a homomorphism and two free resolutions:
$$\mathcal{S} : 0 \longrightarrow R \longrightarrow F \longrightarrow A \longrightarrow 0$$
$$\widetilde{\mathcal{S}} : 0 \longrightarrow \tilde{R} \longrightarrow \tilde{F} \longrightarrow \tilde{A} \longrightarrow 0$$

Dualizing the diagram (7) from the case of the torsion product (April, 23) we obtain
$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}(A,G) \arrow[r, "p*"] & \mathrm{Hom}(F,G) \arrow[r, "i^*"] & \mathrm{Hom}(R,G) \\
0 \arrow[r] & \mathrm{Hom}(\tilde{A},G) \arrow[r, "\tilde{p}*"] \arrow[u, "f^*"] & \mathrm{Hom}(\tilde{F},G) \arrow[r, "\tilde{i}^*"] \arrow[u, "f'^*"] & \mathrm{Hom}(\tilde{R},G) \arrow[u, "f''^*"] \arrow[ul, dotted, "\alpha^*"]
\end{tikzcd}$$
The maps f', f'' exist since F, R are free. f''^* maps the image of $\tilde{i}^*$ to the image of i^*. Hence we get a well defined map
$$\psi(f, \mathcal{S}, \widetilde{\mathcal{S}}) : \mathrm{Hom}(\tilde{R}, G)/\mathrm{image}(\tilde{i}^*) \longrightarrow \mathrm{Hom}(R, G)/\mathrm{image}(i^*).$$
Moreover, $\psi(\tilde{f} \circ f, \mathcal{S}, \widetilde{\widetilde{\mathcal{S}}}) = \psi(f, \mathcal{S}, \widetilde{\mathcal{S}}) \circ \psi(\tilde{f}, \widetilde{\mathcal{S}}, \widetilde{\widetilde{\mathcal{S}}})$ and $\psi(\mathrm{id}, \mathcal{S}, \mathcal{S}) = \mathrm{id}$.
- **Theorem:** For every free resolution $0 \longrightarrow R \xrightarrow{i} F \longrightarrow A \longrightarrow 0$ there is a natural isomorphism
$$Ext(A, G) \longrightarrow \mathrm{Hom}(R, G)/\mathrm{image}(i^*)$$
- **Fact:** In particular, if $f : A \longrightarrow A'$ is a group homomorphism, then we obtain a map
$$f^* : \mathrm{Ext}(A', G) \longrightarrow \mathrm{Ext}(A, G).$$
- **Example:**
  1. A free Abelian, then one can choose $R = 0$ and $\mathrm{Ext}(A, G) = 0$ for all Abelian G.
  2. $A = \mathbb{Z}_n$, consider that the resolution $0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \longrightarrow \mathbb{Z}_n \longrightarrow 0$. Then image$(i^*)$ consists of those homomorphisms which are divisible by n, i.e. $\mathrm{Ext}(\mathbb{Z}_n, G) = G/nG$.

3. $\mathrm{Ext}(\mathbb{Z}_n, G) = 0$ if every element of G is divisible by n. For example $G = \mathbb{Q}, \mathbb{R}, \mathbb{Q}/\mathbb{Z}$.
4. $\mathrm{Ext}(\mathbb{Z}_n, \mathbb{Z}_m) \simeq \mathbb{Z}/\mathrm{gcd}(m,n)\mathbb{Z}$.
5. $\mathrm{Ext}(A_1 \oplus A_2, G) \simeq \mathrm{Ext}(A_1, G) \oplus \mathrm{Ext}(A_2, G)$

- Returning to our topic: Notice that

$$0 \longrightarrow Z_q \xrightarrow{j_q} C_q \xrightarrow{\partial'_q} B_{q-1} \longrightarrow 0$$

is free, exact and split, so the same is true for

$$0 \longrightarrow \mathrm{Hom}(B_{q-1}, G) \longrightarrow \mathrm{Hom}(C_q, G) \longrightarrow \mathrm{Hom}(Z_q, G) \longrightarrow 0. \tag{20}$$

Moreover,

$$0 \longrightarrow B_{q-1} \xrightarrow{i_{q-1}} Z_{q-1} \longrightarrow H_{q-1}(C) \longrightarrow 0$$

is a free resolution of $H_{q-1}(C)$. By the Theorem above, there is a natural isomorphism

$$\psi : \mathrm{Ext}(H_{q-1}(C), G) \longrightarrow \mathrm{Hom}(B_{q-1})/\mathrm{image}(i^*_{q-1})$$

- **Lemma:** The map

$$b : \mathrm{Hom}(B_{q-1}(C), G)/\mathrm{image}(i^*_{q-1}) \longrightarrow H^q(C; G)$$

$$[\varphi] \longmapsto \Bigg[\underbrace{c \longmapsto \varphi(\partial'_q c)}_{\text{not a coboundary, in gen.}} \Bigg] \tag{21}$$

is well defined. Note that φ is defined only on B_{q-1}, not on C_{q-1} (in the latter case the image would be zero).

- **Proof:**

$$\big(\delta_q(b(\varphi))\big)(c) = \varphi(\partial_q \circ \partial_{q-1}(c)) = 0.$$

The representative of $b(\varphi)$ above vanishes on boundaries, it is therefore a cocycle. If $\varphi \in \mathrm{image}(i^*_{q-1})$, i.e. φ extends to all cycles, then φ even extends to C_{q-1} since there is a there is a map $r : C_{q-1} \longrightarrow Z_{q-1}$ such that $r \circ i_{q-1} = \mathrm{id}_{Z_{q-1}}$. Then $\varphi \circ \partial'_q$ coincides with a coboundary.

- **Theorem (Universal coefficient theorem for cohomology):** For free chain complexes C, all $q \in \mathbb{Z}$

$$0 \longrightarrow \mathrm{Ext}(H_{q-1}(C), G) \xrightarrow{\rho} H^q(C, G) \xrightarrow{h} \mathrm{Hom}(H_q(C), G) \longrightarrow 0$$

is split exact. It is also natural (but the splitting is not). Here $\rho = b \circ \psi$.

- **Proof:**
  1. ρ is injective: ψ is an isomorphism, so we have to show that b defined in (21) is injective. Assume $\varphi \circ \partial'_q = \delta_{q-1}\chi$, i.e.

  $$\varphi \circ \partial'_q = \chi \circ \partial_q = \chi \circ j_{q-1} \circ i_{q-1} \circ \partial'_q.$$

  $\partial'_q : C_q \longrightarrow B_{q-1}$ is surjective by definition, hence $\varphi = \chi \circ j_{q-1} \circ i_{q-1} = i^*_{q-1}(\chi \circ j_{q-1})$. Hence $\varphi \in \mathrm{image}(i^*_{q-1})$.
  2. $h \circ \rho = 0$: Trivial, since $(\rho(\varphi))([c_q]) = \varphi(\partial'_q c_q) = 0$ since homology classes are represented by cycles.
  3. $\ker(h) \subset \mathrm{image}(\rho)$: Assume that $[\psi]$ is a cocylce (with values in G) in $\ker(h)$. Then $\langle \psi, z \rangle = 0$ for all cycles. Therefore, $\psi \circ j_q \equiv 0$ so $\psi \in \ker(j^*_q)$. By the exactness of (20) ψ is in the image of ∂'^*_q.
  4. h is surjective. We know that already and we have also constructed a right inverse of h establishing that the sequence splits.

5. The naturality of the sequence is a formal exercise. As in the case of the universal coefficient theorem, the splitting is not natural.

- **Corollary:** Let $f_* : (C, \partial) \longrightarrow (C', \partial')$ be a chain map between chain complexes whose chain groups are free Abelian such that
$$f_* : H_q(C) \longrightarrow H_q(C')$$
is an isomorphism for all q. Then f_* also induces isomorphisms
$$(f_*)^* : H^q(C', G) \longrightarrow H^q(C, G)$$
for all Abelian groups G and all $q \in \mathbb{Z}$.
- **Proof:** This follows from the five-Lemma and the universal coefficient theorem for cohomology:
$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ext}(H_{q-1}(C'), G) \arrow[r, "\rho"] \arrow[d, "(f_*)^*"] & H^q(C', G) \arrow[r, "h"] \arrow[d, "(f_*)^*"] & \mathrm{Hom}(H_q(C'), G) \arrow[r] \arrow[d, "(f_*)^*"] & 0 \\
0 \arrow[r] & \mathrm{Ext}(H_{q-1}(C), G) \arrow[r, "\rho"] & H^q(C, G) \arrow[r, "h"] & \mathrm{Hom}(H_q(C), G) \arrow[r] & 0
\end{tikzcd}$$
Because the outer vertical maps are isomorphisms by assumption, the same is true for the vertical map in the middle.
- There is yet another object we could compare $H^q(C, G)$ with, namely $\mathrm{Hom}_G(H_q(C \otimes G), G)$ where $\mathrm{Hom}_G(\cdot, \cdot)$ denotes G linear maps. We state the facts when $G = K$ is a field.
- Then $C \otimes K$ is a chain complex whose chain groups are vector spaces over K with K-linear boundary operator. The scalar product (only a K-bilinear form, but the terminology is standard)
$$\langle \cdot, \cdot, \rangle : H^q(C, K) \times H_q(C, K) \longrightarrow K$$
$$\left(\varphi, \left[c = \sum_i c_i \otimes k_i\right]\right) \longmapsto \sum_i \varphi(c_i) k_i$$
is well defined.
- **Theorem:** In this situation,
$$H^q(C, K) \longrightarrow \mathrm{Hom}_K(H_q(C \otimes K), K)$$
$$[\varphi] \longmapsto (c \longmapsto \langle \varphi, v \rangle)$$
is an isomorphism of K-vector spaces.
- More generally, one can consider chain complexes whose chain groups are free R-modules, and fix an R-module G. If R is a principal ideal domain, then is an analogue to the version of the universal coefficient theorem for cohomology stated above:
$$0 \longrightarrow \mathrm{Ext}_R(H_{q-1}(C), G) \xrightarrow{\rho} H^q(C, G) \xrightarrow{h} \mathrm{Hom}_R(H_q(C), G) \longrightarrow 0$$
Here R denotes R-linear maps and Ext_R is defined using resolutions of R-modules in terms of free R-modules. The key fact is again that if R is a principal ideal domain, then submodules of free r-modules are again free.

  When R is *not* a principal ideal domain, then one can define a sequence of groups $\mathrm{Ext}^n_R(\cdot, \cdot), n \geq 1$. Our $\mathrm{Ext}(\cdot, \cdot)$ corresponds to the case $n = 1$.

12. Lecture on June, 4 – Properties of cohomology

- We return to topology and translate many statements about homology into cohomoplogy.
- **Coboundary operator for singular cohomology:** Let $\varphi \in C^n(X, G)$ and σ a $n+1$-simplex. By definition of δ^n
$$(\delta^n \varphi)(\sigma) = \sum_{i=1}^{n+1} (-1)^i \varphi(\sigma \circ [v_0, \dots, \widehat{v}_i, \dots, v_{n+1}]).$$
The singular cochain complex computes the singular cohomology $H^*(X; G)$ with values in G.
- $H^0(X; G)$**:** Since $H_{-1}(\cdot;\cdot) = 0$ it follows $H^0(X; G) = \mathrm{Hom}(H_0(X), G)$.
  Singular 0-simplices are points, so a singular 0-cochain can be described using an arbitrary (not necessarily continuous) function $X \longrightarrow G$. In this description, 0-cocyle are functions which are constant on path connected components, i.e.
$$H_0(X; G) \simeq \big\{\{\text{path components of } X\} \longrightarrow G\big\}.$$
- $H^1(X; G)$**:** Since $H_0(X)$ is free, the universal coefficient theorem for cohomology implies
$$H^1(X; G) = \mathrm{Hom}(H_1(X), G) = \mathrm{Hom}(\pi_1(X), G).$$
The last equality follows from the Hurewicz theorem and the fact that G is Abelian.
- **Reduced cohomology:** Is defined by dualizing the augmented singular chain complex of a space. This implies $\widetilde{H}^i(\cdot;\cdot) = H^i(\cdot;\cdot)$ for $i > 0$. By the univ. coeff. theorem $\widetilde{H}^0(X; G) \simeq \mathrm{Hom}(\widetilde{H}^0(X; G))$. This can be interpreted as
$$\{G\text{−valued functions, constant on path components}\}/\{\text{constant } G\text{−valued functions}\}.$$
- **Relative cohomology:** Let (X, A) be a pair of spaces. Since $C_n(X, A)$ is free (it isomorphic to the subgroup of $C_*(X)$ which is generated by simplices which are not contained in A), the exact sequence
$$0 \longrightarrow C_n(A) \longrightarrow C_n(X) \longrightarrow C_n(X, A) \longrightarrow 0$$
splits and its dual sequence
$$0 \longleftarrow C^n(A, G) = \mathrm{Hom}(C_n(A), G) \longleftarrow C^n(X, G) \longleftarrow C^n(X, A; G) \longleftarrow 0$$
is also exact. The dual of the inclusion map $i : A \longrightarrow X$ is the restriction of a cochain to chains in A. Conversely, a cochain $C^n(A, G)$ extends to a cochain in $C^n(X, G)$ by assigning 0 to all singular simplices which are not in A. This provides a splitting of the above sequence. The kernel of i^* consists of cochains on X which vanish on singular simplices in A. Thus $\ker(i^*) = \mathrm{Hom}(C_n(X, A), G) = C^n(X, A; G)$.
- **Long exact sequence of a pair/triple:** Any exact sequence of cochain complexes induces a long exact sequence in cohomology. This applies to singular cochains on a pair (X, A) of spaces
$$\longrightarrow H^n(X, A; G) \longrightarrow H^n(X; G) \longrightarrow H^n(A; G) \xrightarrow{\delta^*} H^{n+1}(X, A; G) \longrightarrow \dots$$
The connecting homomorphism δ^* is defined as follows. Let $\alpha \in H^n(A; G)$ and σ a relative $n+1$-cycle. Then $\delta^*\alpha(\sigma) = \alpha(\partial\sigma)$. Note, that one extends α to a cochain on X (by zero on singular simplices not entirely contained in A) $\partial\sigma$ is a boundary in X but not necessarily in A. In particular, the

connecting homomorphisms in the long exact sequence of a pair for homology and cohomology are compatible in the sense that

(22)
$$\begin{tikzcd} H^n(A;G) \arrow[r, "\delta^*"] \arrow[d, "h"] & H^n(X,A;G) \arrow[d, "h"] \\ \mathrm{Hom}(H_n(A);G) \arrow[r, "(\partial_*)^*"] & \mathrm{Hom}(H_{n+1}(X,A);G) \end{tikzcd}$$

commutes (here h is the map defined in (19)).

An analogous long exact sequence for a triple of spaces $X \supset A \supset B$ is obtained by dualizing the split exact sequence

$$0 \longrightarrow C_n(A,B) \longrightarrow C_n(X,B) \longrightarrow C_n(X,A) \longrightarrow 0.$$

- **Induced maps:** Maps between pairs of spaces induces maps (in the opposite direction) between cohomology groups. The long exact sequence is natural with respect to this. Note $(g \circ f)^* = f^* \circ g^*$ and $\mathrm{id}_X^* = \mathrm{id}_{H^*(X;G)}$.
- **Homotopy invariance:** If $f, g : (X,A) \longrightarrow (Y,B)$ are homotopic, then $f^* = g^*$ on cohomology and the maps f^*, g^* are chain homotopic, i.e. there is $P : C^*(Y,B) \longrightarrow C^{*-1}(X,A)$ such that
$$f^* - g^* = P \circ \delta + \delta \circ P.$$
The proof is dual to the proof for homology.
- **Excision:** Assume that $Z \subset A \subset X$ such that $\overline{Z} \subset \mathring{A}$. Then the inclusion
$$i : (X \setminus Z, A \setminus Z) \longrightarrow (X,A)$$
induces an isomorphism
$$i^* : H^*(X,A) \longrightarrow H^*(X \setminus Z, A \setminus Z).$$
The proof is by the universal coefficient theorem and the five lemma, or one can dualize the proof of excision for homology.
- **Mayer-Vietoris sequence for cohomology:** We give the relative version: Let $(X,Y), (A,C), (B,D)$ be pairs of spaces such that $X = \mathring{A} \cup \mathring{B}$ and $Y = \mathring{C} \cup \mathring{D}$ (in the relative topology).

  We denote the open cover $\{A,B\}$ (resp. $\{C,D\}$) of X (resp. Y) by $\mathfrak{U}$ (resp. $\mathfrak{V}$). The chain complex $C_*^{\mathfrak{U}}(X)$ is generated by singular simplices contained in A or B, etc. We showed last semester that the inclusion
$$C_*^{\mathfrak{U}}(X) \longrightarrow C^*(X)$$
is a chain homotopy equivalence. We dualize this notation and define
$$C^n_{\mathfrak{U},\mathfrak{V}}(X,Y;G) = \ker(i^* : C^n_{\mathfrak{U}}(X;G) \longrightarrow C^n_{\mathfrak{V}}(Y;G).$$
The following diagram commutes and the rows are exact.
$$\begin{tikzcd} 0 \arrow[r] & C^n(X,Y;G) \arrow[r] \arrow[d] & C^n(X;G) \arrow[r] \arrow[d] & C^n(Y;G) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & C^n_{\mathfrak{U},\mathfrak{V}}(X,Y;G) \arrow[r] & C^n_{\mathfrak{U}}(X;G) \arrow[r] & C^n_{\mathfrak{V}}(Y;G) \arrow[r] & 0 \end{tikzcd}$$
and the associated pair of long exact sequences in cohomology. By the five Lemma, the first vertical map induces an isomorphism on cohomology. The

relative Mayer-Vietoris sequence for cohomology is the long exact sequence of the short exact sequence

$$0 \longrightarrow C^n_{\mathfrak{U},\mathfrak{V}}(X,Y;G) \longrightarrow C^n(A,C;G) \oplus C^n(B,D;G) \longrightarrow C^n(A\cap B, C\cap D;G) \longrightarrow 0$$

obtained by dualizing

$$0 \longrightarrow C_n(A\cap B, C\cap D) \longrightarrow C_n(A,C) \oplus C_n(B,D) \longrightarrow C_n^{\mathfrak{U},\mathfrak{V}}(X,Y) \longrightarrow 0.$$

- **Cellular cohomology:** As for homology, the cohomology of a CW-complex can be computed from A CW-decomposition. This is very similar to the case of homology. Consider the diagram

$$\begin{tikzcd}
 & 0 = H^{n-1}(X^{n-2};G) & & & \\
 & H^{n-1}(X^{n-1};G) \arrow[u] \arrow[dr, "\delta^{n-1}"] & & & \\
 {} \arrow[r] & H^{n-1}(X^{n-1},X^{n-2};G) \arrow[u, "j_{n-1}"] \arrow[r, "d^{n-1}"'] & H^n(X^n,X^{n-1};G) \arrow[r, "d^n"] \arrow[dr, "j_n"'] & H^{n+1}(X^{n+1},X^n;G) \arrow[r] & {} \\
 & & & H^n(X^n;G) \arrow[u, "\delta^n"] & \\
 & & & H^n(X^{n+1};G) \simeq H^n(X;G) \arrow[u, "i_n"] & \\
 & & & 0 = H^n(X^{n+1},X^n;G) \arrow[u] &
\end{tikzcd}$$

where the maps d^n, d^{n-1} are defined as the obvious composition. The vertical sequences are exact sequences of pairs. $d^n \circ d^{n-1} = 0$ since $j_n \circ \delta^{n-1}$ is a composition of two consecutive maps from the long exact cohomology sequence of the pair (X^n, X^{n-1}). The row forms the cellular cochain complex, its homology is isomorphic to the cohomology of X, i.e.

$$H^n(X;G) = \frac{\ker(d^n)}{\mathrm{image}(i_n)}$$

1. $H^i(X^n, X^{n-1};G) = 0$ for $i \neq n$ by the universal coefficient theorem.
2. Therefore, $H^i(X^n;G) \longrightarrow H^i(X^{n-1};G)$ is an isomorphism unless $i = n, n-1$.
3. In particular, $H^i(X^n;G) = 0$ for $i \geq n+1$.
4. $H^i(X, X^n;G) = 0$ for $i \leq n$ by the universal coefficient theorem.
5. Therefore, $H^n(X;G) \simeq H^n(X^{n+1};G)$.
6. j_n is surjective, by the univ. coefficient theorem and the fact that $H_{n-1}(X^{n-1})$ is free. $H^n(X^n,X^{n-1};G) \xrightarrow{j_n} H^n(X^n;G) \longrightarrow H^n(X^{n-1};G)$ is a segment of the long exact sequence in cohomology of (X^n, X^{n-1}).

We can conclude:

$$\begin{aligned} H^n(X;G) &= H^n(X^{n+1};G) = \mathrm{image}(i_n) \\ &= \ker(\delta^n) = \frac{\ker(d_n)}{\ker(j_n)} \\ &= \frac{\ker(d^n)}{\mathrm{image}(\delta^{n-1})} \\ &= \frac{\ker(d^n)}{\mathrm{image}(d^{n-1})} \end{aligned}$$

- The cellular cochain complex could have been obtain by dualizing the cellular chain complex. The cellular coboundary operator is the composition in the upper row

$$\begin{tikzcd} H^k(X^k,X^{k-1};G) \arrow[r, "j_k"] \arrow[d, "h"] & H^k(X^k;G) \arrow[r, "\delta^k"] \arrow[d, "h"] & H^{k+1}(X^{k+1},X^k;G) \arrow[d, "h"] \\ \mathrm{Hom}(H_k(X^k,X^{k-1}),G) \arrow[r] & \mathrm{Hom}(H_k(X^k),G) \arrow[r, "(\partial_*)^*"] & \mathrm{Hom}(H_{k+1}(X^{k+1}),X^k,G) \end{tikzcd}$$

The outer vertical maps are isomorphisms by the universal coefficient theorem, the horizontal map on the lower left (like the one in the upper left corner) is the dual of a map induced by the inclusion $X^k \longrightarrow (X^k, X^{k-1})$. Both small diagrams commute. For the right one, see (22), for the left one use the the naturality of the universal coefficient theorem.

Lecture on June, 7 – Products for cohomology, cross-prod.

- **Cohomology cross product:** We assume that G is commutative ring with unit and X a space. For $\varphi \in \mathrm{Hom}(C_p(X),G) = C^p(X;G)$ and $\sigma \in C_r(X)$ we define $\varphi(\sigma) = 0$ if $r \neq p$. For $\psi \in C^q(X;G)$ define $\varphi \times \psi \in \mathrm{Hom}(C_*(X) \otimes C_*(Y), G)$ by

$$(\varphi \times \psi)(c \otimes d) = \varphi(c) \cdot \psi(d). \tag{23}$$

- **Lemma:** $\delta(\varphi \times \psi) = (\delta\varphi) \times \psi + (-1)^p \varphi \times \delta(\psi)$
  by the definition of δ and the tensor product of chain complexes. If φ, ψ are cocycles and $\varphi' = \varphi + \delta\alpha$, then

$$\begin{aligned} \varphi' \times \psi &= (\varphi + \delta\alpha) \times \psi \\ &= \varphi \times \psi + \delta(\alpha \times \psi) \end{aligned}$$

because ψ is a cocyle. This shows that $[\varphi] \times [\psi] \in H^{p+q}(C^*(X) \otimes C^*(Y))$ is well defined.
- **Simple Fact:** If A, B be Abelian groups and G a commutative ring (with unit, see below). Then there is a natural map

$$\begin{aligned} \mathrm{Hom}(A,G) \otimes \mathrm{Hom}(B,G) &\longrightarrow \mathrm{Hom}(A \otimes B, G) \\ \alpha \otimes \beta &\longmapsto \big(a \otimes b \longmapsto \alpha(a) \cdot_G \beta(b)\big). \end{aligned} \tag{24}$$

This is not an isomorphism, in general. The following example can be found at `https://mathoverflow.net/questions/56255/duals-and-tensor-products`.

- **Example:** Let $A = B = \bigoplus_{i\in\mathbb{N}_0} \mathbb{Z}$, $G \in \mathbb{Z}$, with standard generators $e_i, i \in \mathbb{N}_0$. Define

$$\begin{aligned} \xi : A \otimes A &\longrightarrow \mathbb{Z} \\ e_i \otimes e_j &\longmapsto \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases} \end{aligned}$$

This is not in the image of the map (24). Let $\varphi \in (A \otimes A)^*$ Consider

$$\begin{aligned} \Delta_\varphi : A &\longrightarrow A^* \\ b &\longmapsto (a \longmapsto \varphi(a \otimes b)) \end{aligned}$$

If ξ is of the form $\sum_{m=0}^{r} \alpha_i \otimes \alpha_i'$ for suitable α_i, α_i', then the dimension of the image of Δ_ξ is $\leq r$. However, $\Delta_\xi(e_i)$ is the dual of e_i. Thus the map in (24) is not surjective even when A, B are free.

- **Simple fact:** If A is a finitely generated and free Abelian group (finitely gen. free G-module of a ring G), then $A = \mathbb{Z}^r$ and

$$\begin{aligned} (\mathbb{Z}^r \otimes B)^* &\simeq \bigoplus_r B^* \\ &\simeq (\mathbb{Z}^r)^* \otimes B^*. \end{aligned}$$

Thus the map in (24) is an isomorphism if one of the factors A, B is free and finitely generated.

There are examples which show that (24) is not injective for finitely generated A, B.

- **Definition:** Fix an Eilenberg-Zilber equivalence

$$Q : C_*(X \times Y) \longrightarrow C_*(X) \otimes C_*(Y)$$

and consider its dual $Q^* : C^*(X;G) \otimes C^*(Y;G) \longrightarrow C^*(X \times Y;G)$. The cohomology cross product of $[\varphi] \times [\psi] \in H^{p+q}(X \times Y;G)$ is defined by

$$[\varphi] \times [\psi] = [Q^*(\varphi \otimes \psi)].$$

- **Remark:** This is independent of the choice of Q.
- **Properties:** The cohomology cross product satisfies
  1. bilinear: $(\alpha + \alpha') \times \beta = \alpha \times \beta + \alpha' \times \beta$
  2. homogeneous: $(g \cdot \alpha) \times \beta = g(\alpha \times \beta)$ for all $g \in G$.
  3. (skew-)commutuative: $\alpha \times \beta = (-1)^{pq} t^*(\beta \times \alpha)$ with $t : X \times Y \longrightarrow Y \times X$ interchanging the factors and $\alpha \in H^p(X;G), \beta \in H^q(Y;G)$.
  4. associative: $\alpha \times (\beta \times \gamma) = (\alpha \times \beta) \times \gamma$.
  5. natural: $f^*\alpha \times g^*\beta = (f \times g)^*(\alpha \times \beta)$ for continuous maps $f : X \longrightarrow X'$ and $g : Y \longrightarrow Y'$.
  6. unit: Let $1_X \in H^0(X;G)$ be the cocylce which assigns the unit element of the ring G to every 0-simplex. Then $1_X \times \beta = \mathrm{pr}_Y^*\beta$ where $\mathrm{pr}_Y : X \times Y \longrightarrow Y$ is the projection.

These properties follow quite directly from the analogous properties of the homology cross product.

- There is the following simple relationship between the homology and cohomology cross product:

$$\langle \alpha \times \beta, a \times b \rangle = \langle \alpha, a \rangle \cdot \langle \beta, b \rangle \tag{25}$$

- **Proof:** Q is a chain homotopy inverse of P. Hence,

$$\begin{aligned}\langle \alpha \times \beta, a \times b\rangle &= \langle Q^*(\alpha \otimes \beta), P(a \otimes b)\rangle \\ &= \langle \alpha \otimes \beta, QP(a \otimes b)\rangle \\ &= \langle \alpha \otimes \beta, (a \otimes b)\rangle \\ &= \alpha(a) \cdot \beta(b).\end{aligned}$$

- **Theorem:** If one of the following assumptions holds,
  (a) $G = \mathbb{Z}$ and $H_i(X), H_i(Y)$ is free Abelian of finite rank for all $0 \le i \le n$.
  (b) G is a field and the G-vector spaces $H_i(X; G)$ has finite dimension for all $0 \le i \le n$.

  then the cross product induces an isomorphism

$$\begin{aligned}\lambda : \bigoplus_{i+j=n} H^i(X; G) \otimes_G H^j(Y; G) &\longrightarrow H^n(X \times Y; G) \\ \alpha \otimes \beta &\longmapsto \alpha \times \beta.\end{aligned} \tag{26}$$

- **Proof:**

$$\begin{tikzcd} H^n(X \times Y; G) \arrow[r, "f"] & \bigoplus \mathrm{Hom}_G(H_i(X; G) \otimes_G H_j(Y; G), G) \\ \bigoplus H^i(X; G) \otimes H^j(Y; G) \arrow[u, "\lambda"] \arrow[r, "f'"] & \bigoplus \mathrm{Hom}_G(H_i(X; G), G) \otimes_G \mathrm{Hom}_G(H_j(Y; G), G) \arrow[u, "\mu"'] \end{tikzcd}$$

  Here λ is the map from the statement, $\mu(\varphi \otimes \psi)(a \otimes b) = \varphi(a) \cdot \psi(b)$. This map is an isomomorphism by the assumption (finite rank, free). The map λ is injective by the universal coefficient theorem and the assumptions.

  f is defined using the Künneth theorem for homology, after dualizing. This is an isomorphism.

  f' is a sum of tensor products of maps from the universal coefficient theorem in homology. Again, this is an isomorphism by the assumptions.

- **Remark:** The following fact is supposed to illustrate how the cross product in cohomology works.

  For arbitrary $(X, A), Y$ and G the following square commutes:

$$\begin{tikzcd} H^k(A; G) \times H^l(Y; G) \arrow[r, "\delta^* \times \mathrm{id}"] \arrow[d, "\times"] & H^{k+1}(X, A; G) \times H^l(Y; G) \arrow[d, "\times"] \\ H^{k+l}(A \times Y; G) \arrow[r, "\delta^*"] & H^{k+l+1}(X \times Y, A \times Y; G) \end{tikzcd} \tag{27}$$

  This can be checked directly, start with $[\varphi] \in H^k(A; G), [\psi] \in H^l(Y; G)$, i.e. φ is a G-valued cocylce φ on $C_k(A)$. Extend somehow to a G-valued cochain $\overline{\varphi}$ on $C_k(X)$. Then

$$\begin{tikzcd} ([\varphi], [\psi]) \arrow[r, mapsto] & ([\delta\overline{\varphi}], [\psi]) \arrow[d, mapsto] \\ & {[Q^*(\delta\overline{\varphi} \otimes \psi)]} \end{tikzcd}$$

  Since ψ is a cocylce, $\delta\psi = 0$, $Q^*(\overline{\varphi} \otimes \psi)$ is an extension of $Q^*(\varphi \otimes \psi)$ by naturality of Q, and Q^* is a chain map. Therefore,

$$[Q^*(\delta\overline{\varphi} \otimes \psi)] = \delta^*[Q^*(\varphi \otimes \psi)].$$

This shows that the diagram commutes.

LECTURE ON JUNE, 11 – PRODUCTS FOR COHOMOLOGY, CUP-PROD.

- **Cup-product:** We consider a topological space X and the diagonal map
$$\begin{aligned} d: X &\longrightarrow X \times X \\ x &\longmapsto (x,x). \end{aligned}$$
- **Definition:** The cup-product of $\alpha \in H^p(X;G)$ and $\beta \in H^q(X;G)$ is
$$\alpha \cup \beta = d^*(\underbrace{\alpha \times \beta}_{\text{cohom. cross prod.}}) \in H^{p+q}(X;G).$$
- **Properties:** From the properties of the cross product we obtain properties of the cup-product:
  1. bilinear: $(\alpha + \alpha') \cup \beta = \alpha \cup \beta + \alpha' \cup \beta$
  2. homogeneous: $(g \cdot \alpha) \cup \beta = g(\alpha \cup \beta)$ for all $g \in G$.
  3. (skew-)commutative: $\alpha \cup \beta = (-1)^{pq} \beta \cup \alpha$ with $\alpha \in H^p(X;G), \beta \in H^q(Y;G)$.
  4. associative: $\alpha \cup (\beta \cup \gamma) = (\alpha \cup \beta) \cup \gamma$.
  5. natural: $f^*\alpha \cup f^*\beta = f^*(\alpha \cup \beta)$ for continuous maps $f : X \longrightarrow X'$
  6. unit: Let $1_X \in H^0(X;G)$ be the cocylce which assigns the unit element of the ring G to every 0-simplex. Then $1_X \cup \beta = \beta$.
- **Remark:** This is one of the justification for introducing cohomology. The cup-product turns the cohomology of a space into a graded ring.
- **Remark:** There is an explicit formula describing an Eilenberg Zilber equivalence:
$$\begin{aligned} Q : C_n(X \times X) &\longrightarrow \bigoplus_{p+q=n} [C_*(X) \otimes C_*(X)]_n \\ \sigma &\longmapsto \sum_{q=0}^{n} (\mathrm{pr}_1 \circ \sigma \circ [v_0, \ldots, v_q]) \otimes (\mathrm{pr}_2 \circ \sigma \circ [v_q, \ldots, v_n]) \in [C_*(X) \otimes C_*(Y)]_n. \end{aligned}$$

  This leads to an explicit formula for the representative of $[\alpha] \cup [\beta]$: If $[\alpha] \in H_p(X)$ and $[\beta] \in H_q(X)$, then the cup product is represented by the cochain χ with
$$\langle \chi, \sigma \rangle = \sum_{q=0}^{p+q} \langle \alpha, \sigma \circ [v_0, \ldots, v_i] \rangle \langle \beta, \sigma \circ [v_i, \ldots, v_{p+q}] \rangle \tag{28}$$
  for all singular $p+q$ simplices σ.
- **Remark:** By the naturality of Q, one also obtains cup products
$$\begin{aligned} H^*(X;G) \times H^*(X,A;G) &\longrightarrow H^*(X,A;G) \\ H^*(X,A;G) \times H^*(X;G) &\longrightarrow H^*(X,A;G) \\ H^*(X,A;G) \times H^*(X,A;G) &\longrightarrow H^*(X,A;G). \end{aligned}$$
- **Remark:** If A, B are open (or subcomplexes of a CW-complex), then the cup product
$$H^*(X,A;G) \times H^*(X,B;G) \longrightarrow H^*(X, A \cup B; G).$$
  is defined. On the cochain level
$$C^*(X,A;G) \times C^*(X,B;G) \longrightarrow \{\text{cochains vanishing on sums of chains in } A \text{ or } B\}.$$

By the 5-Lemma and the fact that $C_*(A \cup B) \longrightarrow C_*^{\{A,B\}}(A \cup B)$ is a chain homotopy equivalence we get that

$$C^*(X, A \cup B; G) \longrightarrow \{\text{cochains vanishing on sums of chains in } A \text{ or } B\}$$

induces an isomorphism on cohomology.

- **Remark:** The definition we used for the cup-product is difficult to use for explicit computations, and it is generally difficult to compute cup-products. The following two facts are useful:
  1. If $H^n = 0$, then all cup products which land in degree n vanish.
  2. Let n be odd, and $u \in H^n(X)$. Then $u^2 = (-1)^{n^2} u^2 = -u^2$. Thus u^2 is part of the 2-torsion of $H^{2n}(X)$.
- **Remark:** Cup products in $X \times Y$ can be computed once the cup product in $H^i(X; G), H^i(Y; G)$ are known:
- **Lemma:**

$$(\alpha \times \beta) \cup (\alpha' \times \beta') = (-1)^{qr} (\alpha \cup \alpha') \times (\beta \cup \beta'). \tag{29}$$

- **Proof:** Let d_X, d_Y, d be the diagonal map for $X, Y, X \times Y$ and $t : Y \times X \longrightarrow X \times Y$ be the map interchanging the factors. Then

$$\begin{aligned}(\alpha \times \beta) \cup (\alpha' \times \beta') &= d^* Q^*((\alpha \times \beta) \times (\alpha' \times \beta')) \\ &= (-1)^{qr} d^* Q^*(\alpha \times t^*(\alpha' \times \beta) \times \beta') \\ &= (-1)^{qr} d^* (\mathrm{id} \times t \times \mathrm{id})^* Q^*(\alpha \times \alpha' \times \beta \times \beta') \\ &= (-1)^{qr} (d_x \times d_Y)^* Q^*(\alpha \times \alpha' \times \beta \times \beta') \\ &= (-1)^{qr} (\alpha \cup \alpha') \times (\beta \cup \beta').\end{aligned}$$

- **Definition:** The tensor product of graded rings $(A_i), (B_j)$ is $\left(\bigoplus_{i+j=n} A_i \otimes B_j\right)$ with the product

$$(a \otimes b) \cdot (a' \otimes b') = (-1)^{qr} (a \cdot a') \otimes (b \cdot b')$$

when b resp. a' has degree q resp. r. In particular, it is associative.

- **Theorem:** Under one of the assumptions for (26) the map

$$\begin{aligned}\lambda : H^*(X; G) \otimes_G H^*(Y; G) &\longrightarrow H^*(X \times Y; G) \\ \alpha \otimes \beta &\longmapsto \mathrm{pr}_X^* \alpha \cup \mathrm{pr}_Y^* \beta.\end{aligned}$$

is an isomorphism of rings.

- **Proof:** This follows from the corresponding theorem for the cross product, and the previous lemma since

$$\begin{aligned}\mathrm{pr}_X^* \alpha \cup \mathrm{pr}_Y^* \beta &= (\alpha \times 1_Y) \cup (1_X \times \beta) \\ &= (\alpha \cup 1_X) \times (1_Y \cup \beta) \\ &= \alpha \times \beta.\end{aligned} \tag{30}$$

- **Remark:** By the identity it is possible to construct the cohomology cross product from the cup product. This is the approach taken in [Ha] where the cup product is defined via (28).

Lecture on June, 14 – Products for cohomology, Examples

- **Example:** The cohomology of the n-torus (with integer coefficients) is an exterior algebra on $\mathrm{rank}(H^1(T^n;\mathbb{Z})) = n$ generators of degree 1.
- **Fundamental Example 1:** Let $\mathbb{R}^n = \mathbb{R}^i \times \mathbb{R}^j$. The product of two generators on the left in
$$H^i(\mathbb{R}^n, \mathbb{R}^n \setminus \mathbb{R}^j;\mathbb{Z}) \times H^j(\mathbb{R}^n, \mathbb{R}^n \setminus \mathbb{R}^i;\mathbb{Z}) \longrightarrow H^n(\mathbb{R}^n, \mathbb{R}^n \setminus 0;\mathbb{Z})$$
is a generator on the right.
- **Proof:** This is based on (18) on p. 25 and follows then from (29) (this formula also works in relative cohomology, with the same proof).
- **Example:** Let $X = S^n \times S^m$. Then the cohomology ring has two generators, α in degree n, the other, β, in degree m. Their product $\alpha \cup \beta$ is a generator of $H^{n+m}(X;Z)$, and $\alpha^2 = 0$ and $\beta^2 = 0$ by naturality of the cup product and the ring structure of the individual factors.
- **Exercise:** Let X, Y be path connected CW-complexes and x, y base points. Then
$$H^*(X \vee Y;G) \simeq H^*(X;G) \oplus H^*(Y;G) \text{ outside of degree } 0$$
as rings.
- **Consequence:** $S^n \vee S^m \vee S^{n+m}$ and $S^n \times S^m$ have isomorphic (co-)homology groups, if $n, m > 1$ then both spaces are simply connected. However, they are not homotopy equivalent because the ring structures of the cohomology are not isomorphic.

  In particular, the attaching map of the $n+m$-cell in $S^n \times S^m$ to $S^n \vee S^m$ is not null homotopic. There is no map from $S^n \vee S^m \vee S^{n+m}$ to $S^n \times S^m$ inducing an isomorphism in cohomology.
- **Fundamental Example 2:** $H^*(\mathbb{RP}^n;\mathbb{Z}_2) = \mathbb{Z}[\alpha]/\alpha^{n+1}$ (as ring) where $\alpha \in H^1(\mathbb{RP}^n;\mathbb{Z}_2)$ is a generator.
- **Proof:** $\mathbb{Z}_2$-coefficients all over the place. For $i + j = n, i \geq 0$
$$\begin{aligned} \mathbb{RP}^i &= \{[x_0 : \ldots : x_i : 0 \ldots 0]\} \subset \mathbb{RP}^n \\ \mathbb{RP}^j &= \{[0 \ldots : x_i : \ldots : x_n]\} \subset \mathbb{RP}^n \\ U &= \{[x_0 \ldots : x_{i-1} : x_i : x_{i+1} : \ldots : x_n] \mid x_i \neq 0\} \end{aligned}$$

Then $\mathbb{RP}^i \cap \mathbb{RP}^j = [0 : \ldots : 0 : x_i = 1 : 0 : \ldots : 0] = p$, and (U, p) is homeomorphic to $(\mathbb{R}^n, 0)$ via a homeomorphism h. We identify $\mathbb{R}^n = \mathbb{R}^i \times \mathbb{R}^j$ where the coordinates of $\mathbb{R}^i$ are $x_0, \ldots, x_{i-1}$. Consider the diagram

$$\begin{tikzcd} H^i(\mathbb{RP}^n) \times H^j(\mathbb{RP}^n) \arrow[r, "\cup"] & H^n(\mathbb{RP}^n) \\ H^i(\mathbb{RP}^n, \mathbb{RP}^n \setminus \mathbb{RP}^j) \times H^j(\mathbb{RP}^n, \mathbb{RP}^n \setminus \mathbb{RP}^i) \arrow[r, "\cup"] \arrow[u] \arrow[d] & H^n(\mathbb{RP}^n, \mathbb{RP}^n \setminus \{p\}) \arrow[u] \arrow[d] \\ H^i(\mathbb{R}^n, \mathbb{R}^n \setminus \mathbb{R}^j) \times H^j(\mathbb{R}^n, \mathbb{R}^n \setminus \mathbb{R}^i) \arrow[r, "\cup"] & H^n(\mathbb{R}^n, \mathbb{R}^n \setminus 0) \end{tikzcd}$$

whose upward pointing arrows are induced by inclusions while the downward pointing arrows are induced by h. The diagram commutes by naturality of the cup-product.

By the Fundamental Example 1, the cup-product of the generators of the groups in the lower left is a generator of $H^n(\mathbb{R}^n, \mathbb{R}^n \setminus 0)$. We will show that the vertical maps in the diagram above are isomorphisms. This implies the claim.

1. The map $H^n(\mathbb{RP}^n, \mathbb{RP}^n \setminus \{p\}) \longrightarrow H^n(\mathbb{R}^n, \mathbb{R}^n \setminus 0)$ is an isomorphism by excision (remove $\mathbb{RP}^{n-1} = \{x_i = 0\}$ form $\mathbb{RP}^n$).
2. $\mathbb{RP}^{n-1}$ is a deformation retract of $\mathbb{RP}^n \setminus p$. Hence $H^i(\mathbb{RP}^n, \mathbb{RP}^n \setminus p) \simeq H^i(\mathbb{RP}^n, \mathbb{RP}^{n-1})$. The vertical map on the upper right is an isomorphism by the long exact sequence of the pair $(\mathbb{RP}^n, \mathbb{RP}^{n-1})$, note that $H^{n-1}(\mathbb{RP}^n) \longrightarrow H^{n-1}(\mathbb{RP}^{n-1})$ is injective.
3. We now look at the left column and consider

$$\begin{tikzcd}
H^i(\mathbb{RP}^n) \arrow[d, "\text{cell. hom.}"', "\simeq"] & H^i(\mathbb{RP}^n, \mathbb{RP}^{i-1}) \arrow[l, "\text{cell. hom.}"', "\simeq"] \arrow[d, "\text{cell. hom.}"', "\simeq"] & H^i(\mathbb{RP}^n, \mathbb{RP}^n \setminus \mathbb{RP}^j) \arrow[l, "\text{hom. equ.}"', "\simeq"] \arrow[r] \arrow[d] & H^i(\mathbb{R}^n, \mathbb{R}^n \setminus \mathbb{R}^j) \arrow[d, "\text{contract } \mathbb{R}^j"', "\simeq"] \\
H^i(\mathbb{RP}^i) & H^i(\mathbb{RP}^i, \mathbb{RP}^{i-1}) \arrow[l, "\text{cell. hom.}"', "\simeq"] & H^i(\mathbb{RP}^i, \mathbb{RP}^i \setminus p) \arrow[l, "\text{def. equ.}"', "\simeq"] \arrow[r, "\simeq", "\text{exc.} \mathbb{RP}^{i-1}"'] & H^i(\mathbb{R}^i, \mathbb{R}^i \setminus 0)
\end{tikzcd}$$

   The inclusion $(\mathbb{RP}^n, \mathbb{RP}^{i-1}) \longrightarrow (\mathbb{RP}^n, \mathbb{RP}^n \setminus \mathbb{RP}^j)$ is a homotopy equivalence: Note that $\mathbb{RP}^n \setminus \mathbb{RP}^j$ consists of points one of whose first $i-1$ coordinates is non-zero. Thus

$$[x_0 : \ldots : x_{i-1} : x_i : \ldots : x_n] \longmapsto [x_0 : \ldots : x_{i-1} : tx_i : \ldots : tx_n]$$

   is well defined for $t \in [0,1]$.
   The same arguments work when i, j are interchanged. (Multiply the first coordinate entries $0, \ldots, i-1$ by t).

- **Fundamental Example 3:** $H^*(\mathbb{RP}^\infty; \mathbb{Z}_2) \simeq \mathbb{Z}_2[\alpha]$ with $\alpha \in H^1$ the generator.
- **Fundamental Example 4:** The analogous statements with analogous proofs hold for $H^*(\mathbb{CP}^n; \mathbb{Z})$ respectively $H^*(\mathbb{HP}^n; \mathbb{Z})$, except that the generator has degree 2 respectively 4 and similarly for $\mathbb{CP}^\infty, \mathbb{HP}^\infty$.

Lecture on June, 18 – Products for cohomology, Applications and cap product.

- **Observation:** $H^n(\mathbb{RP}^n; \mathbb{Z}_2) \simeq \mathbb{Z}_2 \simeq H^n(S^n; \mathbb{Z}_2)$ and $H^{2n}(\mathbb{CP}^n; \mathbb{Z}) \simeq \mathbb{Z} \simeq H^{2n}(S^{2n}; \mathbb{Z})$. One can hence define degrees between these manifolds (with values in $\mathbb{Z}_2$ or $\mathbb{Z}$ depending on the case). We will see later that this is a property of closed connected manifolds (depending on their orientability).
- **Corollary:** Every map $S^{2n} \longrightarrow \mathbb{CP}^n$ has $\mathbb{Z}$-degree zero.
- **Corollary:** Every map $S^{2n} \longrightarrow \mathbb{RP}^n$ has $\mathbb{Z}_2$-degree zero.
- **Corollary:** A continuous map $f : \mathbb{CP}^n \longrightarrow \mathbb{CP}^n$ has degree a^n with $a \in \mathbb{Z}$. In particular, all continuous maps have non-negative degree if n is even.
- **Proof:** The generator of $H^{2n}(\mathbb{CP}^n; \mathbb{Z})$ is a n-th power.
- **Corollary:** Every map $\mathbb{CP}^n \longrightarrow \mathbb{CP}^n$ has a fixed point when n is even.
- **Proof:** Let $\omega \in H^2(\mathbb{CP}^n; \mathbb{Z})$ be a generator. By the naturality of the cup-product

$$f^*(\omega^k) = (f^*\omega)^k,$$

hence $f^{2k} : H^{2k}(\mathbb{CP}^n;\mathbb{Z}) \longrightarrow H^{2k}(\mathbb{CP}^n;\mathbb{Z})$ is the multiplication with a^k where $f^*(\omega) = a\omega$. Then

$$\begin{aligned} L(f) &= \sum_{k=0}^{n} \big(\mathrm{trace}\,\big(f^* : H^{2k}(\mathbb{CP}^n;\mathbb{Z}) \longrightarrow H^{2k}(\mathbb{CP}^n;\mathbb{Z})\big)\big) \\ &= \sum_{k=0}^{n} a^k. \end{aligned}$$

If $|a| > 1$, then this is always non-zero, the same is true if $|a| = 0$ and $a = 1$. If $a = -1$, then $L(f)$ has an odd number of summands ± 1, hence it is odd and non-vanishing. The claim follows from the Lefschetz fixed point theorem.

- **Reminder:** There are maps from $\mathbb{CP}^n$ to itself without fixed points when n is odd, c.f. 16.
- **Fact:** Let X, Y be connected and x, y be basepoints. Then the cohomolgy ring of $X \vee Y$ is the direct sum of the cohomology rings of X, Y.
- **Consequence:** The attaching map of the the 4-cell e^4 to the 3-skeleton $\mathbb{CP}^1 \simeq S^2$ is not nullhomotopic, i.e. there are maps $\partial e^4 \simeq S^3 \longrightarrow S^2$ which are not homotopic to the constant map. In this case, the map in question is the Hopf map

$$\begin{aligned} S^3 \subset \mathbb{C}^2 &\longrightarrow \mathbb{CP}^1 \\ (z_0, z_1) &\longmapsto [z_0 : z_1]. \end{aligned}$$

If this were null-homotopic, then the cohomology ring of $\mathbb{CP}^2$ would be isomorphic to the cohomology ring of $S^2 \vee S^4$ (as Abelian groups $H^*(S^2 \vee S^4;\mathbb{Z}) \simeq H^*(\mathbb{CP}^2;\mathbb{Z})$).

Considering $\mathbb{HP}^2$ with its standard CW-decomposition one finds a map $h : S^7 \longrightarrow S^4$ which is not null-homotopic. Thus we showed $\pi_3(S^2) \neq 0$ and $\pi_7(S^4) \neq 0$.

- **Definition:** Fix a PID G with unit. Let $\alpha \in C^p(X;G)$ and $c \in C_m(X;G)$. Then the *cap-product* $\alpha \cap c$ is the image of (c, α) under the composition

$$\begin{tikzcd} C_m(X;G) \times C^p(X;G) \arrow[r, "d_* \times \mathrm{id}"] & C_m(X \times X;G) \times C^p(X;G) \arrow[dl, "Q \times \mathrm{id}"] \\ \Big(\bigoplus_{i+j=m} C_i(X;G) \otimes C_j(X;G)\Big) \times C^p(X;G) \arrow[r] & C_{m-p}(X;G). \end{tikzcd}$$

The last arrow is evaluation of the cochain on the first factor. Alternatively, the cap product can be described using (14) as follows (c a singular m-simplex):

(31) $$c \cap \alpha = \langle \alpha, c \circ [v_0, \ldots, v_p] \rangle \cdot c \circ [v_p, \ldots, v_m].$$

- **Properties:**
  1. If $G = \mathbb{Z}$, $c \cap 1 = c$.
  2. The cap product is natural in the sense that the diagram

$$\begin{tikzcd} C_m(X;G) \arrow[d, "f_*"'] & \otimes & C^p(X;G) \arrow[r, "\cap"] & C_{m-p}(X;G) \arrow[d, "f_*"] \\ C_m(Y;G) & \otimes & C^p(Y;G) \arrow[u, "f^*"'] \arrow[r, "\cap"] & C_{m-p}(Y;G). \end{tikzcd}$$

  commutes, i.e. $f_*(c \cap f^*(\alpha)) = (f_* c) \cap \alpha$.

3. Using (31) (or the fact that d_* and Q are chain maps) one obtains

$$\begin{aligned}
(\partial c)\cap\alpha &= \sum_{i=0}^{m}(-1)^i c\circ[v_0,\dots,\widehat{v_i},\dots v_m]\cap\alpha \\
&= \sum_{i=0}^{p}(-1)^i\langle\alpha, c\circ[v_0,\dots,\widehat{v_i},\dots,v_{p+1}]\rangle\cdot c\circ[v_{p+1},\dots,v_m] \\
&\quad + \sum_{i=p+1}^{m}(-1)^i\langle\alpha, c\circ[v_0,\dots,v_p]\rangle\cdot c\circ[v_p,\dots,\widehat{v_i},\dots,v_m] \\
&= \sum_{i=0}^{p+1}(-1)^i\langle\alpha, c\circ[v_0,\dots,\widehat{v_i},\dots,v_{p+1}]\rangle\cdot c\circ[v_{p+1},\dots,v_m] \\
&\quad + \sum_{i=p}^{m}(-1)^i\langle\alpha, c\circ[v_0,\dots,v_p]\rangle\cdot c\circ[v_p,\dots,\widehat{v_i},\dots,v_m] \\
&= \langle\alpha, \partial(c\circ[v_0,\dots,v_{p+1}])\rangle\cdot c\circ[v_{p+1},\dots,v_m] \\
&\quad + (-1)^p\langle\alpha, c\circ[v_0,\dots,v_p]\rangle\cdot\partial(c\circ[v_p,\dots,v_m]) \\
&= c\cap(\delta\alpha) + (-1)^p\partial(c\cap\alpha).
\end{aligned}$$

This means

(32) $$(c\cap\alpha) = (-1)^p((\partial c)\cap\alpha - c\cap\delta\alpha).$$

4. The cap-product gives rise to a well defined operation

$$H_m(X;G)\times H^p(X;G)\longrightarrow H_{m-p}(X;G).$$

If X is path-connected, then the case $m=p$ is the evaluation of the cocycle on the cycle yielding an element in $G=H_0(X;G)$.

5. As in the case of the cup-product, there are relative versions of the cap-product.

$$H_m(X,A;G)\times H^l(X;G)\longrightarrow H_m(X,A;G)$$
$$H_m(X,A;G)\times H^l(X,A;G)\longrightarrow H_m(X;G).$$

Recall that classes in $H^l(X,A;G)$ are represented by cochains vanishing on simplices in A. The cap product

$$C_k(X;G)\otimes C^l(X;G)\longrightarrow C_{k-l}(X;G)$$

restricts to zero on $C_k(A;G)\otimes C^l(X,A;G)$. Recall that $C^*(X,A;G)$ can be identified with those cochains in $C^*(X;G)$ which vanish on $C_*(A)$.) Therefore, there is a cap-product on

$$\frac{C_k(X;G)}{C_k(A;G)}\otimes C^l(X,A;G)\longrightarrow C_{k-l}(X;G).$$

(32) still holds, so we obtain a cap-product

(33) $$H_k(X,A;G)\otimes_G H^l(X,A;G)\longrightarrow H_{k-l}(X;G)$$

If A,B are open, then one also has

$$H_m(X,A\cup B;G)\times H^l(X,B;G)\longrightarrow H_m(X,A;G).$$

6. If one defines the cup product using (28) and the cap product in terms of (31), then one can express a compatibility relation between cap and cup product on the level of (co-)chains.
   Let σ be a singular $k+l$-simplex, $\varphi \in C^l(X;G)$, $\alpha \in C^k(X;G)$
   $$\begin{aligned} \langle \varphi, \sigma \cap \alpha \rangle &= \langle \alpha, \sigma \circ [v_0, \dots, v_k] \rangle \langle \varphi, \sigma \circ [v_k, \dots, v_{k+l}] \rangle \\ &= \langle \alpha \cup \varphi, \sigma \rangle \end{aligned} \tag{34}$$
   This formula means that the map $\alpha\cup : C^l(X;G) \longrightarrow C^{k+l}(X;G)$ coincides with the dual $(\cap\alpha)^*$ of
   $$\cap\alpha : C_{k+l}(X;G) \longrightarrow C_l(X;G).$$
   Passing to homology and cohomology, we obtain the commutative diagram
   $$\begin{tikzcd} H^l(X;G) \arrow[r, "h"] \arrow[d, "\alpha\cup"] & \mathrm{Hom}_G(H_l(X;G),G) \arrow[d, "(\cap\alpha)^*"] \\ H^{k+l}(X;G) \arrow[r, "h"] & \mathrm{Hom}_G(H_{k+l}(X;G),G) \end{tikzcd}$$
7. If h is an isomorphism, for example when G is a field, then the cap product is completely determined in terms of the cup product.

Lecture on June, 21 – Topological manifolds, orientations, fundamental class

- **Reference:** Appendix A of [Mi2], Section 3.3 in [Ha]. The approach to Poincaré-duality taken in [StZ] is slightly more geometric since it relies on simplicial homology and requires manifolds to be simplicial complexes.
- **Definition:** A *topological manifold* of dimension n is a topological Hausdorff space M which is paracompact and locally Euclidian, i.e. for every point $p \in M$ there a neighborhood U and an homeomorphism $h : U \longrightarrow \mathbb{R}^n$ (with $h(p) = 0$).
- One can assume $\mathbb{Z}$-coefficients all over the place. For $K \subset L \subset M$ compact subsets in a manifold M we write
  $$i_{L,K} : (M, M \setminus L) \longrightarrow (M, M \setminus K).$$
- **Reminder:**
  $$H_i(M, M \setminus x) = H_i(\mathbb{R}^n, \mathbb{R}^n \setminus 0) \simeq \begin{cases} \mathbb{Z} & i = n \\ 0 & i \neq n \end{cases}$$
- **Lemma:** Let M be a n-manifold and $K \subset M$ compact. Then $H_i(M, M \setminus K) = 0$ for $i > n$. $\alpha \in H_n(M, M \setminus K)$ is zero iff $i_{K,x*}\alpha = 0 \in H_n(M, M \setminus x)$ for all $x \in K$.
- Proof:
  1. $M = \mathbb{R}^n$, $K \subset M$ a convex set. Fix $B \subset \mathbb{R}^n$ be a large ball containing K. Then ∂B is a deformation retract of $M \setminus x$ and $M \setminus K$. Thus
     $$\rho_{K,y} : H_i(\mathbb{R}^n, \mathbb{R}^n \setminus K) \longrightarrow H_i(\mathbb{R}^n, \mathbb{R}^n \setminus x)$$
     is an isomorphism.

2. Assume $K = K_1 \cup K_2$ such that the lemma is known for $K_1, K_2, K_1 \cap K_2$. From the long exact exact sequence

$$\longrightarrow H_{i+1}(M, M\backslash(K_1 \cap K_2)) \longrightarrow H_i(M, M\backslash K) \longrightarrow H_i(M, M\backslash K_1) \oplus H_i(M, M\backslash K_2) \longrightarrow$$

   This implies the claim (α maps to trivial classes in $H_i(M, M \setminus K_1), \ldots$) By induction, this proves the claim for K a finite union of convex subsets of $\mathbb{R}^n$.

3. K an arbitrary compact subset of $M = \mathbb{R}^n$. Let $\alpha = [\gamma] \in H_i(M, M \setminus K)$. Choose $N \supset K$ such that N is the finite union of compact convex sets and $i_{N,K*}(\alpha') = \alpha$. This is possible since $\partial\gamma$ is supported in a compact set in the complement of K. Cover K by finitely many balls B_j which are disjoint from the support of $\partial\gamma$. Then $\alpha' = [\gamma] \in H_i(M, M \setminus N)$. If $i > n$, then the claim is clear. If $i_{K,x*}\alpha$ vanishes for $x \in B_j$, then $i_{N,y}\alpha' = 0$ for all $y \in B_j$. Thus α' vanishes by what we know, so $\alpha = i_{N,K}\alpha'$ vanishes, too.
4. M arbitrary, $K \subset M$ contained in the domain U of a coordinate system. Then $H_*(U, U \setminus K) \simeq H_i(M, M \setminus K)$ is an isomorphism by excision. The claim follows from the previous step.
5. M arbitrary, K arbitrary: Cover K by finitely many compact pieces contained in charts. The claim follows by induction on the number of pieces, step 2 and 3.

- **Definition:** A *local orientation* $[\mu_x]$ is a generator of $H_n(M, M \setminus x)$.
- Let B_x be a ball around x. If B_x small enough, μ_x is a cycle in $C_*(M, M \setminus B_x)$ and represents a generator of $H_i(M, M \setminus B_x)$. Thus μ_x represents a local orientation for $y \in B_x$ via the isomorphisms
$$H_*(M, M \setminus y) \longleftarrow H_*(M, M \setminus B_x) \longrightarrow H_*(M, M \setminus x)$$
induced by inclusions $i_{B_x,x*} : (M, M \setminus B_x) \longrightarrow (M, M \setminus x), i_{B_x,y*} : (M, M \setminus B_x) \longrightarrow (M, M \setminus y)$. Let $\rho_{x,y} = i_{B_x,y*} \circ i^{-1}_{B_x,x*}$.
- **Definition:** An *orientation* of M is a function $x \longmapsto \mu_x =$ local orientation at x such that for each x there is a neighborhood B_x such that $\mu_y = \rho_{x,y}(\mu_x)$ for all $y \in B_x$. An *oriented manifold* is a manifold with the choice of an orientation.
- **Theorem:** Let M be an oriented n-manifold and $K \subset M$ compact. Then there is a unique class $\mu_K \in H_n(M, M \setminus K)$ such that $i_{K,x*}\mu_K = \mu_x$ with $i_{K,x} : (M, M \setminus K) \longrightarrow (M, M \setminus x)$ the inclusion.
- **Definition:** If M is compact and oriented, then $\mu_M = [M]$ is the *fundamental class* of the oriented manifold M.
- **Proof:** Uniqueness follows from the previous Lemma.

  Existence: If $K \subset B_x$ for $x \in M$ then $i^{-1}_{x*}\mu_x$ is the desired class. If $K = K_1 \cup K_2$ such that μ_1 resp. μ_2 is the desired class for K_1 resp. K_2 then the Mayer Vietoris sequence in homology implies the existence of μ_K:

$$\begin{tikzcd}
0 \arrow[r] & H_n(M, M \setminus K) \arrow[r, "s"] & H_n(M, M \setminus K_1) \oplus H_n(M, M \setminus K_2) \arrow[dll, "t"] \\
H_n(M, M \setminus (K_1 \cap K_2)) \arrow[r] & 0 = H_{n+1}(M, M \setminus K) &
\end{tikzcd}$$

  where s is induced by inclusion, and t is the difference of the inclusions $(M, M \setminus K_i) \longrightarrow (M, M \setminus (K_1 \cap K_2))$. Finally, every compact set is covered by finitely many closed balls B_x as above.

- **Consequence:** Let M be a connected manifold with $H_n(M;\mathbb{Z}) \neq 0$. Then M is $\mathbb{Z}$-orientable: Take a non-zero element in $H_n(M;\mathbb{Z})$. Restricting it to $H_n(M, M \setminus x) \simeq \mathbb{Z}$ we obtain a class which cannot vanish everywhere (by the Lemma on p. 45). But non-vanishing in $\mathbb{Z}$ is independent of x be connectedness, and allows to define a "positive" generator of $H_n(M, M \setminus x) \simeq \mathbb{Z}$. This orients M. , and this shows that every element in $H_n(M;\mathbb{Z})$ is a multiple of the fundamental class.

  Moreover, if M is not compact, then $H_n(M;\mathbb{Z}) = 0$ since otherwise one obtains a class in M which restricts to a non-zero class in $H_n(M, M \setminus x) = \mathbb{Z}$. But every homology class is represented by a singular cycle lying in a compact subset K. This cycle is trivial in $H_n(M, M \setminus x)$ when $x \notin K$. This works for all coefficient groups.

  Thus for connected n-manifolds, $n > 0$ and $M \neq \emptyset$

$$H_n(M;\mathbb{Z}) \simeq \begin{cases} \mathbb{Z} & M \text{ is } \mathbb{Z} - \text{orientable and closed} \\ 0 & \text{otherwise} \end{cases}$$

$$H_n(M;\mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2 & M \text{ is closed} \\ 0 & \text{otherwise} \end{cases}$$

- **Definition:** Let $\gamma \in C^i(M, G)$ be a cochain. It has *compact support* if the is a compact set K such that $\gamma \in C^i(M, M \setminus K; G)$, i.e. γ vanishes on all chains in $M \setminus K$. The cochains with compact support form a subcomplex $C^*_{comp}(M;G)$ of $C^*(M;G)$ whose homology is $H^*_{comp}(M;G)$. If M is compact, then $H^*_{comp}(M;G) = H^*(M;G)$.
- **Remark:** Cohomology with compact support is obtained as a direct limit.
- **Definition/Theorem:** Let I be a directed set, i.e. there is a partial order $\leq$ on I such that for all $\alpha, \beta \in I$ there is $\gamma \in I$ such that $\alpha \leq \gamma$ and $\beta \leq \gamma$. Assume that to each α in I one associates a group G_α and to each pair $\alpha \leq \beta$ one associates a map $f_{\alpha\beta}$ such that $f_{\alpha\gamma} = f_{\beta\gamma} \circ f_{\alpha\beta}$ when $\alpha \leq \beta \leq \gamma$ and $f_{\alpha\alpha} = \text{id}$. For this *directed system* of groups one can define the *direct limit* as follows:

$$\varinjlim_{\alpha} G_\alpha = \left(\bigoplus_{\alpha} G_\alpha \right) / \langle g_\alpha - f_{\alpha\beta}(g_\alpha) \,|\, g_\alpha \in G_\alpha, \alpha \leq \beta \rangle.$$

  Here we view G_α as subgroup of $\oplus_\alpha G_\alpha$.

  For all α there is a map $i_\alpha : G_\alpha \longrightarrow \varinjlim_\alpha G_\alpha$. The direct limit has the following universal property:

  Let A be a group and $h_\alpha : G_\alpha \longrightarrow A$ homomorphisms such that $h_\alpha = h_\beta \circ f_{\alpha\beta}$ for $\alpha \leq \beta$. Then there is a unique map $\psi : \varinjlim_\alpha G_\alpha \longrightarrow A$ such that $\psi \circ i_\alpha = h_\alpha$.
- **Fact:** Let G'_α be a second directed system of groups with morphisms $f'_{\alpha\beta} : G_\alpha \longrightarrow G_\beta$. If $h_\alpha : G_\alpha \longrightarrow G'_\alpha$ are group homomorphisms such that $f_{\alpha\beta} \circ h_\alpha = h_\beta \circ$, then there is a group homomorphism $h : \varinjlim_\alpha G_\alpha \longrightarrow \varinjlim_\alpha G'_\alpha$ such that

$$h \circ i_\alpha = i'_\alpha \circ h_\alpha.$$

- **Example:** Let X be a topological space. Then $I = \{\text{compact subsets of } X\}$ is a directed set (with the partial order given by inclusions, the union of two compact sets is compact). For any coefficient group $K \longmapsto G_K = H_i(K;G)$ with the maps induced by inclusion is a directed set of groups.
- **Lemma:** The inclusions $K \longrightarrow X$ induce an isomorphism $\varinjlim_K H_i(K;G) = H_i(X;G)$.

- **Proof:** Surjectivity: Every class in X is represented by a sing. cycle which is contained in a compact set (as a finite sum of singular simplices). Injectivity: similar argument.
- **Remark:** For $K \subset L$ recall the inclusion of pairs $i_{L,K} : (M, M \setminus K) \longrightarrow (M, M \setminus L)$. With the same proof as in the previous lemma:
- **Lemma:** The maps $K \longrightarrow X$ induce an isomorphism
$$\varinjlim_{K} H^i(X, X \setminus K) \longrightarrow H^i_{comp}(X).$$
- **Example:** We want to compute $H^*_{comp}(\mathbb{R}^n; \mathbb{Z})$. It is enough to consider the closed balls B_k around the origin with integer radius k. Note that
$$H^i(\mathbb{R}^n, \mathbb{R}^n \setminus B_k) \longrightarrow H^i(\mathbb{R}^n, \mathbb{R}^n \setminus B_{k+1})$$
is an isomorphism for all k. Thus
$$H^i_{comp}(\mathbb{R}^n; \mathbb{Z}) \simeq \begin{cases} \mathbb{Z} & i = n \\ 0 & i \neq n. \end{cases}$$
- **Warning:** Let $X \longrightarrow Y$ be continuous. This does not induce a map $f^* : H^*_{comp}(Y) \longrightarrow H^*_{comp}(X)$, in general. However, this is the case if f is *proper*, i.e. preimages of compact sets are compact. Also, compactly supported cohomology is not a homotopy invariant. It is only invariant under proper homotopies.
- **Definition:** If M is an oriented, closed manifold, then
$$\begin{aligned} H^n_{comp}(M; G) &\longrightarrow G \\ [\varphi] &\longmapsto \langle \varphi, [M] \rangle \end{aligned}$$
is defined directly. To generalize this to noncompact manifolds, pick a representative φ with support in $H^*(M, M \setminus K; G)$ and evaluate it on the fundamental class $[M] \in H_n(M, M \setminus K)$. This is well defined: Let L, L' be compact sets such that $[\varphi] \in H^n(M, M \setminus (L \cap L'))$ and $i_L : (L \cap L') \longrightarrow L$ the inclusion etc. Then
$$\begin{aligned} \langle i_L^*[\varphi], \mu_L \rangle &= \langle [\varphi], i_{L*}\mu_L \rangle \\ &= \langle [\varphi], \mu_{L \cap L'} \rangle = \langle i_{L'}^*[\varphi], \mu_{L'} \rangle \end{aligned} \tag{35}$$
by the uniqueness of the fundamental class.
- **Definition:** Let X be a space and G a commutative ring with unit. Then G-orientablity and the fundamental classes $[M, M \setminus K] \in H_n(M, M \setminus K; G)$ are defined as in the case $G = \mathbb{Z}$. Details can be found in [Ha], p. 234ff.

  The case $G = \mathbb{Z}_2$ is particularly important: Every manifold is $\mathbb{Z}_2$-orientable since $\mathbb{Z}_2$ is generated by the only non-zero element.

Lecture on June, 25 – Poincaré duality

- **Propaganda:** It is difficult to overstate the importance of the following theorem (and its non-compact analogue). The proof we will give requires passing through non-compact manifiolds.
- **Theorem (Poincaré Duality, compact case):** If M is a closed G-orientable n-manifold with fundamental class $[M] \in H_n(M; G)$, then the map
$$\begin{aligned} D : H^k(M; G) &\longrightarrow H_{n-k}(M; G) \\ [\alpha] &\longmapsto [M] \cap [\alpha] \end{aligned}$$
is an isomorphism for all k.

- This can be generalized to non-compact manifolds. For this one has to extend the definition of the map D.

  For compact sets $K \subset L \subset M$ and coefficients in G (a PID with unit) consider the commutative diagram

$$\begin{array}{ccccc} H_n(M, M \setminus L) & \times & H^k(M, M \setminus L) & \xrightarrow{\cap} & H_{n-k}(M) \\ \downarrow{\scriptstyle i_*} & & \uparrow{\scriptstyle i^*} & & \| \\ H_n(M, M \setminus K) & \times & H^k(M, M \setminus K) & \xrightarrow{\cap} & H_{n-k}(M) \end{array}$$

  We use the system of orientation classes $\mu_K \in H_n(M, M \setminus K)$. Then $i_*\mu_L = \mu_K$ by uniqueness, and $i_*(\mu_L) \cap \alpha = \mu_L \cap i^*\alpha$.

  If $[\alpha] \in H^*_{comp}(M)$, then one fixes a representative α whose support is contained in a fixed compact set K and $i_L : (M, M \setminus L) \longrightarrow (M, M \setminus K)$ for all L which contains K.

  The cochain α represents the class $\alpha_L = i^*_L[\alpha] \in H^*(M, M \setminus L)$.

  This class is independent from L: Let L' be another compact set. Then the above diagram implies

$$\mu_K \cap \alpha = i_*\mu_L \cap \alpha = \mu_L \cap i^*(\alpha) = \mu_L \cap \alpha$$

  This is very similar to (35).

  Thus $\mu_L \cap \alpha_L$ represents a well defined element in $H_{n-k}(M)$. For $[\alpha] \in H^k_{comp}(M)$, this defines $D(\alpha) = \mu_L \cap \alpha_L$.
- **Theorem (Poincaré Duality):** If M is a G-oriented n-manifold with fundamental classes $\mu_K \in H_n(M, M \setminus K; G)$, then the map

$$\begin{aligned} D : H^k_{comp}(M; G) &\longrightarrow H_{n-k}(M; G) \\ [\alpha] &\longmapsto \mu_{support(\alpha)} \cap [\alpha] \end{aligned}$$

  is an isomorphism for all k.
- The technical heart of the proof is the following lemma. We suppress the coefficients in the notation.
- **Lemma:** Let $M = U \cup V$ be the union of two open sets. Then there are Mayer-Vietoris sequences such that the diagram

$$\begin{array}{ccccccc} \dots H^k_{comp}(U \cap V) & \longrightarrow & H^k_{comp}(U) \oplus H^k_{comp}(V) & \longrightarrow & H^k_{comp}(M) & \xrightarrow{\delta^*} & H^{k+1}_{comp}(U \cap V) \dots \\ \downarrow{\scriptstyle D_{U\cap V}} & & \downarrow{\scriptstyle D_U \oplus D_V} & & \downarrow{\scriptstyle D_M} & & \downarrow{\scriptstyle D_{U\cap V}} \\ \dots H_{n-k}(U \cap V) & \longrightarrow & H_{n-k}(U) \oplus H_{n-k}(V) & \longrightarrow & H_{n-k}(M) & \xrightarrow{\partial_*} & H_{n-k-1}(U \cap V) \dots \end{array}$$

  commutes up to sign depending only on k. (The lower part is the standard Mayer Vietoris sequence in homology. The map $H^k_{comp}(U \cap V) \longrightarrow H^k_{com}(U)$ is induced be extending cochains to homomorphisms $C^*(U)$ which do vanish on all singular simplices which are not contained in $U \cap V$, etc. To see exactness use compactly supported cochains in $\mathrm{Hom}(C^{\{U,V\}}_*(M); G)$ to compute $H^*(M; G)$.)
- **Proof:** Let $K \subset U$ and $L \subset V$ be compact. Then the Mayer-Vietoris sequence of the open covering $(M, M \setminus K), (M, M \setminus L)$ of $(M, M \setminus (K \cup L))$ is the top

row of the following diagram

$$\begin{tikzcd}
H^k(M, M \setminus (L \cap K)) \arrow[r] \arrow[d, "exc."', "\simeq"] & H^k(M, M \setminus K) \oplus H^k(M, M \setminus L) \arrow[r] \arrow[d, "exc."', "\simeq"] & H^k(M, M \setminus (K \cup L)) \arrow[dd, "\mu_{K \cup L} \cap"] \\
H^k(U \cap V, (U \cap V) \setminus (K \cap L)) \arrow[r] \arrow[d, "\mu_{K \cap L} \cap"] & H^k(U, U \setminus K) \oplus H^k(V, V \setminus L) \arrow[d, "\mu_K \cap \oplus \mu_L \cap"] & \\
H_{n-k}(U \cap V) \arrow[r] & H_k(U) \oplus H_{n-k}(V) \arrow[r] & H_{n-k}(M)
\end{tikzcd}$$

The diagram extends further to the left and right. The two smaller squares on the left clearly commute, the same is true for the larger square on the right. Here is the square to the right of the above diagram:

$$\begin{tikzcd}
H^k(M, M \setminus (K \cup L)) \arrow[r, "\delta", "M.V.conn.homo"'] \arrow[dd, "\mu_{K \cup L} \cap"] & H^{k+1}(M, M \setminus (K \cap L)) \arrow[d, "exc."', "\simeq"] \\
& H^k(U \cap V, (U \cap V) \setminus (K \cap L)) \arrow[d, "\mu_{K \cap L} \cap"] \\
H_{n-k}(M) \arrow[r, "\partial", "M.V.-conn.homo."'] & H_{n-k-1}(U \cap V)
\end{tikzcd}$$

1. Description of δ: The top sequence comes from the short exact sequence

$$0 \longrightarrow \underbrace{C^*(M, (M \setminus K) + (M \setminus L))}_{\substack{\text{cochains vanishing on simpl.} \\ \text{in } M \setminus K, M \setminus L}} \longrightarrow \begin{matrix} C^*(M, M \setminus K) \\ \oplus \\ C^*(M, M \setminus L) \end{matrix} \longrightarrow C^*(M, \underbrace{(M \setminus K) \cap (M \setminus L)}_{= M \setminus (K \cup L)}) \longrightarrow 0.$$

   This is exact, the first chain complex computes $H^*(M, (U \setminus K) \cap (V \setminus L))$ (five lemma, subdivision) To find $\delta[\varphi]$ write $\varphi = \varphi_{M \setminus K} - \varphi_{M \setminus L}$, then $\delta \varphi_{M \setminus K}$ represents $\delta[\varphi]$.
2. Description of ∂: This is a connecting homo. Decompose
3. How to obtain $\mu_K, \mu_{K \cap L}$ from $\mu_{K \cup L}$: $U \setminus L(!), U \cap V, V \setminus K$ is an open cover of $M = U \cup V$ ($U \setminus K, \ldots$ is not). By subdivision, one can choose

$$\alpha_{U \setminus L} \in C_n(U \setminus L), \alpha_{V \setminus K} \in C_n(V \setminus K), \alpha_{U \cap V} \in C_n(U \cap V)$$

   such that the sum α represents $\mu_{K \cup L}$.
   3.1 $\alpha_{U \setminus L}, \alpha_{V \setminus K}$ lie in the complement of $K \cap L$. Hence, by uniqueness (p. 46) of the fundamental class in $H_n(M, M \setminus (K \cap L))$, $\alpha_{U \cap V}$ represents $\mu_{K \cap L}$.
   3.2 $\alpha_{U \setminus L} + \alpha_{U \cap V}$ represents $\mu_K \in H_n(M, M \setminus K)$, again by uniqueness of the fundamental classes.

We now check the commutativity (up to sign) of the diagram above. Let $[\varphi] \in H^k(M, M \setminus (K \cap L))$.

- First right, then down: Let $\varphi = \varphi_{M \setminus K} - \varphi_{M \setminus L}$. Then the composition of the first two arrows is [the restriction of $\delta \varphi_{M \setminus K}$ to chains in $U \cap V$].

Thus we get as final result

$$\begin{aligned}\varphi \longmapsto\ & \mu_{K\cap L} \cap [\delta\varphi_{M\setminus K}] \\ &= [\alpha_{U\cap V} \cap \delta\varphi_{M\setminus K}] \\ &= [(\partial\alpha_{U\cap V}) \cap \varphi_{M\setminus K} - \partial(\underbrace{\alpha_{U\cap V} \cap \varphi_{M\setminus K}}_{\in C_*(U\cap V)})] \\ &= [(\partial\alpha_{U\cap V}) \cap \varphi_{M\setminus K}].\end{aligned}$$

– First down, then right: The downward arrow maps $[\varphi]$ to $[\alpha \cap \varphi]$. Then

$$\partial[\alpha\cap\varphi] = \partial[\overbrace{\underbrace{\alpha_{U\setminus L}}_{\in C_*(M,M\setminus L\cup M\setminus(K\cup L))} \cap \underbrace{\varphi}_{\in C_k(M,M\setminus(K\cup L))}}^{\in C_*(M,M\setminus U)} + \overbrace{\underbrace{(\alpha_{V\setminus K}+\alpha_{U\cap V})}_{\text{repr. } \mu_K \in H_n(M,M\setminus(K\cup L)\cup M\setminus L)} \cap \underbrace{\varphi}_{\in C_k(M,M\setminus(K\cup L))}}^{\in C_*(M,M\setminus L)}]$$

$$\begin{aligned}&= [\partial\left(\alpha_{U\setminus L}\cap\varphi\right)] \\ &= (-1)^k[\partial\alpha_{U\setminus L}\cap\varphi] \qquad \text{since } \delta\varphi = 0 \\ &= (-1)^k[\partial\alpha_{U\setminus L}\cap(\varphi_{M\setminus K}-\varphi_{M\setminus L})] \\ &= (-1)^k[\partial\alpha_{U\setminus L}\cap\varphi_{M\setminus K}] \qquad \text{since } \varphi_{M\setminus L} \text{ vanishes on chains in } U\setminus L \\ &= (-1)^k[\partial(\underbrace{(\alpha_{U\setminus L}+\alpha_{U\cap V})}_{repr.\mu_K} - \alpha_{U\cap V})\cap\varphi_{M\setminus K}] \qquad \text{since } \varphi_{M\setminus K} \text{ vanishes on chains in } M\setminus K, \\ &\qquad\qquad \text{like } \partial(\alpha_{U\cap V}+\alpha_{U\setminus L}) \\ &= (-1)^{k+1}[\alpha_{U\setminus L}\cap\varphi_{M\setminus K}], \qquad \text{finally !}\end{aligned}$$

So the diagram commutes up to sign depending only on k.

- **Proof of Poincaré-duality:** We use various forms of induction.
  1. By the previous lemma and the five lemma: If the theorem holds for open sets $U, V, U\cap V$, then it holds for $U\cup V$.
  2. Assume that $M = \cup_{i\in\mathbb{N}} U_i$ where $U_i \subset U_{i+1}$ are open subsets for which Poincare duality holds, i.e. $D_i : H^k_{comp}(U_i) \longrightarrow H_{n-k}(U_i)$ is an isomorphism. Then D_M is an isomorphism as direct limit of D_i:
     By excision:
     $$\begin{aligned}H^k_{comp}(U_i) &= \varinjlim_{K\subset U_i \text{compact}} H^k(U_i, U_i\setminus K) \\ &= \varinjlim_{K\subset U_i \text{compact}} H^k(M, M\setminus K)\end{aligned}$$
     Therefore, there are maps $H^k_{comp}(U_i) \longrightarrow H^k_{comp}(U_{i+1})$ for all i and we can form the direct limit
     $$\varinjlim_i H^k_{comp}(U_i) = H^k_{comp}(M)$$
     since the direct limit on the right goes over all compact sets in M.

$$\begin{tikzcd} H^k_{comp}(U_i) \arrow[r] \arrow[d, "\simeq"', "D_i"] & H^k_{comp}(U_{i+1}) \arrow[r] \arrow[d, "\simeq"', "D_{i+1}"] & \dots \\ H^{n-k}(U_i) \arrow[r] & H^{n-k}(U_{i+1}) \arrow[r] & \dots \end{tikzcd} \qquad \begin{tikzcd} \varinjlim_i H^k_{comp}(U_i) \arrow[r, equal] \arrow[d, "\simeq"', "\varinjlim_i D_i"] & H^k_{comp}(M) \arrow[d, "D_M"] \\ \varinjlim_i H_{n-k}(U_i) \arrow[r, equal] & H_{n-k}(M) \end{tikzcd}$$

These are the tools, now we want to conclude:

1. $M = \mathbb{R}^n$: $\mathbb{R}^n$ is non-compact, but topologically tame, i.e. it is the interior of a manifold with boundary, this makes this case relatively simple. Identify $\mathbb{R}^n \simeq D^n$ (the interior of $\overline{D}^n$), and look at

$$\begin{array}{ccccc} H_n(D^n, D^n \setminus K) & \times & H^k(D^n, D^n \setminus K) & \xrightarrow{\cap} & H_{n-k}(D^n) \\ \downarrow & & \uparrow & & \downarrow \\ H_n(\overline{D}^n, \overline{D}^n \setminus K) & \times & H^k(\overline{D}^n, \overline{D}^n \setminus K) & \xrightarrow{\cap} & H_{n-k}(\overline{D}^n). \end{array}$$

   Coefficients are suppressed, vertical maps are induced by inclusions, and K is a compact ball in the interior D^n (deviating from the convention that D^n denotes a closed ball) of the closed ball. (It suffices to consider compact balls because every compact set in D^n is contained in such a ball.) Vertical maps are isomorphisms by excision (or homotopy invariance).
   Moreover, the left-most map maps μ_K onto the generator $[\overline{D}^n, \partial\overline{D}^n]$ because this class has the defining property of fundamental classes.
   The only interesting case in the bottom line is $k = n$, in this case the cap product reduces to evaluation and $\alpha \longmapsto \langle \alpha, [\overline{D}^n, \partial\overline{D}^n] \rangle$ defines an isomorphism

$$H^k(D^n, D^n \setminus K) \longrightarrow H_0(D^n) = G$$

2. M an arbitrary open set in $\mathbb{R}^n$: M not topologically tame in general. However, M is the countable union of open (metric) balls. The intersection of a finite collection of such balls is either empty or convex and open (hence it is homeomorphic to a ball). By the first tool, Poincaré-duality holds for all finite unions of the balls. By the second tool, Poincaré-duality holds for M.
3. M arbitrary: M can be covered by countably many charts (manifolds are paracompact/second countable). By the previous step, Poincaré-duality holds for all open sets in chart domains. Hence it holds for M (use the second tool).
   As is observed in [Ha], one can drop the paracompactness assumption and use Zorn's Lemma to prove Poincaré duality even in that case.

Lecture on June, 28 – Applications of Poincaré duality

- **Reference:** A good source for applications of duality is [Br].
- **Consequence:** Let M be a (path-)connected, closed $\mathbb{Z}$-oriented manifold of dimension n. Then $H^n(M) \simeq H_0(M) = \mathbb{Z}$. The fundamental class is a preferred generator when an orientation of M is fixed.
- **Consequence:** For every closed n-manifold $H^n(M; \mathbb{Z}_2) = \mathbb{Z}_2$.
- **Consequence:** $\mathbb{RP}^k$ is not orientable for k even. Viewing $\mathbb{RP}^n$ as quotient of an oriented sphere by the orientation preserving antipodal map, one obtains orientations on $\mathbb{RP}^k$ for k odd.
- **Consequence:** Take coefficients in $\mathbb{Q}$ or any other field. Then $b_k(M) = b_{n-k}(M)$. In particular, many finite CW-complexes are not homotopy equivalent to compact manifolds.

- **Consequence:** Let M be a closed manifold. Then $\chi(M) = 0$. One uses $\mathbb{Z}_2$-coefficients to compute $\chi(M)$, since every manifold admits a $\mathbb{Z}_2$-orientation. A reminder on χ is on p. 13.
- **Consequence/sketch:** This requires that you know what a manifold with boundary is. Assume that $\mathbb{RP}^2$ is the boundary of a compact manifold X with boundary. Gluing two copies of X one obtains a closed topological 3-manifold M with
$$\chi(M) = 2\chi(X) - \chi(\mathbb{RP}^2) = 2\chi(X) - 1.$$
using $\mathbb{Z}_2$-coefficients. Thus $\chi(M)$ is odd, but it should be zero. This works for all $\mathbb{RP}^k$ with k even.
- **Consequence:** If M is closed and $\mathbb{Z}$-oriented, then $H^n(M;\mathbb{Z}) = \mathbb{Z}$ by Poincaré-duality and connectedness.

  If M is non-orientable, then $H_{n-1}(M;\mathbb{Z})$ must contain 2-torsion by the universal coefficient theorem for homology and $H_n(M;\mathbb{Z}_2) = \mathbb{Z}_2$. Then $H_{n-1}(M,\mathbb{Z}_2) \neq 0$ and hence $H^1(M;\mathbb{Z}_2) \neq 0$. By the Hurewicz theorem, $\pi_1(M)$ must contain a (normal) subgroup of index 2 since $\mathrm{Hom}(\pi_1(M),\mathbb{Z}_2) = H^1(M;\mathbb{Z}_2)$. This can be seen more geometrically using the orientation cover $\overline{M} \longrightarrow M$, where
$$\overline{M} = \{(x, \text{generator of } H_n(M, M \setminus x)\} \longrightarrow M$$
is the map forgetting the second entry. The topology on $\overline{M}$ is the coarsest topology so that the map is continuous. Then $\overline{M}$ is a two fold covering and it is trivial iff M is $\mathbb{Z}$-orientable.
- **Duality in manifolds with boundary:** Let V be a manifold with boundary M.
  1. In a n-manifold with boundary, every point has a coordinate chart which is homeomorphic to a relative open set in $\{x_0 \leq 0\} \subset \mathbb{R}^n$.
  2. The boundary of a manifold is well defined, and a closed subset. The boundary of a manifold of dimension n is a $n-1$-manifold. This is consistent with the convention that $\emptyset$ is a manifold of any dimension.
  3. ∂V has a neighbourhood which is homeomorphic to $\partial V \times (-\varepsilon, 0]$ (when ∂V is paracompact). Such a neighbourhood is a *collar*, it deformation retracts onto ∂V.
  4. V is orientable if and only if $V \setminus \partial V$ is orientable.
  5. If M is compact with boundary and $x \in V \setminus \partial V$, then there is a compact set containing x which is disjoint from the boundary, for example $V \setminus \times[-\varepsilon/N, 0]$ for N big enough. Moreover, the inclusion $(V, \partial V \times (-\varepsilon, 0]) \longrightarrow (V, \partial V = \partial V \times \{0\})$ induces an isomorphism on relative homology.
  6. In particular, one can define and construct fundamental classes as on p. 45ff. with respect to points in the interior of M.
  7. One can check that $H^i_{comp}(V \setminus \partial V) \simeq H^i(V, \partial V)$ for all coefficients.
  8. From the Poincaré-duality theorem, one obtains that the cap product with the relative fundamental class $[V] \in H^n(V, \partial V)$ is a natural (with respect to orientation preserving maps) isomorphism
$$H^k(V, \partial V) \longrightarrow H_{n-k}(V).$$
  9. More generally, assume that $\partial V = A \cup B$ is the union of two submanifolds with boundary such that $\partial A = \partial B = A \cap B$. By the existence of collar neighbourhoods there are rel. open neighbourhoods $U, W \subset \partial V$ of A, B

which deformation retract on A, B and $A \cap B$ is a deformation retract of $U \cap W$.

10. The following diagram commutes:

$$\begin{tikzcd}
\dots H^k(V,\partial V) \arrow[r] \arrow[dd, "\simeq"', "{[V]\cap}"] & H^k(V,A) \arrow[r] \arrow[dd, "{[V]\cap}"] & H^k(\partial V, A) \arrow[r, "\delta^*"] \arrow[d, "\simeq"', "exc."] & H^{k+1}(V,\partial V)\dots \arrow[dd, "\simeq"', "{[V]\cap}"] \\
 & & H^k(B,\partial B) \arrow[d, "\simeq"', "{[B]\cap}"] & \\
\dots H_{n-k}(V) \arrow[r] & H_{n-k}(V,B) \arrow[r, "\partial_*"] & H_{n-k-1}(B) \arrow[r] & H_{n-k-1}(M)\dots
\end{tikzcd}$$

The top row is the exact sequence of the triple $(V, \partial V, A)$. The bottom corresponds to the pair (V, B). The second vertical arrow from the left is an isomorphism by the five-Lemma.

- **Theorem:** Let M be a closed $2k$-manifold, F a field, and V an F-oriented $2n+1$-manifold with boundary such that $\partial V = M$. Then

$$\dim(\ker(i_* : H_k(M) \longrightarrow H_k(V))) = \dim(\mathrm{im}(i^* : H^k(V) \longrightarrow H^k(M))) = \frac{\dim(H^k(M))}{2}.$$

In particular, $H^k(M)$ has even dimension. Moreover, the cup product of classes in $\mathrm{im}(i^*)$ vanishes provided that M is connected.

- **Proof:** the diagram

$$\begin{tikzcd}
H^k(V) \arrow[r, "i^*"] & H^k(M) \arrow[r, "\delta^*"] \arrow[d, "{[M]\cap}"] & H^{k+1}(V,M) \arrow[d, "{[V]\cap}"] \\
 & H_k(M) \arrow[r, "i_*"] & H_k(V)
\end{tikzcd}$$

commutes. This follows from (32) (and interpreting i_*). The upper row is part of the exact cohomology sequence of (V, M), the lower part is the homology exact sequence of (V, M). Then $\ker(\delta^*) = \mathrm{im}(i_*) \simeq \ker(i_*)$, the isomorphism comes from Poincaré duality.

Note that i^* is the dual of i_*, so these maps have the same rank (if A is a matrix of rank r, then so is A^T). Therefore,

$$\begin{aligned}
\mathrm{rank}(i^*) &= \dim(\ker(\delta^*)) = \dim(H_k) - \dim(\mathrm{im}(i_*)) \\
&= \dim(H_k(M)) - \mathrm{rank}(i^*)
\end{aligned}$$

This implies the second equality above. The first also follows since $\ker(i_*) \simeq \ker(\delta^*)$. Now consider two classes $i^*\alpha, i^*\beta$ in the image if $i^* : H^k(V) \longrightarrow H^k(M)$. Then

$$\delta^*(i^*(\alpha) \cup i^*(\beta)) = \delta^* i^*(\alpha \cup \beta) = 0.$$

Moreover, $\delta^* : H^n(M) \longrightarrow H^{n+1}(V, M)$ is injective, because

$$\begin{tikzcd}
H^n(M) \arrow[r, "\delta^*"] \arrow[d, "{[M]\cap}"] & H^{n+1}(V,M) \arrow[d, "{[V]\cap}"] \\
H_0(M) \arrow[r, "i_*"] & H_0(V)
\end{tikzcd}$$

commutes and the lower horizontal map is an isomorphism.

- **Corollary:** If the closed, connected manifold M^{2k} bounds a compact manifold, then $\chi(M)$ is even (take coefficients in $\mathbb{Z}_2$ to compute the Euler characteristic).
- **Example:** $\mathbb{RP}^{2k}, \mathbb{CP}^{2k}, \mathbb{HP}^{2k}$ do not bound compact manifolds.
- **Definition:** Let M be a closed oriented manifold of dimension $n = 2k$. Then the cup-product induces a bilinear form

$$H^k(M;\mathbb{Z}) \times H^k(M;\mathbb{Z}) \longrightarrow \mathbb{Z}$$
$$([\alpha],[\beta]) \longmapsto \langle \alpha \cup \beta, [M] \rangle.$$

It is symmetric when k is even, and antisymmetric when k is odd.

We now replace $\mathbb{Z}$ by coefficients in a field (e.g. $\mathbb{Q}$ or $\mathbb{Z}_2$). Then the above bilinear form takes values in the field, and it is non-degenerate by Poincaré duality since
  - $\langle \alpha \cup \beta, [M] \rangle = \langle \alpha, [M] \cap \beta \rangle$ by (34), and
  - $H^k(M;F) = \mathrm{Hom}_F(H_k(M;F), F)$ by the universal coefficient theorem for cohomology.

This bilinear form is the *intersection form*.
- **Consequence:** Non-degenerate, anti symmetric bilinear form exist only on even dimensional $\mathbb{Q}$-vector spaces. Hence, the Euler characteristic of $\mathbb{Z}$-orientable closed manifolds in dimension $4l+2$ is even.
- **Reminder:** If the characteristic of a field is $\neq 2$, then the set of quadratic and bilinear forms are isomorphic via polarization. Symmetric bilinear form over $\mathbb{R}$ are classified up to isomorphism by the dimension of the underlying vector spacet, the nullity (=dimension of kernel), and the signature σ

$$\sigma = (\text{dim. of maximal subspace where the form is positive definite})$$
$$- (\text{dim. of maximal subspace where the form is negative definite})$$

Recall that symmetric real matrices can be diagonalized (over $\mathbb{R}$).
- **Definition:** Let M be a closed oriented $4k$-dimensional oriented manifold. The *signature* $\sigma(M)$ of M is the signature of its intersection form.
- **Remark:** If one changes the orientation of M, i.e. replacing $[M]$ by $-[M]$, then positive and negative definite subspaces of $H^{2k}(M;\mathbb{Z})$ are interchanged and the signature changes its sign: $\sigma(\overline{M}) = -\sigma(M)$. $\overline{M}$ is the same manifold but has the opposite orientation. The signature is an invariant of *oriented manifolds*.
- **Example:** $\mathbb{CP}^2$ has two orientations and if $\omega \in H^2(\mathbb{CP}^2;\mathbb{Z})$ is a generator, then $\omega \cup \omega$ is a generator of $H^4(\mathbb{CP}^2;\mathbb{Z})$. The question is which one. If $\langle \omega\cup\omega, [\mathbb{CP}^2]\rangle = 1$, then the signature is 1 and -1 otherwise. In the second case it is common to denote the manfold by $\overline{\mathbb{CP}^2}$.

The signature of $S^2 \times S^2$ is 0. Unlike $\mathbb{CP}^2$, $S^2 \times S^2$ admits an orientation reversing homeomorphism.
- **Theorem (Thom):** If $M^{4k} = \partial V^{4k+1}$ is connected, V is compact, and orientable, then

$$\sigma(M) = 0.$$

- **Proof:** We use real coefficients. Then there are subspaces $W^+, W^- \subset H^{2k}(M;\mathbb{R})$ on which the intersection form is definite and $W^+ \oplus W^- = H^{2k}(M;\mathbb{R})$. Let $r = \dim(W^+)$. If $\dim(H^{2n}(M)) = 2n$, then $\dim(W^-) = 2n - r$. By the Theorem, there is a subspace $L \subset H^{2n}(M)$ of dimension n such that the restriction of the intersection form to U vanishes. Therefore,

$$\dim(U \cap W^+) = \{0\} \quad \Rightarrow 2n \leq \dim(U + W^-) \leq n + 2n - r \Rightarrow r \leq n.$$
$$\dim(U \cap W^-) = \{0\} \quad \Rightarrow 2n \leq \dim(U + W^+) \leq n + r \Rightarrow n \leq r.$$

Hence, $r = n$ and $\sigma(M) = 0$.

- **Examples:** I assume that you know what the connected sum of two (oriented, connected) manifolds of equal dimension is. (If not, think of a manifold version of the one-point union).
  - $\mathbb{CP}^2 \# \mathbb{CP}^2$ has signature 2, so it does not bound an oriented manifold. Note $\chi(\mathbb{CP}^2 \# \mathbb{CP}^2) = 4$.
  - $\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$ has signature 0 and Euler characteristic 4. It bounds a 5-manifold: Take $(\mathbb{CP}^2 \setminus B_\varepsilon(x)) \times [0,1]$. This shows that $M \# \overline{M}$ always bounds when M is oriented.
  - $M = \mathbb{CP}^2 \# \mathbb{CP}^2$ bounds a non-orientable manifold. Let $(M \times [0,1]) \# \mathbb{RP}^2 \times S^3$ take an orientation reversing arc connecting the two boundary components and drill out a tubular neighborhood of this arc. The result is a manifold V which bounds M. This had nothing to do with the precise form of $\mathbb{CP}^2$. or dimension 4.
- **Application of non-compact Poincaré duality:** Let U be an open set in $\mathbb{R}^3$. Then $H_1(U);\mathbb{Z})$ has no torsion.
- **Proof:**

$$\begin{aligned}
H_1(U;\mathbb{Z}) &\simeq H^2_{comp}(U;\mathbb{Z}) && \text{Poincaré-duality}\\
&= \varinjlim_K H^2(U, U \setminus K;\mathbb{Z}) && \text{definition}\\
&= \varinjlim_K H^2(\mathbb{R}^3, \mathbb{R}^3 \setminus K;\mathbb{Z}) && \text{excision of } \mathbb{R}^3 \setminus \overline{U}\\
&= \varinjlim_K H^1(\mathbb{R}^3 \setminus K;\mathbb{Z}) && \text{long exact sequence of } (\mathbb{R}^3, \mathbb{R}^3 \setminus K).
\end{aligned}$$

The direct limit of torsion free groups (like $H^1(\cdot;\mathbb{Z})$) is again torsionfree. We will see later, that open sets on $\mathbb{R}^4$ can have torsion in H_1.

- Poincaré-Alexander-Lefschetz duality generalizes Poincaré-duality. It is useful in the study of complements of compact sets in manifolds.
- **Reference:**[Br], section VI.8. and Appendix D.
- **Definition:** Let $L \subset K \subset M$ be compact subsets of a manifold. Then

$$\check{H}^p(K, L; G) := \varinjlim \{H^p(U, V; G) \,|\, (K, L) \subset (U, V), U, V \text{ open}\}.$$

The partial order is given by $(U, V) \geq (U', V') \Leftrightarrow (U', V') \supset (U, V)$, the maps for the directed system are induced by inclusions.

This group is well defined, i.e. it does actually not depend on how K is embedded in the manifold M. Moreover, if K, L are CW-complexes, then $\check{H}^*(K, L)$ coincides with singular cohomology, but it does not in general. (The topologists sin-curve is connected, but not path connected, leading to $H^0 \not\simeq \check{H}^0$ or consider wild knots in $\mathbb{R}^3$)

- **Duality map:** This is again the composition of capping with a certain fundamental class, and various isomomorphisms (based on excision and subdivision)
  - There is a cap product

$$\frac{C_n(V) + C_n(U \setminus L)}{C_n(U \setminus K)} \otimes C^p(U, V) \longrightarrow C_{n-p}(U \setminus L, U \setminus K).$$

given by $(b+c)\cap f = c\cap f$.

- Since $\{V, U\setminus L\}$ is an open cover of U,

$$H_*\left(\frac{C_n(V)+C_n(U\setminus L)}{C_n(U\setminus K)}\right) \simeq H_*(U, U\setminus K) \simeq H_*(M, M\setminus K).$$

  (First, subdivision, then excision).
- By excision in homology $H_{n-p}(U\setminus L, U\setminus K) \simeq H_{n-p}(M\setminus L, M\setminus K)$.
- Thus we get a (co-)homology cap product

$$H_n(M, M\setminus K)\otimes H^p(U,V) \longrightarrow H_{n-p}(M\setminus L, M\setminus K).$$

  In order to compute it, one has to decompose the representative μ of a class in $H_n(M, M\setminus K)$ into a sum

$$\mu = b+c+d \in C_n(V)+C_n(U\setminus L)+C_n(U\setminus K).$$

  This reflects the first isomorphism in the previous item.
- Assume that $A\subset M$ is a compact set containing K. Then there is a fundamental class $\mu_A\in H_n(M, M\setminus A)$ given by the orientation of M and we get a map

$$\mu_A\cap : H^p(U,V) \longrightarrow H_{n-p}(M\setminus L, M\setminus K).$$

  This is compatible with inclusions of (U,V). Passing to the direct limit we get a map

$$\mu_A\cap : \check{H}^p(K,L) \longrightarrow H_{n-p}(M\setminus L, M\setminus K).$$

- **Theorem (Poincaré-Alexander-Lefschetz duality):** Let M be an oriented n-manifold, $L\subset K\subset M$ compact subsets. Then the cap product

$$\mu\cap : \check{H}^p(K,L) \longrightarrow H_{n-p}(M\setminus L, M\setminus K)$$

  defined above is an isomorphism.
- **Lemma (Reduction to $L=\emptyset$):** The following diagram with exact rows commutes up to sign depending only on p.

$$\begin{tikzcd}
\dots\check{H}^p(K,L) \arrow[r] \arrow[d] & \check{H}^p(K) \arrow[r] \arrow[d] & \check{H}^p(L) \arrow[r, "\delta^*"] \arrow[d] & \check{H}^{p+1}(K,L)\dots \arrow[d] \\
\dots H_{n-p}(M\setminus L, M\setminus K) \arrow[r] & H_{n-p}(M\setminus K) \arrow[r] & H_{n-p}(M\setminus L) \arrow[r, "\partial_*"] & H_{n-p-1}(M\setminus L, M\setminus K)\dots
\end{tikzcd}$$

  Vertical maps are cap products with the orientation class. Therefore, if Poincaré-Alexander-Lefschetz duality holds for $L=\emptyset$, then it holds for pairs by the five lemma.
- **Proof:** The top row is exact, by the general fact that direct limits preserve exactness. For commutativity, we consider the left most square.

  Let μ be the fundamental class and decompose

$$\mu = b+c+d \in C_n(V)+C_n(U\setminus L)+C_n(M\setminus K)$$

  where $\mu\in C_n(M, M\setminus A)$ represents the orientation class for a compact set such that $U\subset A$. This decomposition allows, to compute the cap-product with the fundamental class $\check{H}^{p+1}(K,L)\longrightarrow H_{n-p-1}(M\setminus L, M\setminus K)$. To compute the cap product with the fundamental class for $(L,\emptyset)$, one uses the decomposition

$$\mu = 0+b+(c+d)\in C_n(\emptyset)+C_n(V\setminus\emptyset)+C_n(M\setminus L).$$

Let $f \in C^p(V)$ represent a class in $\check{H}^p(L)$. In particular, $\delta f = 0$ (on V). Then (first right, then down)

$$[\mu \cap \delta f] = [c \cap \delta f] \in H_{n-p-1}(M \setminus L, M \setminus K).$$

But we also have

$$\begin{aligned}\partial_*[\mu \cap f] &= (-1)^p \Bigg[\underbrace{\partial\mu}_{\in C_{n-1}(M\setminus K)} \cap f + \mu \cap \delta f \Bigg] \\ &= [c \cap \delta f] \in H_{n-p-1}(M \setminus L, M \setminus K).\end{aligned}$$

- The following lemma corresponds to the Lemma on p. 49 used for induction in the proof of Poincaré duality.
- **Remark:** $\check{H}^*$ is more complicated than H^*: For example, a map between compact sets does not automatically yield a map between neighborhoods in some manifold.
- **Lemma:** Let K, L be compact subsets in the n-manifold M and μ represents a fundamental class in $H_n(M, M \setminus (K \cup L))$. Then there is a diagram

$$\begin{tikzcd}
\check{H}^p(K \cup L) \arrow[r] \arrow[d] & \check{H}^p(K) \oplus \check{H}^p(L) \arrow[r] \arrow[d] & \check{H}^p(K \cap L) \dots \arrow[d] \\
H_{n-p}(M, M \setminus (K \cup L)) \arrow[r] & H_{n-p}(M, M \setminus K) \oplus H_{n-p}(M, M \setminus L) \arrow[r] & H_{n-p}(M, M \setminus (K \cap L))
\end{tikzcd}$$

$$\begin{tikzcd}
\dots \check{H}^p(K \cap L) \arrow[r, "\delta^*"] \arrow[d] & \check{H}^{p+1}(K \cup L) \arrow[d] \\
\dots H_{n-p}(M, M \setminus (K \cap L)) \arrow[r, "\partial_*"] & H_{n-p-1}(M, M \setminus (K \cup L))
\end{tikzcd}$$

with exact rows, and which commutes up to sign depending on p only. The vertical maps are cap products.
- **Proof:** The top sequence is the limit of Mayer-Vietoris sequences coming from the short exact sequence of cochain complexes

$$0 \longrightarrow \mathrm{Hom}(C_*(U){+}C_*(V), G) \longrightarrow \mathrm{Hom}(C_*(U), G){\oplus}\mathrm{Hom}(C_*(V), G) \longrightarrow \mathrm{Hom}(C_*(U{\cap}V), G) \longrightarrow 0$$

for open neighborhoods (U, V) of (K, L). We consider the last square. Let $f \in C^p(U{\cap}V; G)$ represent a class in $\check{H}^p(K{\cap}L)$ and μ represent a fundamental class in $H_n(M, M \setminus (U \cup V))$.

The subdivision theorem appears in the maps in the diagram. We therefore have to decompose a representative $\delta^*[f]$: f comes from $f{\oplus}0 \in C^n(U){\oplus}C^n(V)$. Then $\delta^*[f]$ is represented by $h \in \mathrm{Hom}(C^*(U) + C^*(V))$ with $h(u+v) = \delta f(u)$. One extends h arbitrarily to $C^*(M)$.

We decompose μ as follows.

$$\mu = b + c + d + e \in C_n(U \cap V) + C_n(U \setminus L) + C_n(V \setminus K) + C_n(M \setminus (K \cup L)).$$

Then the class $[\mu] \cap \delta^*[f]$ is represented by

$$\begin{aligned}(b + c + d + e) \cap h &\sim (b + c + d) \cap h \qquad \mathrm{mod}\ C_{n-p}(M \setminus (K \cup L)) \\ &= c \cap h \quad \text{since } \delta f = h \text{ vanishes on } C_*(V).\end{aligned}$$

The lower row also involves some decomposition of a chain with respect to an open covering: $M \setminus (K \cap L)$ is viewed as union of $M \setminus K$ with $M \setminus L$ and the long exact sequence comes from

$$0 \longrightarrow \frac{C_*(M)}{C_*(M \setminus (K \cup L))} \longrightarrow \frac{C_*(M)}{C_*(M \setminus K)} \oplus \frac{C_*(M)}{C_*(M \setminus L)} \longrightarrow \frac{C_*(M)}{C_*(M \setminus K) + C_*(M \setminus L)} \longrightarrow 0$$

To compute $\partial_*(\mu \cap f)$ one has to decompose $\mu \cap f$ according to the second non-trivial map, and then applies ∂ to obtain a chain in the left most chain group. The first step results in $(\mu \cap f, 0)$. The second step yields

$$\begin{aligned}(-1)^p(\partial(\mu \cap f)) &= (\partial\mu) \cap f - \mu \cap \delta f \\ &= -\partial\mu) \cap f - c \cap \delta.\end{aligned}$$

$\partial\mu$ is a chain in the complement of $U \cup W$ while f has support in $U \cap V$. Thus the first summand does not contribute.

- **Proof of Poincaré-Alexander-Lefschetz duality:**
  - By the first Lemma after the statement of the theorem (+ five Lemma) we may assume $L = \emptyset$.
  - The statement is true for a point in $x \in \mathbb{R}^n$. Then $\check{H}_*$ vanishes in non-zero degree, while in degree zero it is $\varinjlim_\varepsilon H^0(B_\varepsilon(x)) = \mathbb{Z}$ generated by a constant cochain 1. Let an orientation class in $H_n(\mathbb{R}^n, \mathbb{R}^n \setminus 0)$. Then $1 \longmapsto \mu \cap 1 = \mu$ which generates $H_{n-0}(\mathbb{R}^n, \mathbb{R}^n \setminus 0)$
  - $\check{H}^*(U) = H^*(U)$ for all convex subsets (we did not check that $\check{H}$ is homotopy invariant).
  - The case when K is a finite union of convex sets follows from the previous Lemma.
  - If $M \supset K_1 \supset K_2 \subset \ldots$ is a sequence of compact sets as above such that $\cap_i K_i = K$. For each i one choose a descending sequence $U_{i,j}$ such that $\cap U_{i,j} = K_i$ (for example a $1/j$-neighborhood for some metric on M, topological manifolds are metrizable [Qu]). Instead of $U_{i,j}$ consider $U_{i_0,j} = \cap_{i \le i_0} U_{i,j}$ for all j. This is a family of neighborhoods of K. Using this and general properties of direct limits one shows that

    $$\varinjlim_i \check{H}^*(K_i) \simeq \check{H}^*(K).$$

    The same inductive step which allowed passage do direct limits/compactly supported cohomology now shows that the statement is true for arbitrary compact sets $K \subset \mathbb{R}^n$.
  - The claim for general manifolds follows in the same fashion as for Poincaré-duality.
- **Application:** The following map is an embedding

  $$\begin{aligned}\mathbb{RP}^2 &\longrightarrow \mathbb{R}^4 \\ [x_0 : x_1 : x_2] &\longmapsto (x_0^2 - x_1^2, x_0x_1, x_1x_2, x_2x_0).\end{aligned}$$

  The first homology of the complement of the image is

  $$\check{H}^2(\mathbb{RP}^2) = H^2(\mathbb{RP}^2; \mathbb{Z}) \simeq H_2(\mathbb{R}^4, \mathbb{R}^4 \setminus \mathbb{RP}^2; \mathbb{Z}) = H_1(\mathbb{R}^4 \setminus \mathbb{RP}^2; \mathbb{Z}).$$

  For this note that the $\mathbb{R}^4$ is orientable, so we may use $\mathbb{Z}$-coefficients. The last equality follows from the exact sequence of the pair $(\mathbb{R}^4, \mathbb{R}^4 \setminus \mathbb{RP}^2)$. By the universal coefficient theorem for cohomology

$$H^2(\mathbb{RP}^2;\mathbb{Z}) \simeq \mathrm{Hom}(H_2(\mathbb{RP}^2,\mathbb{Z});\mathbb{Z}) \oplus \mathrm{Ext}(H_1(\mathbb{RP}^2;\mathbb{Z}),\mathbb{Z}) \simeq \mathrm{Ext}(\mathbb{Z}_2,\mathbb{Z}) \simeq \mathbb{Z}_2$$

This shows $H_1(\mathbb{R}^4 \setminus \mathbb{RP}^2;\mathbb{Z}) \simeq \mathbb{Z}_2$, in particular there is an open set in $\mathbb{R}^4$ whose first homology is torsion. This is of course independent of the form of the embedding $\mathbb{RP}^2 \longrightarrow \mathbb{R}^4$.

- **Corollary (Alexander duality):** Let A be a compact set in $\mathbb{R}^n$. Then

$$\widetilde{H}_q(\mathbb{R}^n \setminus A; G) \simeq \check{H}^{n-q-1}(A;G).$$

- **Proof:** By Poincaré-Alexander-Lefschetz duality $\check{H}^{n-q-1}(A;G) \simeq H_{q+1}(\mathbb{R}^n \setminus A;G)$. By the reduced homology sequence of the pair $(\mathbb{R}^n, \mathbb{R}^n \setminus A)$ all inclusion maps $\widetilde{H}_{q+1}(\mathbb{R}^n, \mathbb{R}^n \setminus A) = H_{q+1}(\mathbb{R}^n, \mathbb{R}^n \setminus A)$ are isomorphisms because $\widetilde{H}_i(\mathbb{R}^n) = 0$.
- **Remark:** If one considers only embeddings, where A has a basis of neighborhoods U_i such that $A \subset U_i$ is a strong deformation retract, then $\check{H}^*$ coincides with H^*. This is the case for A a smooth manifold which is a smooth submanifold in M (this is not true in the topological catagory).

  The homology of $\mathbb{R}^n \setminus A$ is completely determined by the topology of A, not how A is embedded (as locally flat submanifold). This is not true for the fundamental groups, for example.
- **Corollary (Generalized Jordan curve theorem):** Assume that M is connected, orientable, closed with $H_1(M;\mathbb{Z})$. Let $A \subset M$ be a proper closed (hence compact) subset. Then $\check{H}^{n-1}(A;\mathbb{Z})$ is a free Abelian group whose rank is the number of components of $M^n \setminus A$.
- **Proof:**

$$\begin{aligned} \check{H}^{n-1}(A) &\simeq H_1(M, M \setminus A) && \text{by duality} \\ &\simeq \widetilde{H}_0(M \setminus A) && \text{long exact homology sequence}, \widetilde{H}_0(M) = 0, H_1(M) = 0 \end{aligned}$$

- **Application:** No non-orientable (smooth) $(n-1)$-manifold A (smoothly) embeds into a compact, orientable (smooth) n-manifold with $H_1(M;\mathbb{Z}) = 0$ since $H^{n-1}(A;\mathbb{Z}) \simeq \mathbb{Z}_2$ is not free Abelian.

Lecture on July, 1 – Intersections

- We want to (partially) interpret the cup product and Poincaré-duality for *smooth* manifolds. For this we need a few fact (similar to the collar theorem). After addition of adjectives (like *locally flat*) the statement can be transferred to topological manifolds.
- **Theorem (tubular neighborhood theorem):** Let $N \subset M$ be a smooth submanifold of codimension k. Then there is an embedded submanifold with boundary W of dimension n and a smooth map $\mathrm{pr} : W \longrightarrow N$ such that each point $x \in N$ there is an open neighborhood $U \subset N$ and a diffeomorphism $\varphi_U : \mathrm{pr}^{-1}(U) \longrightarrow U \times \overline{D}^k$ such that $\varphi(U) = U \times \{0\}$ and

$$\begin{tikzcd} \mathrm{pr}^{-1}(U) \arrow[rr, "\varphi"] \arrow[dr, "\mathrm{pr}"'] & & U \times \overline{D}^k \arrow[dl] \\ & U & \end{tikzcd}$$

commutes. W is a *tubular neighborhood of* N. It is a smooth disk bundle. If $\mathrm{pr}' : W' \longrightarrow N$ is another tubular neighborhood, then there is a diffeomorphism ψ of M such that $\psi(W) = W'$ and $\mathrm{pr}' \circ \psi = \mathrm{pr}$.

- **Remark:** This is an example of a disc bundle, i.e. a map $\mathrm{pr} : W \longrightarrow N$ which satisfies a local triviality axiom as above around every point in the base. A disc bundle can be defined over an arbitrary space. Given two local trivializations over open sets U, V uch that $U \cap V$ one obtains a transition map into $\mathrm{Homeo}(D^k, \partial D_k)$ (homeo's of pairs, not relative to the boundary). A bundle is *orientable* if one can choose a bundle atlas such that all transition functions preserve an orientation of the fiber $(D^k, \partial D^k)$. An *orienation* is the choice of such an atlas. The fiber of this bundle is homeo. to $(D^k, \partial D^k)$. If N and the fiber a smooth, then the bundle is smooth if one can choose an bundle atlas whose transition functions take values on $\mathrm{Diffeo}(D^k, \partial D^k)$ and depends smoothly on the base point.
- **Remark:** Orientations matter. The tubular neighborhood is a disc bundle over N whose fiber is the closed disc. The total space W of the bundle and the base N are oriented, then so is the fiber using the local product structure with the fiber first convention. This is consistent with the outward normal first convention for the orientation of the boundary of an oriented manifold.
- **Remark:** A disc bundle W is homotopy equivalent to the base space N once a section $s : N \longrightarrow W$ is fixed. If N is a CW-complex, this can always be done.
- **Definition:** If N, N' are two smooth submanifolds of codimension k, k' in M, then they are transverse if $T_pN + T_pN' + T_pM$ for all $p \in (N \cap N')$. Then $N \cap N'$ is a smooth submanifold of codimension $k + k'$.
- **Definition:** Let M be compact, oriented, connected and denote the inverse of $\mu\cap$ with
$$PD : H^j(M, \partial M) \longrightarrow H_{n-j}(M) \text{ or}$$
$$PD : H^j(M) \longrightarrow H_{n-j}(M, \partial M)$$
inverses of the Poincaré duality map, i.e.
$$[M] \cap PD(\sigma) = \sigma.$$
The intersection product
$$\bullet : H_i(M) \otimes H_j(M) \longrightarrow H_{n-i-j}(M)$$
$$\bullet : H_i(M, \partial M) \otimes H_j(M) \longrightarrow H_{n-i-j}(M, \partial M)$$
$$\bullet : H_i(M, \partial M) \otimes H_j(M, \partial M) \longrightarrow H_{n-i-j}(M, \partial M)$$
is defined as
$$PD(a \bullet b) = PD(a) \cup PD(b).$$
- **Properties:** This is associative and $a \bullet b = (-1)^{(n-|a|)(n-|b|)} b \bullet a$.
- **Definition:** Let $\mathrm{pr} : W \longrightarrow N$ be an oriented disk bundle (not necessarily a tubular neighborhood) and $s : N \longrightarrow W$ a section (exists always when the base is a CW-complex), i.e. $\mathrm{pr} \circ s = \mathrm{id}_N$. We assume that N is oriented. The *Thom class* of the disc bundle $\mathrm{pr} : W \longrightarrow N$ is
$$\tau = PD_W(s_*[N]) \in H^k(W, \partial W).$$
This implies
$$[W] \cap \tau = i_*[N]. \tag{36}$$

- This is less explicit than it looks. Another interpretation of the Thom-class is as follows. It is the unique class τ which restricts to a generator of $H^k(D^k, \partial D^k)$. If the bundle is oriented, one can construct the Thom class in the same way as the fundamental class of an oriented manifold. One has to show that the result of this construction and the above definition coincide. Wince $[W]\cap$ is an isomorphism, it suffices to verify that the result of the construction satisfies $[W] \cap \tau = [N]$. If one views of $[W]$ as sum of cross products of generators $H_k(D^k, \partial D^k)$ and $H_n(U, \partial U)$ where U is the domain of a trivialization of the bundle, then $[M] \cap \tau$ restrict to the generator of $H_n(N, N \setminus x)$ representing the local orientation of N.
- **Defintion:** Let $f : N \longrightarrow M$ be a map from a compact oriented n-manifold N into the compact oriented connected m-manifold M which maps ∂N to ∂M. Define

  $$f^! : H^{n-p}(N) \longrightarrow H^{m-p}(M) \qquad f_! : H^{n-p}(N, \partial N) \longrightarrow H^{m-p}(M, \partial M)$$

  as $f^! = PD_M \circ f_* \circ PD_N^{-1}$. The maps

  $$f_! : H^{m-p}(M) \longrightarrow H^{n-p}(N) \qquad f_! : H_{m-p}(M, \partial M) \longrightarrow H^{n-p}(N, \partial N)$$

  are $f_! = PD_N^{-1} \circ f^* \circ PD_M$.
- **Theorem (Thom isomorphism):** Let $\mathrm{pr} : W \longrightarrow N$ be a k-disc bundle over the closed, connected oriented n-manifold N and a section i. Then the Thom isomorphism is the composition

  $$H^p(N) \xrightarrow{\mathrm{pr}} H^p(W) \xrightarrow{\cup\tau} H_{p+k}(W, \partial W)$$

  and it coincides with $i^!$.
- **Proof:** Let $\beta = i^*\alpha$

  $$\begin{aligned} i^!\beta &= PD_W i_* PD_N^{-1}(\beta) \\ &= PD_W i_*([N] \cap \beta) \\ &= PD_W i_*([N] \cap i^*\alpha) \\ &= PD_W(i_*[N] \cap \alpha) \\ &= PD_W(([W] \cap \tau) \cap \alpha) \text{ by (36)} \\ &= PD_W([W] \cap (\alpha \cup \tau)) \text{ by (34)} \\ &= \alpha \cup \tau \\ &= \mathrm{pr}^*\beta \cup \tau. \end{aligned}$$

  This is an isomorphism since $i^!$ is a composition of isomorphisms.
- So far we discussed N as a submanifold inside a disc bundle. Now we turn to $i_N^W : N^n \longrightarrow W^w$, where everything is smooth and oriented. Moreover we assume that $N \cap \partial W = \partial N$ and N is transverse to ∂W. We write $[N]_W = i^W_{N*}[N]$ and $PD_W : H_n(W, \partial W) \longrightarrow H^{w-n}(W)$.
- **Definition:** The Thom class of $N \subset W$ is

  $$\tau^W = PD_W[N]_W \in H^{w-n}(W)$$

- **Remark:** This is the image of the previous Thom class under

  $$H^{w-n}(tube, \partial tube) \simeq H^{w-n}(W, W \setminus tube) \longrightarrow H^{w-n}(W).$$

  The first isomorphism is given by excision. Moreover,

  $$[N]_W = [W] \cap \tau_N^W.$$

- The connection with the intersection product on homology is as follows. Let K, N properly embedded manifolds in W. Then

$$
\begin{aligned}
[K]_W \bullet [N]_W &= [W] \cap (PD_W[K]_W \cup PD_W[N]_W) \\
&= [W] \cap (\tau_K^W \cup \tau_N^W).
\end{aligned}
\tag{37}
$$

- To interpret this more geometrically, we need to know the following plausible fact about tubular neighborhoods: Let K, N be two submanifolds meeting transversely along $K \cap N$ with the standard precautions a the boundary. Then one can choose the tubular neighborhood of K in W so that the restriction of ν_K^W to $K \cap N$ <u>is</u> the tubular neighborhood of $K \cap N$ in N. By the characterization of the Thom class, this implies $i_N^{W*}\tau_K^W = \tau_{N\cap K}^N$.
- **Remark:** This picks out the orientation convention for transverse intersections of oriented manifolds (which result in oriented manifolds).
- **Theorem:** Assume that K, N intersect transversely with the usual precautions at the boundary and orientation/compactness assumptions

$$\tau_{K\cap N}^W = \tau_K^W \cup \tau_N^W.$$

this is equivalent to

$$[K \cap N]_W = [K]_W \bullet [N]_W.$$

- **Proof:**

$$
\begin{aligned}
[K \cap N]_W &= i_{K\cap N*}^W [K \cap N] \\
&= i_{N*}^W i_{K\cap N*}^N [K \cap N] \\
&= i_{N*}^W ([N] \cap \tau_{K\cap N}^N) \\
&= i_{N*}^W ([N] \cap i_N^{W*}\tau_K^W) \\
&= [N]_W \cap \tau_K^W \\
&= ([N] \cap \tau_N^W) \cap \tau_K^W \\
&= [N] \cap (\tau_K^W \cup \tau_N^W) \\
&= [K]_W \bullet [N]_W
\end{aligned}
$$

- **Warning:** Sign conventions differ from book to book. I have tried to stay consistent in these notes, but one has to be aware of the fact that many other notes/books use different sign conventions.
- **Example:** Products of spheres. The cohomology ring structure can be determined geometrically.
- **Example:** $\mathbb{CP}^2, \mathbb{RP}^2$.
- **Warum nicht gleich so?:** The above works only when the ambient space is a manifold. More subtly, not every cohomology class has a Poincaré-dual which is the (multiple of) the the image of the fundamental class of a closed submanifold (under inclusion), c.f. [Th]
- **Definition:** Let N^n be a smooth submanifold of W^w (both oriented) with N transverse to ∂W and $\partial N \subset \partial W$. The *Euler class* of the normal bundle of N in W is

$$\chi_N^W = i_N^{W*}\tau_N^W \in H^{w-n}(N). \tag{38}$$

i.e. the restriction of the Thom class to N. In particular, if $H^{w-n}(W) = 0$, then $\chi_N^W = 0$.

- Let T be a closed tubular neighborhood of N in W. Then

$$\begin{tikzcd}
 & H^{w-n}(W, W \setminus T) \arrow[r] \arrow[d, "exc."', "\simeq"] & H^{w-n}(W) \ni \chi_N^W \arrow[d] & \\
\tau_N^T \in & H^{w-n}(T, \partial T) \arrow[r] & H^{w-n}(T) \arrow[r] \arrow[d, "\simeq"', "i_N^{W*}"] & H^{w-n}(\partial T) \\
 & & H^{w-n}(N) \arrow[ur, "\mathrm{pr}^*"] &
\end{tikzcd}$$

is commutative, the lower row is part of the long exact sequence of $(T, \partial T)$. Now assume that there is section $s : N \longrightarrow \partial T$ such that $\mathrm{pr} \circ s = \mathrm{id}_N$. Then the inclusion $\partial T \longrightarrow T$ is injective in cohomology, so the map $H^{w-n}(T, \partial T) \longrightarrow H^{w-n}(T)$ is the zero map. Then χ vanishes.
- **Theorem:** If there is a non-zero section of the normal bundle of N in W (both oriented), then $\chi_N^W = 0$.
- **Fact:** The tangent bundle of a smooth manifold N is isomorphic to the normal bundle of the diagonal in $N \times N$: Fix a Riemannian metric on N and consider the product metric on M. Tangent vectors to the diagonal have the form (v, v). Normal vectors have the form $(v, -v)$.

  Let $\mathrm{pr} : W \longrightarrow \Delta$ be tubular neighborhood of the diagonal. The fibers of pr have a tangent space at the intersection of the fiber with the diagonal, and one can choose a diffeomorphism of a neighborhood of the zero section which fixes the diagonal and whose differential is the projection of the tangent space of the fiber at the zero section to $T_{(p,p)}\Delta^\perp$.

  When one proves the tubular neighborhood theorem, one shows among other things the following: Fix a Riemannian metric on M. Consider $\nu_N^M = (TN)^\perp$. This is a family of subvectorspaces of TM perpendicular to TN. The tubular neighborhood is diffeomorphic to

$$\{v \in \nu_N^M \mid \|v\| \leq 1\}.$$

  and zero-vectors correspond to points in N.

  We obtain the following: The tubular neighborhood of $N \subset N \times N$ is isomorphic to the tubular neighborhood of N in ν_N^M which is in turn isomorphic to a tubular neighborhood of $N \subset TN$ (note that the latter is a smooth $2n$-manifold (like $N \times N$).
- **Goal:** The current goal is to describe the Euler class explicitly. We did not show, that the chohomology of a closed smooth manifold is finitely generated. This can be proved in many ways, but the slickest way might be to use cellular homology and the fact that closed manifolds have the homotopy type of finite CW-complexes. For smooth manifolds this can be shown using Morse theory [Mi]. The claim is still true for topological manifolds, but the proof is different.
- **Theorem:** Let N be a smooth, closed n-manifold ($\partial N = \emptyset$). Fix a field F of coefficients and fix a basis α_i of $H^*(N; F)$. The dual basis is denoted by α_i^*, i.e.

$$\langle \alpha_i^* \cup \alpha_j, [N] \rangle = \delta_{ij}. \tag{39}$$

Then the Thom class of the normal bundle of N in $N \times N$ (or the Thom class of the normal bundle in $N \times N$/tangent bundle $TN \longrightarrow N$) is

$$\tau_N^{N\times N} = (-1)^n \sum_i (-1)^{|\alpha_i|} \alpha_i^* \times \alpha_i \in H^n(N \times N).$$

For the Euler class, we get

$$\chi = d^*\tau = (-1)^n \sum_i (-1)^{|\alpha_i|} \alpha_i^* \cup \alpha_i.$$

Moreover,

$$\langle \chi, [N] \rangle = (-1)^n \chi(N) = \text{Eulercharacteristic of } M.$$

- The factor $(-1)^n$ is not nice, and it does not appear in other sources. The difference seems to come from the convention (23). However, it does not bother too much, since $\chi(M) = 0$ for odd n.
- **Proof:** By the Künneth formula, there are coefficients x_{ij} such that

$$\tau = \sum_{i,j} x_{ij} \alpha_i^* \times \alpha_j \in H^n(N \times N; F)$$

On $N \times N$ we pick the product orientation $[N \times N] := [N] \times [N]$. On the one hand

$$\begin{aligned}
\langle (\alpha_i \times \alpha_j^*) \cup \tau, [N \times N] \rangle &= \langle \alpha_i \times \alpha_j^*, [N \times N] \cap \tau \rangle \quad \text{by (34)} \\
&= \langle \alpha_i \times \alpha_j^*, d_*[N] \rangle \quad \text{by (36)} \\
&= \langle d^*(\alpha_i \times \alpha_j^*), [N] \rangle \\
&= \langle \alpha_i \cup \alpha_j^*, [N] \rangle \\
&= (-1)^{|\alpha_i|(n-|\alpha_j|)} \langle \alpha_j^* \cup \alpha_i, [N] \rangle \\
&= (-1)^{|\alpha_i|(n-|\alpha_j|)} \delta_{ij}.
\end{aligned}$$

On the other hand

$$\begin{aligned}
\langle (\alpha_i \times \alpha_j^*) \cup \tau, [N \times N] \rangle &= \left\langle (\alpha_i \times \alpha_j^*) \cup \left(\sum_{k,l} x_{kl} \alpha_k^* \times \alpha_l \right), [N] \times [N] \right\rangle \\
&= \sum_{k,l} x_{kl} (-1)^{(n-|\alpha_k|)(n-|\alpha_j|)} \left\langle (\alpha_i \cup \alpha_k^*) \times \left(\alpha_j^* \cup \alpha_l \right), [N] \times [N] \right\rangle \quad \text{by (29)} \\
&= \sum_{k,l} x_{kl} (-1)^{(n-|\alpha_k|)(n-|\alpha_j|)+|\alpha_i|(n-|\alpha_k|)} \langle \alpha_k^* \cup \alpha_i, [N] \rangle \langle \alpha_j^* \cup \alpha_l, [N] \rangle \quad \text{by (23)} \\
&= x_{ij} (-1)^{(n-|\alpha_i|)(n-|\alpha_j|)+|\alpha_i|(n-|\alpha_i|)}
\end{aligned}$$

Comparing coefficients, one gets $x_{ij} = (-1)^{n+|\alpha_i|} \delta_{ij}$. The formula for χ and the Euler characteristic follow immediately.
- Consider a map $f : N \longrightarrow N$ (N closed, oriented) and its graph

$$\Gamma(f) = \{(x, f(x)) \,|\, x \in N\}$$

with the orientation $[\Gamma] = ((1 \times f) \circ d)_*[N]$. As before, $[N \times N] = [N] \times [N]$. The intersection number $[\Gamma] \cdot [\Delta]$ of the diagonal Δ with Γ is $\varepsilon_*([\Gamma] \bullet [\Delta])$. Recall that $\varepsilon : C_0(N) \longrightarrow \mathbb{Z}$ is the augmentation map $\varepsilon(\sum_p c_p \cdot p) = \sum_p c_p$.

  Note that $\Gamma \cap \Delta$ are the fixed points of f.
- **Theorem:** $L(f) = [\Gamma] \cdot [\Delta]$.

- **Proof:** Let $\gamma = PD([\Gamma])$ in $N \times N$, and τ the Poincare dual of the diagonal (=Thom class of TN)

$$
\begin{aligned}
[\Gamma]\cdot[\Delta] &= \varepsilon_*([\Gamma]\bullet[\Delta]) \quad \text{by (37)}\\
&= \varepsilon_*([N\times N]\cap(\gamma\cup\tau))\\
&= \langle \gamma\cup\tau, [N\times N]\rangle\\
&= (-1)^n\langle \tau, [N\times N]\cap\gamma\rangle \quad \text{by (34)}\\
&= (-1)^n\langle \tau, [\Gamma]\rangle\\
&= (-1)^n\langle \tau, (1\times f)_* d_*[N]\rangle\\
&= (-1)^n\langle d^*(1\times f)^*\tau, [N]\rangle\\
&= \sum_i (-1)^{|\alpha_i|}\left\langle d^*(1\times f)^*\left(\alpha_i^*\times\alpha_i\right), [N]\right\rangle \quad \text{by previous Thm.}\\
&= \sum_i (-1)^{|\alpha_i|}\left\langle \left(\alpha_i^*\cup f^*\alpha_i\right), [N]\right\rangle \quad \text{by Defn. of cup-prod.}\\
&= \sum_i (-1)^{|\alpha_i|}\left\langle \alpha_i^*\cup\left(\sum_j f_{ij}\alpha_j\right), [N]\right\rangle\\
&= \sum_i (-1)^{|\alpha_i|} f_{ii} \quad \text{by (39)}\\
&= L(f).
\end{aligned}
$$

- This reproves the Lefschetz fixed point theorem for closed manifolds and allows to interpret the Lefschetz number more geometrically. It is sad to have $\chi(M) = (-1)^n L(\mathrm{id})$, though.
- **Fact:** N smooth, closed, orientable, $f : N \longrightarrow N$ smooth such that Df_p does not have 1 as an eigenvalue for any fixed point $p \in Fix(f)$ of f. Then

$$L(f) = \sum_{p\in Fix(f)} \mathrm{sign}(\det(I - Df_p)).$$

- The hypothesis on the eigenvalues implies that $Df_p(T_pN)\cap T_pN = 0$ for all fixed points of f. Hence $(1\times f)(N) = \Gamma$ is transverse to the diagonal. $\Gamma\cap\Delta$ is a collection of points and we have to determine the sign assigned to each intersection point by an appropriate orientation convention. This is best done in exmples in each dimension n.
- **Proposition:** If W^{n+k} is an oriented k-disk bundle over N, then the self intersection class $[N]_W \bullet [N]_W$ is the image of the Poincaré-dual of the Euler class of the bundle under the inclusion $i_N^W : N \longrightarrow W$.
- **Proof:** By computation:

$$
\begin{aligned}
[N]_W\bullet[N]_W &= [W]\cap(\tau_N^W\cup\tau_N^W)\\
&= ([W]\cap\tau_N^W)\cap\tau_N^W\\
&= i_{N*}^W[N]\cap\tau_N^W\\
&= i_{N*}^W([N]\cap i_N^{W*}\tau_N^W)\\
&= i_{N*}^W([N]\cap\chi_N^W).
\end{aligned}
$$

- **Fact:** Assume that $k = n$ and N is connected. Then if $[N]_W\bullet[N]_W = 0$, then it is possible to find a section $s : N \longrightarrow W$ such that $s(N)\cap N = \emptyset$

- **Summer reading:** [Ro] is a classic in low dimensional geometric topology, centering mostly around knots and links in S^3. [Sa] is a great introduction to the Casson invariant for homology 3-spheres. Several parts and some of the appendices of Chapter 2 and 3 of [Ha], for example about H-spaces. Finally, any book by Milnor, for example [Mi, Mi2] or the more basic [Mi3]

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[1]there are slight differences between the printed and the online versions