Ludwig-Maximilians-Universität München
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Lecture in the winter term 2017/18

# Topology 1

*Please note:* These notes summarize the content of the lecture, many details and examples are omitted. Sometimes, but not always, we provide a reference for proofs, examples or further reading. I will not attempt to give the first reference where a theorem appeared. Some proofs might take two lectures although they appear in a single lecture in these notes. Changes to this script are made without further notice at unpredictable times. If you find any typos or errors, please let me know.

## 1. Lecture on Oct. 16 - Basic notions from point set topology

- **Reference:** Chapter 1 in [Jä]
- **Definition:** Let $X$ be a set. A *topology* on $X$ is a subset $\mathcal{T} \subset \{\text{subsets of } X\} = P(X)$ such that
  - $\emptyset, X \in \mathcal{T}$,
  - $U, V \in \mathcal{T}$ implies $U \cap V \in \mathcal{T}$, and
  - $U_i \in \mathcal{T}$ for $i \in I$ ($I$ is some index set) implies $(\cup_i U_i) \in \mathcal{T}$.

  A set $U \in \mathcal{T}$ is called *open.* $A \subset X$ is *closed* if $X \setminus A$ is open.
- **Examples:** $\mathcal{T} = \{\emptyset, X\}$ and $P(X)$ are topologies. If $(X, d)$ is a metric space, then
$$\mathcal{T}_d = \{U \subset X \,|\, \text{for all } u \in U \text{ there is } \delta > 0 \text{ such that } B_\delta(u) \subset U\}$$
  is a topology. Here $B_\delta(u) = \{v \in X \,|\, d(u,v) < \delta\}$ is the $\delta$-ball around $u$.
- **Definition:** A topological space $(X, \mathcal{T})$ is *Hausdorff* if for all $u \neq v$ there are open sets $U \ni u$ and $V \ni v$ such that $U \cap V = \emptyset$. It is *connected* if for all $U, V \in \mathcal{T}$ such that $U \cup V = X$ and $U \cap V = \emptyset$ it follows that $U = \emptyset$ or $V = \emptyset$.

  A subset $\mathcal{U} \subset \mathcal{T}$ is an open cover of $X$ if $\cup_{U \in \mathcal{U}} U = X$. $X$ is *compact* if for every open cover $\mathcal{U}$ there is a finite collection $U_1, U_2, \ldots, U_n \in \mathcal{U}$ such that $U_1 \cup \ldots \cup U_k = X$.
- **Examples:** Every topology which is induced by a metric is Hausdorff. $[0,1]$ is compact and connected in the standard topology (i.e. the one induced by the metric $d(x,y) = |x-y|$).
- **Proof:** Hausdorff is clear. For $U, V$ open and disjoint in $[0,1]$ such that $U \cup V = [0,1]$ assume $0 \in U$ and $V \neq \emptyset$ consider $x_0 = \sup\{t \,|\, [0,t) \subset U\}$. Then $x_0 \notin U$ but $x_0 \in V$ leads to a contradiction to the choice of $x_0$.

  In order to show that $[0,1]$ is compact we show the following: If $\mathcal{U}$ is an open cover of $[0,1]$, then there is $\delta > 0$ such that
$$\forall x \in [0,1] \exists U \in \mathcal{U} : (x - \delta, x + \delta) \cap [0,1] \subset U.$$
  If this is not true, then there is a sequence $x_n$ such that $(x_n - 1/n, x_n + 1/n) \not\subset U$ for all $U \in \mathcal{U}$. But $(x_n)$ has a convergent subsequence because $[0,1]$ is bounded an complete. The limit point of such a subsequence lies in one set $U\mathcal{U}$. This leads to a contradiction to the choice of the sequence $(x_n)$. Using $\delta$ one can extract a finite subcover from $\mathcal{U}$.

- **Definition:** Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces and $f : X \longrightarrow Y$ a map. $f$ is *continuous* if for all $V \in \mathcal{T}_Y$ the set $f^{-1}(V)$ is open in $X$.
- **Theorem (Heine-Borel)**: A set $K \subset \mathbb{R}^n$ with the topology induced by the Euclidean distance is compact if it is closed and bounded.
- **Remark:** When $\mathcal{T}_X$ and $\mathcal{T}_Y$ are induced by metrics, then this definition is equivalent to the $\varepsilon$-$\delta$-definition of continuity.
- **Definition:** Let $A \subset (X, \mathcal{T})$. Then one defines the *subspace topology* on $A$ by
$$\mathcal{T}_A = \{V \subset A \,|\, \text{there is } U \in \mathcal{T} \text{ such that } U \cap A = V\}.$$
This is the smallest topology on $A$ for which the inclusion map $i : A \hookrightarrow X$ is continuous.
- **Definition:** Let $\sim$ be an equivalence relation on $(X, \mathcal{T})$. Then the *quotient topology* on $X/\sim$ is
$$\mathcal{T}_\sim = \{V \subset X/\sim \,|\, \mathrm{pr}^{-1}(V) \in \mathcal{T}\}$$
where $\mathrm{pr} : X \longrightarrow X/\sim$ is the quotient map. The quotient topology is the largest topology for which pr is continuous.
- **Lemma:** If $(X, \mathcal{T})$ is compact and $A \subset X$ is closed, then $A$ is compact.
- **Proof:** If $\mathcal{V} \subset \mathcal{T}_A$ is an open cover of $A$, then for all $V \in \mathcal{V}$ there is $U(V) \in \mathcal{T}$ such that $U(V) \cap A = V$. Then $\{U(V) \,|\, V \in \mathcal{V}\} \cup \{X \setminus A\}$ is an open cover of $X$. From the compactness we get a finite subcover of $X$ and from that a finite subcover of $A$.

## 2. Lecture on Oct. 19 - Basic notions from point set topology

- **Lemma:** Let $K$ be compact, $X$ any topological space and $f : K \longrightarrow X$ continuous. Then $f(K)$ is compact (viewed as subspace of $X$).
- **Proof:** An from an open cover of $f(K)$ we obtain an open cover of $K$.
- **Proposition:** Let $X$ be a compact Hausdorff space and $K \subset X$ compact. Then $K$ is a closed subset of $X$.
- **Proof:** Let $x \in X \setminus K$. We will find an open $U_x \subset X \setminus K$. For all $a \in K$ there are disjoint open sets $V(x, a) \ni a$ and $U(a) \ni x$. Then $V(x, a) \cap K, a \in A$ is an open cover of $K$ and we get an open subcover $V(x, a_1), \ldots, V(x, a_k)$ of $K$. Then $U_x = \cap_i U(a_i)$ is as desired and
$$X \setminus K = \bigcup_{x \in X \setminus K} U_x$$
is open.
- **Definition:** $f : X \longrightarrow Y$ is a *homeomorphism* if $f$ is bijective, continuous and $f^{-1}$ is also continuous.
- **Theorem:** Assume that $X$ is compact and $Y$ is Hausdorff. If $f : X \longrightarrow Y$ is continuous and bijective, then $f^{-1}$ is continuous.
- **Proof:** $f^{-1}$ continuous iff[1] $f(A)$ is closed for all closed $A \subset X$. But $A$ is compact, hence $f(A)$ is compact and $f(A)$ is closed.
- **Example:** Consider
$$f : [1, 2) \cup [3, 4] \longrightarrow [1, 3]$$
$$x \longmapsto \begin{cases} x & x < 2 \\ x - 1 & x \geq 2. \end{cases}$$

[1] *iff* means *if and only if*

- **Notation:** $D^n = B_1(0) \subset \mathbb{R}^n$, $S^n = \{x \in \mathbb{R}^{n+1} \,|\, \|x\| = 1\}$, $I = [0,1]$.
- **Question:** Which of these spaces are homeomorphic to each other?
- **Example:** $D^1$ is not homeomorphic to $D^n$ for $n \geq 2$. Removing one point from $D^1$ results in a non-connected space while this is not the case for $D^n$.
- **Definition:** Let $B \subset X$ be a subset. Then

$$\mathring{B} = \bigcup_{U \subset B \text{ open}} U \text{ is the interior of } B$$

$$\overline{B} = \bigcap_{A \supset B \text{ closed}} A \text{ is the closure of } B$$

$$\partial B = \overline{B} \setminus \mathring{B} \text{ is the boundary of } B.$$

- **Example:** $\partial D^n = S^{n-1}$.
- **Terminology:** Let $\mathcal{T}, \mathcal{T}'$ be two topologies on $X$. Then $\mathcal{T}$ is *finer* than $\mathcal{T}'$ if $\mathcal{T} \supset \mathcal{T}'$ (then $\mathcal{T}'$ is *coarser*).
- **Definition:** Let $(X, \mathcal{T})$ be a topological spaces. A collection of open sets $U_j, j \in J$ is a subbasis of $\mathcal{T}$ if $\mathcal{T}$ is the smallest/coarsest topology containing $U_j$ for all $j$. It is a basis if every open set is a union of some of the sets $U_j$.
- **Definition:** Let $(X_i, \mathcal{T}_i), i \in I$ be a family of topological spaces. Then the *product topology* on $\prod_i X_i$ has the subbasis

$$U_{i,V_i} = \mathrm{pr}_i^{-1}(V_i), \text{ with } i \in I \text{ and } V_i \in \mathcal{T}_i.$$

Here $\mathrm{pr}_i : \prod_i X_i \longrightarrow X_i$ is the projection. The product topology is the coarsest topology for which all projection maps are continuous.
- **Theorem (Tychonoff, c.f. Chapter 10 in** [Jä]**):** If all $X_i, i \in I$, are compact, then $\prod_i X_i$ is compact with the product topology.
- **Lemma:** Let $X, Y$ be spaces and $\sim_X, \sim_Y$ equivalence relations. If $f(x) \sim_Y f(x')$ for all $x \sim_X x'$, then the map

$$f : X/\sim_X \longrightarrow Y/\sim_Y$$

is well defined and continuous.
- **Examples of common equivalence relations:** If $A \subset X$ is a subset then $\sim$ defined by $a \sim a'$ for all $a, a' \in A$ and $x \sim x$ for all $x \in X$ defines an equivalence relation. We write $X/A := X/\sim$.

  If $f : X \longrightarrow Y$ is a map, then $x \sim x' \Leftrightarrow f(x) = f(x')$ defines an equivalence relation.
- **Definition:** Let $\sim$ be an equivalence relation on $X$ and $A \subset X$. Then $A^* = \{x \in X \,|\, x \sim a \text{ for some } a \in A\}$ is the *saturation* of $A$.
- **Propostion:** Let $\sim$ be an equivalence relation on the space $X$ and $A \subset X$ be such that $A^* = X$. Then

$$k : A/\sim \longrightarrow X/\sim$$

is a homeomorphism if and only if the saturation of every open (closed) set in $A$ is open (closed) in $X$
- **Warning:** Let $X = [0,1]$, $\sim$ generated by $0 \sim 1$ and $A = [0,1)$. Then $A^* = X$ and $A/\sim \simeq A$ is not homeomorphic to $X/\sim$. Note that $U = [0, 1/2) \subset A$ is open in $A$ but $U^* = U \cup \{1\}$ is not open in $X$.

## 3. Lecture on Oct. 23 - More on quotients and gluing of spaces

- **Reference:** Sections 1.3, 1.6 in [StZ], Section I.13 in [Br]
- **Remark:** Consider $C \subset X$ closed. Then the restriction
$$f : X \setminus C \longrightarrow (X/C) \setminus \{[C]\}$$
of the quotient map pr is a homeomorphism.
- **Proof:** The map $f$ is continuous, injective, surjective. Now let $A \subset X \setminus C$ be closed, and $B \subset X$ closed such that $B \cap (X \setminus C) = A$. Since $B \cup C$ is closed in $X$ and
$$\mathrm{pr}^{-1}\left(\mathrm{pr}(B) \cup \{[C]\}\right) = B \cup C,$$
$\mathrm{pr}(B) \cup \{[C]\}$ is closed in $X/C$. Hence, $\mathrm{pr}(B) \cap (X/C) \setminus \{[C]\} = f(A)$ is closed in $(X/C) \setminus \{[C]\}$.
- **Definition:** A *pair of spaces* $(X, A)$ is a pair of a topological spaces and a subset $A \subset X$ with the subset topology. A map $f : (X, A) \longrightarrow (Y, B)$ is a map of pairs of spaces if $f : X \longrightarrow Y$ is a map and $f(A) \subset B$ and it is continuous if $f : X \longrightarrow Y$ is. We abbreviate $(X, X)$ with $X$.
- **Remark:** Usually, $A$ will be closed.
- **Definition:** A map $f : (X, A) \longrightarrow (Y, B)$ is a *relative homeomorphism* if $f : X/A \longrightarrow Y/B$ is a homeomorphism.
- **Definition:** Let $f : A \longrightarrow Y$ be a continuous map and $A \subset X$ be closed. Then $Y \cup_f X = (Y \cup X)/\sim$ with $f(a) \sim a$ for all $a \in A$ is the result of *attaching $X$ to $Y$ using $f$*.
- **Proposition:** In this situation the map $j_Y : Y \longrightarrow Y \cup_f X$ is an embedding (i.e. a homeomorphism onto its image). The same is true for $X \setminus A \longrightarrow Y \cup_f X$.
- **One main point of the proof:** $j_Y$ is closed: Let $i_Y : Y \longrightarrow Y \cup X$ be the inclusion and $\mathrm{pr} : Y \cup X \longrightarrow Y \cup_f X$ the projection. Let $B \subset Y$ be closed, then
$$\mathrm{pr}^{-1}(\mathrm{pr} \circ i_Y(B)) = B + \underbrace{f^{-1}(B)}_{\subset A} \subset Y \cup X$$
is closed since $A$ is closed in $X$. Therefore, the image $\mathrm{pr} \circ i_Y(B)$ is closed.
- **Important special case of the previous definition:** $(X, A) = (D^n, S^{n-1} = \partial D^n)$. In this case the map $\mathring{D}^n \longrightarrow Y \cup_f D^n$ is denoted by $e^n$. This case is referred to as attachment of a $n$-cell.
- **Notation:** Let $f : X \longrightarrow Y$ be continuous. Then the *mapping cylinder of $f$* is
$$M_f = \big(Y \cup X \times [0, 1]\big)/(f(x) \sim x \times \{0\}).$$
The *mapping cone* is $C_f = M_f/(M \times \{1\})$. If $X$ is a space, then the *suspension* $\Sigma X$ of $X$ is $X \times [-1, 1]/\sim$ with
$$(x, -1) \sim (x', -1) \text{ for all } x, x' \in X$$
$$(x, 1) \sim (x', 1) \text{ for all } x, x' \in X.$$
- **Reference:**
- **Important example:** Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$, $n = 1, 2, 3, \ldots$. Then
$$\mathbb{KP}^n = \mathbb{K}^{n+1} \setminus \{0\}/\sim \text{ with } x \sim x' \Longleftrightarrow x = \lambda x', \lambda \in \mathbb{K}.$$
The quotient topology is induced by a metric: Equip the real vector space $\mathbb{K}^n$ with a Euclidean scalar product. Let $d([x], [y])$ be the angle between the

subspaces $[x], [y] \subset \mathbb{K}^n$ (the angle taking values in $[0, \pi)$) The maps

$$\begin{aligned} \mathbb{K}^n &\longrightarrow \mathbb{K}^{n+1} \\ (x_1, \dots, x_n) &\longmapsto (x_1, \dots, x_n, 0) \end{aligned}$$

induce embeddings $\mathbb{KP}^{n-1} \longrightarrow \mathbb{KP}^n$ (i.e. homeomorphism onto their image) and

$$\begin{aligned} \varphi : \mathbb{K}^n &\longrightarrow \mathbb{KP}^n \setminus \mathbb{KP}^{n-1} \\ (x_1, \dots, x_n) &\longmapsto [(x_1, \dots, x_n, 1)] \end{aligned}$$

is a homeomorphism onto its image (compute the inverse). One can identify the real vector space $\mathbb{K}$ with $\mathbb{R}^d$ (with $d = \dim_{\mathbb{R}}(\mathbb{K}) \in \{1, 2, 4\}$) and $\mathbb{K}^n$ with the interior of a $dn$-ball via

$$\begin{aligned} G : \mathring{D}^{dn} &\longrightarrow \mathbb{K}^n \\ x &\longmapsto \frac{x}{1 - \|x\|}. \end{aligned}$$

The composition $e = h \circ G$ map extends to the *closed* $dn$-ball as follows

$$\begin{aligned} F : \overline{D}^{dn} &\longrightarrow \mathbb{KP}^n \\ x &\longmapsto [(x, 1 - \|x\|)]\,. \end{aligned}$$

The restriction of $F$ to $\partial D^{dn} = S^{dn-1}$ is denoted by $f$. Then the map

$$\begin{aligned} \mathbb{KP}^{n-1} \cup_f \overline{D}^{dn} &\longrightarrow \mathbb{KP}^n \\ [x] &\longmapsto \begin{cases} [(x, 0)] & \text{if } x \in \mathbb{KP}^{n-1} \\ F(x) & \text{if } x \in D^{dn} \end{cases} \end{aligned}$$

is well defined, continuous, injective and surjective. Since both sides are compact/Hausdorff, it is a homeomorphism. Thus, we can think of the space $\mathbb{KP}^n$ as $\mathbb{KP}^{n-1}$ with a $dn$-ball attached via $f$.

- **Example:** Let $\mathbb{K} = \mathbb{C}$ and $n = 1$. Then $\mathbb{CP}^0$ has exactly one point ($\mathbb{C}$ itself). Thus the attaching map $f : D^2 \longrightarrow \mathbb{CP}^0$ is determined and we can think of $\mathbb{CP}^1$ as union of a point with a closed disc attached to it in the obvious way.

  The complement of a point in $S^2$ is an open disc. Therefore $\mathbb{CP}^1$ and $S^2$ are homeomorphic as spaces. You should try to figure out analogous statements for $\mathbb{RP}^1$ and $\mathbb{HP}^1$.

## 4. Lecture on Oct. 26 - More on quotients and gluing of spaces

## 5. Lecture on Oct. 30- CW-Complexes

- **Reference:** [StZ], I.4.1
- **Definition:** Let $X$ be a topological space. A *cellular decomposition* of $X$ is a collection

$$\mathcal{Z} = \{e_i : \mathring{D}^{n(i)} \longrightarrow X \mid i \in I\}$$

of homeomorphisms onto their image such that the images, the *open cells*, are pairwise disjoint and cover $X$. '$n_i$ is the dimension of the cell $e_i$. The $n$-skeleton is

$$X^n = \bigcup_{n(i) \leq n} \text{image}(e_i).$$

The *boundary of cell* is $\dot{e} := \overline{e} \setminus e$.

- **Warning:** open cells are not open subsets in general, the boundary of a cell is not the point set theoretic boundary of a cell (in general).
- **Warning:** It is not yet clear, that the dimension of a cell is well defined. It will be shown much later, that this is the case.
- **Notation:** When it matters, I will try to indicate if a ball is open or closed by writing $\mathring{D}$ or $\overline{D}$. However, if we write $S^n \subset D^{n+1}$ or $S^n = \partial D^{n+1}$ it is clear, that we mean the closed ball.
- **Definition:** Let $X$ be a top. space with cellular decomposition and $e \subset X$ one of its $n$ -cells. A map $F : D^n \longrightarrow X$ is a *characteristic map* of $e$ if
  - $F(S^{n-1}) \subset X^{n-1}$, and
  - $e^{-1} \circ F|_{\mathring{D}^n} : \mathring{D}^n \longrightarrow \mathring{D}^n$ is a homeomorphism.

  $F|_{S^{n-1}}$ is the *gluing map* of $e$.
- **Lemma:** Let $X$ be Hausdorff and $F : D^n \longrightarrow X$ a characteristic map. Then $\overline{e} = F(D^n)$ and $\dot{e} = F(S^{n-1})$.
- **Auxiliary exercise:** $f : X \longrightarrow Y$ is continuous iff $f(\overline{A}) \subset \overline{f(A)}$ for all $A \subset X$.
- **Proof of Lemma:** $F(\mathring{D}^n) = e$ and continuity of $F$ implies $F(D^n) \subset \overline{F(\mathring{D}^n)} = \overline{e}$. Moreover, $F(D^n)$ is compact. Since $X$ is Hausdorff, $F(D^n)$ is also closed and contains $e$. Therefore $F(D^n) = \overline{e}$.

  By definition $F(S^{n-1}) = F(D^n \setminus \mathring{D}^n) \supset F(D^n) \setminus F(\mathring{D}^n) = \overline{e} \setminus e = \dot{e}$. Since $F(\mathring{D}^n)$ and $F(\partial D^n)$ are disjoint, the $\supset$ is actually an equality.
- **Definition:** A *CW-complex* is a Hausdorff space $X$ with a cellular decomposition and a characteristic map $F$ for each cell such that
  (C) for every cell $e$, the closure $\overline{e}$ meets only finitely many other cells,
  (W) a set $A \subset X$ is closed if and only if $A \cap \overline{e}$ is closed for each cell (this is the weak topology).

  A CW-complex is finite if there are finitely many cells and has dimension $n$ if $X^n \neq X^{n-1}$ and $X^m = X^n$ for all $m \geq n$.
- **Examples:**
  - $S^n$ has a CW-decomposition with one 0-cell and one $n$-cell.
  - $S^n$ contains $S^{n-1}$ and the complement of this equatorial sphere is a union of two open $n$-balls which admit characteristic maps. Thus $S^n$ admits a cell decomposition with two $n$-cells and all cells in a given CW-decomposition of $S^{n-1}$ which has dimension $n-1$. In particular $S^1 \subset S^2 \subset S^3 \ldots$ provides a CW-decomposition of $S^\infty = \bigcup_n S^n$(equipped with the weak topology).
  - $\mathbb{KP}^n$ from above is finite CW-complex and the union $\mathbb{KP}^\infty = \bigcup_n \mathbb{KP}^n$ with the weak topology is also a CW-complex.
- **Remark:** (C) and (W) are automatic if the the cell decomposition is finite, i.e. if there are finitely many cells. The weak topology is the finest which is consistent with the natural topology on cells.
- **Lemma:** Let $P$ be a set in a CW-complex $X$ such that for each cell $e$, at most one point of $P$ is in $e$. Then $P$ is has the discrete topology and $P$ is closed in $X$.
- **Proof:** Let $Q \subset P$. Then $Q \cap \overline{e}$ is a finite collection of points. Since points in Hausdorff spaces are closed, $Q \cap \overline{e}$ is closed, hence $Q$ is closed (in $P$ and as a subset of $X$).
- **Proposition:** Let $A \subset X$ be a compact subset of a CW-complex. Then $A$ is contained in a finite union of cells.

- **Proof:** For each cell $e$ of $X$ for which $e \cap A \neq \emptyset$ pick $p_e \in e \cap A$. Then $P = \{p_e \,|\, e \text{ a cell of } X\}$ is closed in $X$. Hence $P = A \cap P$ is also compact. Since $P$ has the discrete topology it must be finite.

## 6. Lecture on Nov. 2 - CW-complexes, Homotopy

- **Sublemma:** Each cell of a CW-complex $X$ is contained in a finite union of cells $A$ such that if $e \subset A$, then $\overline{e} \subset A$.
- **Proof:** By induction on the dimension of $e$: If $n = 0$ then $A$ is discrete and $e = \overline{e}$ for all 0-cells. If $\dim(e) = n$, then $\dot{e} \subset X^{n-1}$ is contained in a finite union of cells of dimension $\leq n-1$.
- **Corollary:** A compact subset in a CW-complex is contained in a finite union of cells $A$ which is itself a CW-complex (because it is finite).
- **Definition:** Let $X$ be a CW-complex and $A \subset X$ a union of cells of $X$ which is a CW-complex (with the induced cellular decomposition from $X$ and the subspace topology). Then $A$ is a *subcomplex* and $(X, A)$ is a *CW-pair*.
- **Proposition:** Let $A \subset X$ be a union of cells. Then the following are equivalent:
  (1) $A$ is a subcomplex.
  (2) $A \subset X$ is closed.
  (3) for all cells $e \subset A$, $\overline{e} \subset A$
- **Proof:** Everything is clear if $A$ is a finite subcomplex.
  (2)⇒(3): obvious.
  (3)⇒(1): $A$ is Hausdorff, the cellular decomposition of $A$ is induced by $X$ and so are the corresponding characteristic maps (which are well defined since the image of the characteristic map of a cell $e \subset A$ is $\overline{e} \subset A$). (C) follows from the analogous property of $X$.

  Let $B \subset A$ be a subset with the property that $B \cap \overline{e}$ is closed for all cells $e \subset A$. Let $f \subset X$ be a cell and $A_f$ a finite subcomplex containing $f$. Then

$$\overline{f} \cap B = \bigcup_{e \subset A_f \cap A \text{ a cell}} \overline{f} \cap \underbrace{\overline{e} \cap B}_{\text{closed/compact in } \overline{e}}.$$

  This is closed as a finite union of closed sets, i.e. $B$ is closed in $X$ and hence in $A$.

  Conversely, if $B \subset A$ is closed, then there is $B' \subset X$ closed with $B' \cap A = B$. Then for every cell $e$ in $A$

$$\overline{e} \cap B' = \overline{e} \cap B$$

  is closed/compact in $\overline{e}$.

  (1)⇒(2): Let $f \subset X$ a cell. Then the same argument as before shows that $\overline{f} \cap A$ is a finite union of compact sets (the computation uses $\overline{e} \subset A$ for all cells $e$ in $A$). Hence $A$ is closed.
- **Corollary:** $X^n$, finite unions and finite intersections of subcomplexes of $X$ are subcomplexes of $x$.
- **Remark:** Let $X$ be a CW-complex, $X' \subset X$ a subcomplex, and $e \subset X \setminus X'$ a $n$-cell such that $\dot{e} \subset X$. Let $F_e$ be a characteristic map for $e$. Then $X' \cup e \subset X$

is a subcomplex and

$$\begin{aligned} X' \cup_{f_e} \overline{D}^n &\longrightarrow X' \cup e \\ [x' \in X'] &\longmapsto x' \in X' \\ [y \in \overline{D}^n] &\longmapsto F_e(y) \end{aligned}$$

is well-defined, continuous and bijective. It is also closed.

- **Definition:** Continuous $f, g : X \longrightarrow Y$ are *homotopic* if there is a continuous map $H : X \times [0,1] \longrightarrow Y$ such that $H(\cdot, 0) = f(\cdot)$ and $H(\cdot, 1) = g(\cdot)$. $H$ is called *homotopy.*
- **Examples:** Every map into $\mathbb{R}^n$ or out of $\mathbb{R}^n$ is homotopic to a constant map.

## 7. Lecture on Nov. 6 - Homotopy, Degree of maps $S^1 \longrightarrow S^1$

- **Reference:** [StZ] I.2.2.
- **Notation:** $E : \mathbb{R} \longrightarrow S^1, t \longmapsto e^{2\pi i t}$. $S^1$ is a group, $f(1) \cdot E(\cdot)$ refers to that group structure.
- **Lemma:** Let $f : S^1 \longrightarrow S^1$ be continuous. Then there is a unique continuous function $\varphi : I \longrightarrow \mathbb{R}$ such that $\varphi(0) = 0$ and $f(E(t)) = f(1)E(\varphi(t))$, i.e. the following diagram commutes.

(1)
$$\begin{tikzcd} ([0,1], 0) \arrow[r, "\varphi"] \arrow[d, "E"] & (\mathbb{R}, 0) \arrow[d, "f(1)\cdot E"] \\ S^1 \arrow[r, "f"] & S^1 \end{tikzcd}$$

- **Proof:** We fix $\log : S^1 \setminus \{-1\} \longrightarrow (-\pi, \pi)$ a (part of a ) branch of the inverse function of $\exp : \mathbb{C} \longrightarrow \mathbb{C}, z \longrightarrow e^z$. There is a finite decomposition $t_0 = 0 < t_1 < t_2 < \ldots < t_k = 1$ of $[0,1]$ such that

$$f(E([t_i, t_{i+1}])) \subset S^1 \setminus \{-f(t_i)\}.$$

One defines $\varphi$ inductively:

$$\begin{aligned} \varphi(t) &= \frac{1}{2\pi i} \log\left(\frac{f(t)}{f(t_0)}\right) && \text{when } t \in [t_0, t_1] \\ \varphi(t) &= \varphi(t_1) + \frac{1}{2\pi i} \log\left(\frac{f(t)}{f(t_1)}\right) && \text{when } t \in [t_1, t_2] \\ \varphi(t) &= \varphi(t_2) + \frac{1}{2\pi i} \log\left(\frac{f(t)}{f(t_2)}\right) && \text{when } t \in [t_2, t_3] \\ &\vdots \end{aligned}$$

where $\varphi(t_1)$ is computed from the first line, $\varphi(t_1)$ is computed in the second line etc.

A direct computation shows that $\varphi$ has the desired property. This shows existence.

Let $\hat{\varphi}, \varphi$ be two solutions. Then $f(1)E(\varphi(t)) = f(1)E(\hat{\varphi}(t))$ and hence $\varphi(t) - \hat{\varphi}(t)$ is continuous, takes values in the integers and $\varphi(0) - \hat{\varphi}(0) = 0$. Since $[0,1]$ is connected $\varphi - \hat{\varphi} \equiv 0$. This shows uniqueness.

- **Remark:** $\varphi(1) \in \mathbb{Z}$ since $f(1)E(\varphi(1)) = f(\varphi(1)) = f(\varphi(0)) = f(1)E(\varphi(0))$.

- **Definition:** The *degree* of $f : S^1 \longrightarrow S^1$ is
$$\deg(f) = \varphi(1)$$
where $\varphi : I \longrightarrow \mathbb{R}$ is the function from the previous Lemma.
- **Examples:**
  - $f = \mathrm{id}$, then $\varphi(t) = t$ and $\deg(f) = 1$.
  - $f = z_0$, constant, then $\varphi(t) = 0$ and $\deg(f) = 0$.
  - $f_n(z) = z^n, n \in \mathbb{Z}$,then $\varphi_n(t) = nt$ and $\deg(f_n) = n$.
- **Lemma:** $f, g : S^1 \longrightarrow S^1$ are homotopic iff $\deg(f) = \deg(g)$.
- **Proof:** $\Leftarrow$: Let $\varphi_f, \varphi_g$ be the functions from the previous lemma associated to $f, g$. Fix a path $\gamma : [0,1] \longrightarrow S^1$ such that $\gamma(0) = f(1)$ and $\gamma(1) = g(1)$. Then

(2)
$$\begin{aligned} &H : S^1 \times [0,1] \longrightarrow S^1 \\ &\qquad (z,s) \longmapsto \gamma(s) \cdot E\left(s\varphi_g(t) + (1-s)\varphi_f(t)\right) \text{ with } E(t) = z, t \in [0,1]. \end{aligned}$$

is well defined since
$$\begin{aligned} E\left(s\varphi_g(1) + (1-s)\varphi_f(1)\right) &= E(s\deg(f) + (1-s)\deg(g)) = 1 \\ E\left(s\varphi_g(0) + (1-s)\varphi_f(0)\right) &= E(0) = 1. \end{aligned}$$
Moreover, $H$ is continuous and
$$\begin{aligned} H(\cdot, 0) &= \gamma(0)E(\varphi_f(t)) = f(1)E(\varphi_f(t)) = f(t) \\ H(\cdot, 1) &= \gamma(1)E(\varphi_g(t)) = g(1)E(\varphi_g(t)) = g(t). \end{aligned}$$

$\Rightarrow$: Assume that $f, g$ are homotopic via $H : S^1 \times [0,1] \longrightarrow S^1$. We will construct
$$\phi : [0,1] \times [0,1] \longrightarrow \mathbb{R}$$
such that $H(1,s)E(\phi(t,s)) = H(E(t), s)$. This is done using a finite covering $[t_i, t_{i+1}] \times (s_0 - \delta(s_0), s_0 + \delta(s_0))$ of $[0,1] \times \{s_0\} \subset [0,1]^2$ with a small $\delta(s_0) > 0$ such that
$$t_0 = 0 < t_1 < t_2 < \ldots < t_k = 1$$
such that $H([t_i, t_{i+1}] \times (s_0, -\delta(s_0), s_0 + \delta(s_0))) \subset S^1 \setminus \{-H(t_i, s_0)\}$. Using the formulas we obtain $\phi_{\delta(s_0), s_0}$ on a $\delta(s_0)$-neighborhood of $s_0$. Recall that $\phi(t, s_0)$ is uniquely determined by $H(t, s_0)$ for fixed $s_0 \in [0,1]$. Hence the functions $\phi_{\delta(s_0), s_0}$ can be glued to form a continuous function $\phi$ on $[0,1] \times [0,1]$. Then
$$\begin{aligned} \deg(f) &= \varphi_f(1) = \phi(1,0) \\ &= \phi(1,s) = \phi(1,1) = \varphi_g(1) = \deg(g) \end{aligned}$$
since $\phi(1,s) = \deg(H(\cdot, s))$ is continuous (left-hand side) and an integer (right-hand side), and therefore constant.
- **Theorem:** Let $F : \overline{D}^2 \longrightarrow \overline{D}^2$ be continuous. Then $F$ has a fixed point, i.e. there is $x \in \overline{D}^2$ such that $F(x) = x$.
- **Proof:** Assume not. Then
$$\begin{aligned} &G : \overline{D}^2 \longrightarrow S^1 \\ &\qquad x \longmapsto (\text{ray from } f(x) \text{ through } x) \cap S^1 \end{aligned}$$
is continuous and $G(x) = x$ (see Figure 1). Then $G|_{S^1 = \partial D^2} = \mathrm{id}_{S^1}$ has degree 1. But $G|_{S^1 = \partial D^2}$ is homotopic to a constant map via
$$H(z,t) = G(tz) \text{ with } z \in S^1, t \in [0,1]$$

so the degree of $G$ should be 0.

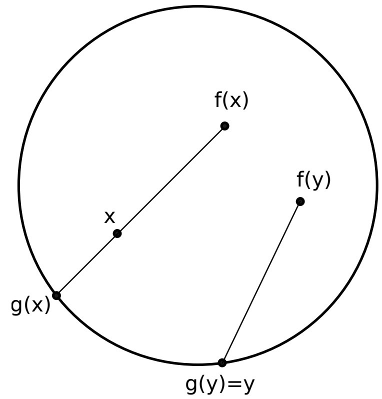


FIGURE 1. Construction of a retraction $G : \overline{D}^2 \longrightarrow S^1$ in the proof of Brouwers fixed point theorem.

- Other applications include the fundamental theorem of algebra. The details can be found in [StZ], Satz 2.2.9, p.56.
- **Reference:** Other classical proofs of the fundamental theorem of algebra use the Liouville theorem or other theorems from complex analysis.

  There are also proofs using Galois theory (c.f. for example p.62ff in [Cox]). These also use analysis to show that polynomials of odd degree and real coefficients have zeroes in $\mathbb{R}$.

## 8. Lecture on Nov. 9 - Borsuk-Ulam, homotopy extension property

- **Reference:** [StZ], I.2.2
- **Lemma:** Let $f : S^1 \longrightarrow S^1$ be continuous such that $f(-x) = -f(x)$. Then $\deg(f)$ is odd.
- **Proof:** The function $\varphi$ used above can be extended to $\mathbb{R}$ such that the diagram

$$\begin{tikzcd} (\mathbb{R},0) \arrow[r, "\varphi"] \arrow[d, "E"] & (\mathbb{R},0) \arrow[d, "f(1)\cdot E"] \\ S^1 \arrow[r, "f"] & S^1 \end{tikzcd}$$

  commutes. Then

$$\begin{aligned} f(-E(t)) &= f(E(t+1/2)) = f(1)E(\varphi(t+1/2)) \\ -f(E(t)) &= -f(1)E(\varphi(t)) = f(1)E(\varphi(t)+1/2). \end{aligned}$$

  Hence the continuous function $\varphi(t+1/2) - (\varphi(t)+1/2)$ is an integer $k$ and constant. Then

$$\begin{aligned} \deg(f) = \varphi(1) &= \varphi(1/2) + 1/2 + k = \varphi(0) + 1/2 + k + 1/2 + k \\ &= 2k+1. \end{aligned}$$

- **Theorem(Borsuk-Ulam):** Let $g : S^2 \longrightarrow \mathbb{R}^2$ be continuous. Then there is $p \in S^2$ such that $g(p) = g(-p)$.

- **Proof:** Assume not and consider
$$\begin{aligned} g_1 : S^2 &\longrightarrow S^1 \\ p &\longmapsto \frac{g(p)-g(-p)}{\|g(p)-g(-p)\|}. \end{aligned}$$
Consider the embedding $\psi : D^2 \longrightarrow S^2, \psi(x,y) = (x, y, \sqrt{1-x^2-y^2})$. The restriction of the composition $f = g \circ \psi$ to $\partial D^2 = S^1$ is odd. So $\deg(f)$ is odd. However, $f$ extends to $D^2$, so $\deg(f) = 0$.
- **Reference:** [StZ]
- **Definition:** $A \subset X$ (or the pair $(X, A)$) is a *retract* if there is a continuous map $r : X \longrightarrow A$ such that $r(a) = a$ for all $a \in A$. $r$ is called a *retraction.*
- **Example:** $\{0\} \subset [0, 1]$ is a retract, $\partial I \subset I = [0, 1]$ is not. $\partial D^2 = S^1 \subset D^2$ is not a retract.
- **Proposition:** Let $A \subset X$. The following are equivalent:
  1. $f : A \longrightarrow Y$ extends to $X$.
  2. $(Y \cup_f X, Y)$ is a retract.
- **Lemma:** Assume $A \subset X$ is a retract and $X$ is Hausdorff. Then $A \subset X$ is closed.
- **Proof:** We show that $X \setminus A$ is open, let $x \in X \setminus A$. Then $x \neq f(x) \in A$, so there are $x \in U \subset X$ open, $f(x) \in V \subset X$ which are disjoint. Let $r : X \longrightarrow A$ be a retraction. Then $r^{-1}(V) \cap U$ is an open set containing $x$. Moreover, $r$ moves every point of $r^{-1}(V) \cap U$ into $V$, so $r^{-1}(V) \cap U$ is open and disjoint from $A$. Then $X \setminus A$ is open.
- **Definition:** $(X, A)$ has the *homotopy extension property* (HEP) if for all spaces $Y$ and maps $h, F$ as in the diagram, there is a map $H : X \times I \longrightarrow Y$ such that the diagram commutes
$$\begin{tikzcd}
A \times 0 \arrow[r] \arrow[d] & A \times I \arrow[d] \arrow[ddr, "h"] & \\
X \times 0 \arrow[r] \arrow[drr, "F"'] & X \times I \arrow[dr, dashed, "H"] & \\
& & Y.
\end{tikzcd}$$
Maps without names are inclusions.
- **Lemma:** Let $A \subset X$ be closed[2]. The following are equivalent.
  1. $(X, A)$ has the HEP.
  2. $(A \times I) \cup (X \times 0)$ is a retract of $X \times I$.
- **Proof:** (1) $\Rightarrow$ (2): Apply the diagram above to $Y = (A \times I) \cup (X \times 0)$ and $F, h$ the inclusions. The HEP yields a maps $H =: r$ which is a retraction.
  (2) $\Rightarrow$ (1): Use the retraction to define
$$H(x,t) = \begin{cases} h(r(x,t)) & \text{if } r(x,t) \in A \times I \\ F(r(x,t)) & \text{if } r(x,t) \in X \times 0. \end{cases}$$
The closedness of $A$ is used to prove continuity.
- **Example:** $A = S^n \subset D^{n+1} = X$ has the HEP. A possible retraction as in the lemma is indicated in Figure 2.

[2] This is our standard convention.

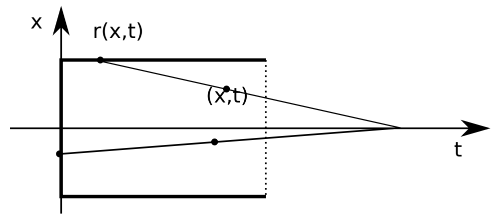


FIGURE 2. Construction of a retraction $D^{n+1} \times I \longrightarrow (D^{n+1} \times 0) \cup (S^n \times I)$

- **Example:** $X = I$ and $A = \{0\} \cup \{n^{-1} \,|\, n = 1, 2, \ldots\}$ does not have the HEP. There is no retraction as in the above Lemma.

## 9. Lecture on Nov. 13 - Homotopy extension property, deformation retractions

- **Definition:** A continuous map $f : X \longrightarrow Y$ is a *homotopy equivalence* if there is a continuous map $g : Y \longrightarrow X$ such that $g \circ f$ is homotopic to $\mathrm{id}_X$ and $f \circ g$ is homotopic to $\mathrm{id}_Y$. Then $g$ is a *homotopy inverse* of $f$ and $X, Y$ are *homotopy equivalent spaces.*
- **Remark:** Homotopy equivalence is an equivalence relation.
- **Examples:**
  - Homeomorphisms are homotopy equivalences.
  - Every convex (or star shaped) subset of $\mathbb{R}^n$ if homotopy equivalent to one point.
  - $S^1 \hookrightarrow \mathbb{R}^2 \setminus \{(0,0)\}$ is a homotopy equivalence.
  - $S^1$ is not homotopy equivalent to $\mathbb{R}^2$ (or any other contractible space): Assume there is a homotopy equivalence $f : S^1 \longrightarrow \mathbb{R}^2$ with $g$ a homotopy inverse. Then since $g \circ f$ is homotopic to $\mathrm{id}_{S^1}$, so $\deg(g \circ f) = 1$. However, $f$, and hence $g \circ f$ is homotopic to a constant map. This implies $\deg(g \circ f) = 0$.
- **Definition:** A space is *contractible* if it is homotopy equivalent to a point.
- **Theorem:** Let $(X, A)$ have the HEP and $A$ contractible. Then the quotient map $X \longrightarrow X/A$ is a homotopy equivalence.
- **Proof:** Fix a homotopy $h : A \times I \longrightarrow A$ with $h(a, 0) = a$ and $h(a, 1) = a_0$ for all $a$ and a suitable $a_0 \in A$. Apply HEP to $Y = X, F : X \times 0 \longrightarrow X, F(x, 0) = x$ and $h$ from above. We obtain a homotopy $H : X \times I \longrightarrow X$ such that $H(x, 0) = x$, $H(a, t) \in A$ and $H(a, 1) = a_0$. Then

$$\begin{aligned} g : X/A &\longrightarrow X \\ [x] &\longmapsto H(x, 1) \end{aligned}$$

  is well defined and a homotopy inverse of $f$, the required homotopies are provided by $H$.
- **Definition:** $A \subset X$ is a *deformation retract* if there is a homotopy

$$H : X \times I \longrightarrow X$$

  such that $H(\cdot, 0) = \mathrm{id}_X$, $H(a, t) = a$ and $H(x, 1) \in A$ for all $x, a, t$.
- **Examples:**
  - $S^n \times I \cup D^{n+1} \times 0 \subset D^{n+1} \times I$ is a deformation retract see Figure 2).
  - $S^1 \subset \mathbb{R}^2 \setminus \{(0,0)\}$ is a deformation retract.

- **Theorem:** Let $(X, A)$ be a CW-pair. Then $X \times 0 \cup A \times I$ is a deformation retract of $X \times I$.
- **Proof:** $X^{n+1} \times I \cup X \times 0$ is obtained from $(X^n \cup A^{n+1}) \times I \cup X \times 0$ by attaching a copies of $D^{n+1} \times [0,1] \simeq D^{n+2}$ along the subspace $\partial D^{n+1} \times I \cup D^{n+1} \times 0$ for each $n+1$-cell in $X^{n+1}$ which is not in $A^{n+1}$. Since $\partial D^{n+1} \times I \cup D^{n+1} \times 0 \subset D^{n+1} \times I$ is a deformation retract,

$$\big( \big(X^n \cup A^{n+1}\big) \times I \cup X \times 0\big) \subset X^{n+1} \times I \cup X \times \{0\}$$

is also a deformation retract. Let

$$h_{n+1} : \big(X^{n+1} \times I \cup X \times \{0\}\big) \times J \longrightarrow X^{n+1} \times I \cup X \times \{0\}$$

be the corresponding deformation retraction, such that

$$h_{n+1}(x, 1) \in \big(X^n \cup A^{n+1}\big) \times I \cup X \times 0,$$

so that $h_{n+1}$ moves points $(x, t)$ for $x \in e$ when $e$ is a $n+1$-cells of $X$ which is not in $A$. Then

$$\begin{aligned} h_{n+1}((x,0), s) &= (x, 0) && \text{for all } x \in X \\ h_{n+1}((x,t), s) &= (x, t) && \text{if } (x,t) \in A \times I. \end{aligned}$$

Then the map $H : (X \times I) \times J \longrightarrow X \times I$ defined by

$$((x,t), s) \longmapsto \begin{cases} (x,t) & \text{if } s = 0 \\ (x,t) & \text{if } x \in X^n \text{ and } s \leq 2^{-(n+1)} \\ h_n\left((x,t), 2^{n+1}s - 1\right) & \text{if } x \in X^n \text{ and } 2^{-(n+1)} \leq s \leq 2^{-n} \\ h_{n-1}\left(h_n((x,t),1), 2^n s - 1\right) & \text{if } x \in X^n \text{ and } 2^{-n} \leq s \leq 2^{-(n-1)} \leq 1 \\ h_{n-2}\left(h_{n-1}((x,t),1), 2^{n-1} - 1\right) & \text{if } x \in X^n \text{ and } 2^{-(n-1)} \leq s \leq 2^{-(n-2)} \leq 1 \\ \vdots & \end{cases}$$

is continuous (even at $s = 0$). In order to see this note that $H$ is the identity on the $k$-skeleton for $s$ close enough to 0. A map from $X \times I \times J$ is continuous if and only if it is continuous on the $k$-skeleton $(X \times I \times J)^k$ for all $k$. Therefore, $H$ is continuous. The homotopy $H$ establishes that $X \times 0 \cup A \times I$ is a deformation retract of $X \times I$.

Figure 3 is an attempt to illustrate what $H$ is doing. The two thickened dots at the bottom are elements of $X^0 \setminus A^0$ and the arrows indicate what happens to $e \times I$ for such a zero cell for $s \in [1/2, 1]$. Moreover, $H((x,t), s) = (x, t)$ for $x \in e \subset X^0$ and $s \leq 1/2$.

The thickened horizontal line in $X \times 0$ at the level $s = 1/2$ represents $e \times 0$ for a 1-cell $e$ in $X^1 \setminus A^1$. The thickened, somewhat diagonal lines (one dashed one solid) represent $\dot{e} \times I$. The three arrows indicate how $H$ (and $h_1$ could act on points $(x,t)$ for $s \in [1/4, 1/2]$ when $x \in e$. For $s \in [1/4, 1/2]$ points of the form $(x_0, t)$ with $x_0 \in X^0$ do not move.

- **Corollary:** CW-pairs $(X, A)$ have the HEP.
- **Remark:** It would have sufficed to show that $X \times 0 \cup A \times I$ is only a retract of $X \times I$, or, and this is not difficult, to show directly that $(X, A)$ has the HEP: This can be done by induction over the skeleta using the fact that $(D^n, S^{n-1})$ has the homotopy extension property.

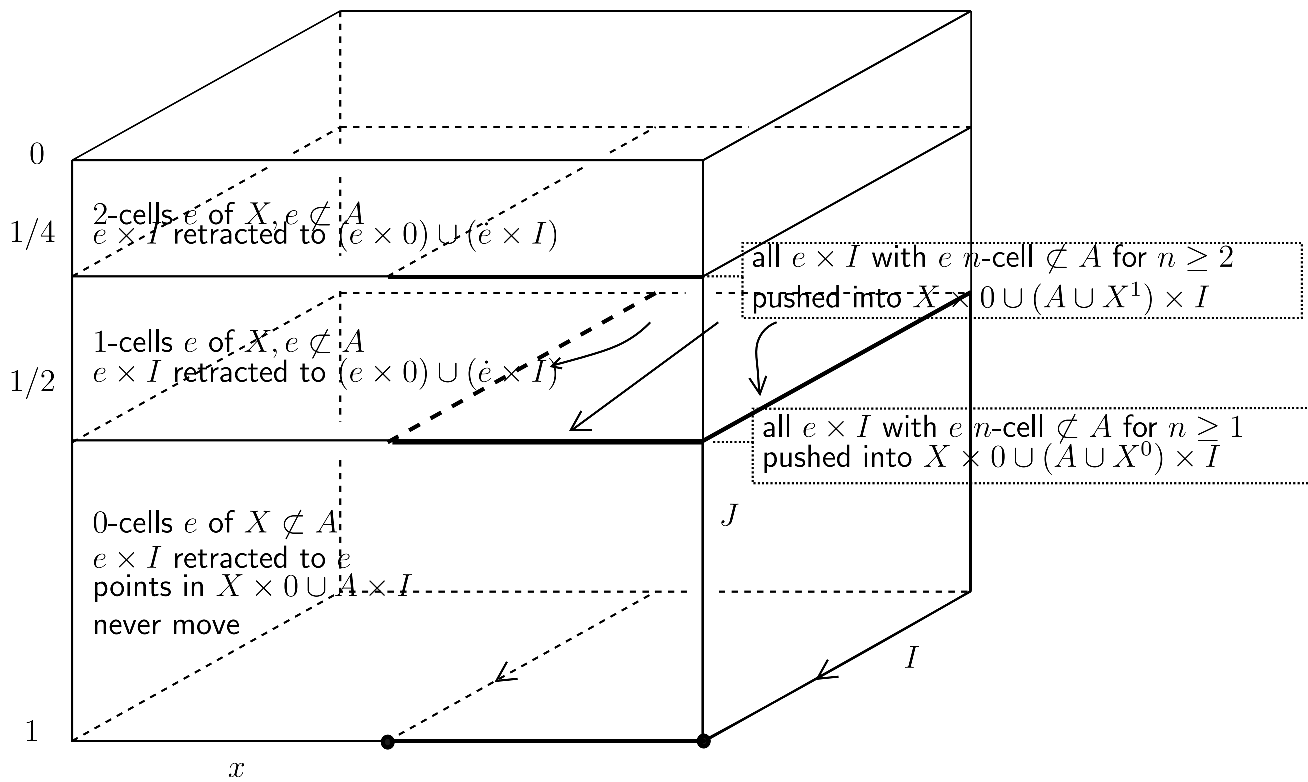


FIGURE 3. What does $H$ do?

- **Warning:** We did *not* show that $(X \times I, X \times 0 \cup A \times I)$ is a CW-pair when $(X, A)$ is a CW-pair.

  In general, the product of two CW-complexes with the product topology is *not* a CW-complex (the weak topology being finer than the product topology), see [Do] when the obvious cell decomposition and the obvious characteristic maps are used. However, if one of the complexes is compact (i.e. finite), like $I$, or only locally compact, then the product topology and the weak topology on the product coincide.
- **For your information:** A Hausdorff space is locally compact if every point has a compact neighborhood (this implies that every neighborhood contains a compact neighborhood), and a CW complex is locally compact iff every point has a neighborhood that meets only finitely many closures of other cells.

  The proof of these facts is not difficult but we omit them anyways (see [Sch], III.3.3. or [Do]).

## 10. Lecture on Nov. 16 - Homotopy equivalences

- **Notation:** $I = [0,1]$ and if there are intervals with different roles, I will try to denote one of them by $I$, the other one by $J$.
- **Theorem:** Let $A \subset X$ be a CW-pair and $f, g : A \longrightarrow Y$ homotopic maps. Then $Y \cup_f X$ and $Y \cup_g X$ are homotopy equivalent spaces.
- **Proof:** Let $H : A \times I \longrightarrow Y$ be a homotopy. Then the space

$$Y \cup_H (X \times I)$$

contains $Y \cup_f X = Y \cup_f (X \times 0)$ and $Y \cup_g X = Y \cup_g (X \times 1)$ as subspaces. By the previous theorem $X \times 0 \cup A \times I$ is a deformation retract of $X \times I$. Let $H_{def} : (X \times I) \times J \longrightarrow (X \times I)$ be the corresponding homotopy relative to $X \times 0 \cup A \times I$. The inclusion

$$\iota_f : Y \cup_f X \hookrightarrow Y \cup_H (X \times I)$$

has a homotopy inverse

$$\begin{aligned} G_f : Y \cup_H (X \times [0,1]) &\longrightarrow Y \cup_f X = Y \cup_H (X \times 0 \cup A \times I) \\ [y] &\longmapsto [y] \\ [(x,t)] &\longmapsto [H_{def}((x,t),1)] \end{aligned}$$

Then $G_f \circ \iota_f = \mathrm{id}_{Y \cup_f X}$ and $\iota_f \circ G_f$ is homotopic to the identity of $Y \cup_H (X \times I)$ via $H_{def}$. This homotopy is relative to $Y$.
- **Example:** The last theorem together with the fact that $X \longrightarrow X/A$ is a homotopy equivalence when $(X, A)$ has the HEP and $A$ is contractible can be used to produce interesting homotopy equivalences. For example, let $K = T^2 \cup_f D^2$ where $f : \partial D^2 \longrightarrow S^1 \times \{0\} \subset S^1 \times S^1 = T^2$ is a homeomorphism. Then $K$ is homotopy equivalent to the 1-point union $S^1 \vee S^2$.

## 11. Lecture on Nov. 20 - Cellular approximation theorem

- **Definition:** Let $X, Y$ be CW-complexes and $f : X \longrightarrow Y$ continuous. Then $f$ is *cellular* if $f(X^n) \subset Y^n$ for all $n$.
- **Warning:** It is hard to overstate the importance of the following theorem! Unfortunately, the proof is a bit more intricate than one might hope.
- **Theorem (Cellular approximation theorem):** Every continuous map $F : X \longrightarrow Y$ between CW-complexes is homotopic to a cellular map. The homotopy can be chosen relative to a subcomplex $A$ of $X$ where $F$ is already cellular.
- **Remark:** The analogous statement holds for maps of pairs $F : (X, A) \longrightarrow (Y, B)$ . (Apply the cellular approximation theorem to $A$ and use HEP of $(X, A)$, then apply the relative version of the cellular approximation theorem to the remaining cells.)
- **Corollary:** If $f : S^n \longrightarrow S^m$ is continuous and $m > n$, then $f$ is homotopic to a constant map.
- **Proof of Corollary:** Equip $S^m$ with a CW-structure with cells of dimension 0 and $m$, and $S^n$ with a CW-structure with cells of dimension $\leq n$. The $n$-skeleton of $S^m$ is then a point, so the corollary follows from the cellular approximation theorem.

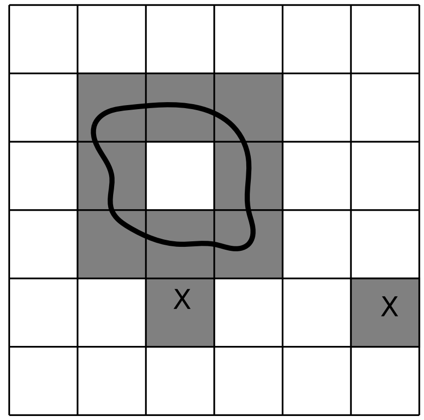


FIGURE 4. Decomposition of $I^n$ into smaller squares, $X$ is shaded, $F^{-1}(0 \in e_Y)$ is the curved loop

- The proof of the cellular approximation theorem is a mildly weird induction. To better motivate the proof, we introduce auxiliary statements. This is useless from a logical point of view but I hope it helps to digest it.
- **(0)-Approximation:** Cellular approximation for 0-dimensional complexes.
- **Proof of (0)-Approximation:** Let $x$ be a 0-cell of $X \setminus A$. $F(x)$ is contained in a $k$-cell of $Y$, and because closures of cells are path-connected (they are images of closed balls) there is a path from $F(x)$ to the $k-1$-skeleton of $Y$. Inductively, we find a path from $F(x)$ to $Y^0$. This can be done for all 0-cells of $X^0 \setminus A$ and we obtain $h : (X^0 \cup A) \times [0,1] \longrightarrow Y$ with $h(x_0, 0) = F(x_0)$, $h(x_0, 1) \in Y^0$, and $h(a,t) = F(a)$ for all $a \in A$. Apply the HEP of CW-pairs to obtain the desired homotopy.
- **(n)-Approximation:** Cellular approximation theorem for complexes $X$ of dimension $\leq n$.
- **(n)-Clean up:** Let $I = [-1, 1]$ and $f_0 : I^n \longrightarrow I^m$ be continuous with $m > n$. Then $f_0$ is homotopic relative to $\partial I^n$ to a map $f_1$ such that $0 \notin f_1(I^n)$.

  The same is true when the source $X$ of $f_0$ is a union of closed subcubes of a subdivision of $I^n$ be small cubes when $f_0(\partial X)$ does not meet 0.
- **Remark:** (n)-Clean up implies (l)-Clean up for all $l \leq n$. Moreover $(D^n, \partial D^n)$ and $(I^m, \partial I^m)$ are homeomorphic, we will identify the two whenever this is convenient.
- **(n)-Clean up and (n-1)-Approximation $\Rightarrow$ (n)-Approximation:** Let $F : X \longrightarrow Y$ be a continuous map of CW-complexes and $\dim(X) \leq n$ which is cellular on $A$. By (n-1)-Approximation, we may assume that $F$ is cellular on $X^{n-1}$ (relative $A$). Let $e \subset X \setminus A$ be a $n$-cell, $G_e$ a characteristic map. Because $F(\overline{e})$ is compact, it hits only finitely many cells of $Y$. Let $e_Y \subset Y$ be a cell with maximal dimension $m$ among the cells meeting $F(\overline{e})$.

  Using (n)-Clean up we homotope $F$ (relative to $X^n$) so that the resulting map $F_1$ does no longer meet the open cell $e_Y$. For this consider the closed subset $F^{-1}(0 \in e_Y)$ of $e$. If one decompses $D^n \simeq I^n$ into sufficiently fine squares (as in Figure 4, then there is a subcomplex $X$ of $I^n$ which gets mapped into the interior of the $m$-cell $e_Y$, so $F$ can be viewed as a map from $X$ to $I^m$. Now one can really apply (n)-Clean up.

Let $Y_{red}$ be a subcomplex of dimension $m+1$ containing $F(\overline{e})\setminus e_Y$ and consider $Y_{red} \cup_\lambda e_Y$ where $\lambda : \partial D^m \longrightarrow Y_{red}$ is a gluing map for $e_Y$. By (n)-Clean up we may assume that $F(G_e(D^n))$ does not meet $0 \in e_Y$. Since $\dot{e}_Y$ is a deformation retract of $\overline{e}_Y \setminus 0$ we can homotope $F|_{\overline{e}}$ relative to $X \setminus e$ (this is a CW-complex) so that the resulting map $F_1$ does no longer meet $e_Y$.

After finitely many steps we have homotoped $F$ relative to the complement of $e$ so that the result maps $e$ to $Y^n$. This can be done for all $n$-cells of $X$ at the same time.

- **(n-1)-Approximation $\Rightarrow$ (n)-Clean up:** Notice that the Corollary above for maps $f : S^{n-1} \longrightarrow S^l$ with $l > n-1$ follows from (n-1)-Approximation.

  Let $f : I^n \longrightarrow I^m$ be continuous such that $0 \notin f(\partial I^n)$. Since $f(\partial I^n)$ is closed there is a cube $Q_0$ around 0 which is disjoint from $f_0(\partial I^n)$. After a reparametrization we may assume $Q = [-1/3, 1/3]^m$. Let $U = (-2/3, 2/3)^m$ and $V = I^m \setminus Q$. This is an open cover of $I^m$, so $f^{-1}(U), f^{-1}(V)$ is an open cover of $I^n$. Such a cover has a positive Lebesgue-number, so there is $N$ such that the $1/N$-cube neighborhood around each point of $I^m$ is contained in one of the sets $f^{-1}(U), f^{-1}(V)$.

  Let $X^n = I^n$ be the regular CW-structure on $I^n$ whose 0-skeleton consists of the points $k_1/N, \dots k_n/N$. Using (n-1)-Approximation (and the way this is proved, namely using a deformation retraction of $[-2/3, 2/3]^m \setminus \{0\}$ to $\partial[-2/3, 2/3]^m$ instead of a deformation retraction $\overline{e}_Y \setminus 0$ to $\dot{e}_Y$ we may assume that the $n-1$-skeleton of $X$ is mapped to $V$ and the deformation does not affect cells of $X$ which were mapped entirely to $V$. The new map is called $f'$.

  Let $k = (k_1, \dots, k_l)$ be the corner of a cell $e$ of $X$ (with $\sum k_i$ minimal among such corners) such that $f'(e) \not\subset V$. We arranged $f'(\dot{e}) \subset V$

  Then, by the first sentence of this proof, $f'(\dot{e})$ is **homotopic to a constant map** inside of $V$ since $V$ is homotopy equivalent to $S^{m-1}$ and $\dot{e}$ is a $n-1$-dimensional complex, and $m > n$ by assumption. So there is an extension $f_1 : \overline{e} \longrightarrow V$ of the restriction of $f'$ to $\dot{e}$. Moreover, $f_1$ **and** $f'$ **are homotopic relative to** $\dot{e}$ **inside** $I^m$ because $I^m$ is convex.
- **Proof of the cellular approximation theorem:** This is a formality. Notice

$$\begin{aligned}
&\text{(0)-Approximation relative } A \Rightarrow \text{(1)-Clean up} \\
&\Rightarrow \text{(1)-Approximation relative } A \cup X^0 \Rightarrow \text{(2)-Clean up} \\
&\Rightarrow \text{(2)-Approximation relative } A \cup X^1 \dots
\end{aligned}$$

The resulting homotopies $h_i : X \times [0,1] \longrightarrow Y$ relative to $X^{i-1} \cup A$ such that $h_i(\cdot, 0) = h_{i-1}(\cdot, 1)$ (with $h_0(\cdot, 0) = F$, note $X^{-1} = \emptyset$) and $h(X^i, 1) \subset Y^i$ can now be assembled in a similar way as in the proof of the main theorem from the previous lecture:

$$H : X \times [0,1) \longrightarrow Y$$

$$(x,t) \longmapsto \begin{cases} h_0(x, 2t - 2(2^0 - 1)) & \text{if } t \in [0, 1/2] \\ h_1(x, 2^2 t - 2(2^1 - 1)) & \text{if } t \in [1/2, 3/4] \\ h_2(x, 2^3 t - 2(2^2 - 1)) & \text{if } t \in [3/4, 7/8] \\ \vdots & \\ h_i(x, 2^{i+1} t - 2(2^i - 1)) & \text{if } t \in \left[\frac{2^i - 1}{2^i}, \frac{2^{i+1}-1}{2^{i+1}}\right]. \\ \vdots & \end{cases}$$

If $x \in X^n$, then $(x,t)$ is constant for $t \geq \frac{2^{n+1}-1}{2^{n+1}}$. Thus we can extend $H$ from $X \times [0,1)$ to $X \times [0,1]$ by

$$H(x,1) = h_n(x,1) \text{ for } x \in X^n.$$

Because $X$ has the weak topology with respect to cells/skeleta the extension $H$ is continuous everywhere. $H(\cdot,1)$ is cellular.

- **Remark:** To see an example of a surjective map $I \longrightarrow I^2$ look at Exercise 12 of Chapter 5 in [Wa].
- This concludes the discussion of general properties of CW-complexes. We now look for tools to distinguish topological spaces up to homotopy equivalence. One such tool we applied already are sets of homotopy classes of maps $S^1$ into a space. When one adds a base point and considers homotopies relative to the base point one obtains a much richer structure.

## 12. Lecture on Nov. 23 - Homotopy groups

- **Definition:** For $x_0 \in X$, the pair $(X, x_0)$ a called a pointed space and $x_0$ is the basepoint. Continuous maps between pairs of spaces of this form are called *pointed*.
- **Definition:** Let $x_0 \in X$. The *fundamental group* of $(X, \{x_0\})$ is

$$\pi_1(X,x_0) = \{\gamma : ([0,1],\{0,1\}) \longrightarrow (X,\{x_0\}) \text{ continuous}\}/\sim$$

where $\alpha \sim \beta$ if these maps are homotopic as maps of pairs. If $[\alpha],[\beta] \in \pi_1(X,x_0)$, we define

$$\begin{aligned} \alpha * \beta : ([0,1],\{0,1\}) &\longrightarrow (X,\{x_0\}) \\ t &\longmapsto \begin{cases} \alpha(2t) & \text{if } t \in [0,1/2] \\ \beta(2t-1) & \text{if } t \in [1/2,1]. \end{cases} \end{aligned} \tag{3}$$

This path depends on the choice of representatives, but the (relative) homotopy class of the path $\alpha * \beta$ is well defined.

- **Theorem:** This defines a group structure on $\pi_1(X,\{x_0\})$.
- **Remark:** If one replaces $([0,1],\{0,1\})$ by $(S^1, z_0)$ where $z_0$ is any point of $S^1$ all this can be reformulated. Often one uses $z_0 = 1 \in S^1 \subset \mathbb{C}$.
- **Examples:**
  - If $x_0$ is a deformation retract of $X$, then $\pi_1(X,x_0) = \{1\}$ is trivial.
  - Let $\alpha,\beta : (S^1,z_0) \longrightarrow (S^1,x_0)$ be continuous. If these maps have different degree viewed as maps $S^1 \longrightarrow S^1$, then they can't be homotopic relative to base points. If $\deg(\alpha) = \deg(\beta)$, then $\alpha \sim \beta$ are homotopic relative to basepoints. The proof of the second lemma on Nov. 6 produces a homotopy relative to basepoints when the auxiliary path $\gamma$ is chosen constant. Moreover, in view of the definition (3) one can compute the degree of $\alpha * \beta$ since

$$\varphi_{\alpha*\beta}(t) = \begin{cases} \varphi_\alpha(2t) & \text{if } t \in [0,1/2] \\ \deg(\alpha) + \varphi_\beta(2t-1) & \text{if } t \in [1/2,1] \end{cases}$$

    We obtain $\deg(\alpha * \beta) = \varphi_{\alpha*\beta}(1) = \deg(\alpha) + \deg(\beta)$. Thus

$$\begin{aligned} \deg : \pi_1(X,x_0) &\longmapsto \mathbb{Z} \\ [\alpha] &\longmapsto \deg(\alpha) \end{aligned}$$

    is a group isomorphism.

- The loop $\alpha(t) = [\cos(\pi t) : \sin(\pi t) : 0], t \in [0,1]$ represents an element of $\pi_1(\mathbb{RP}^2, [1:0:0])$ such that $\alpha * \alpha$ is trivial in $\pi_1(\mathbb{RP}^2, [1:0:0])$. It is not obvious that $\alpha$ is non-trivial (but this is true).

- **Definition:** Let $k \in \{1, 2, \ldots\}$. The $k$-th homotopy group $\pi_k(X, x_0)$ of $(X, x_0)$ is
$$\pi_1(X, x_0) = \{\gamma : (I^k, \partial I^k) \longrightarrow (X, \{x_0\}) \text{ continuous}\}/\sim .$$
The product $[\alpha][\beta] = [\alpha * \beta]$ is by
$$\begin{aligned} \alpha * \beta : (I^k, \partial I^k) &\longrightarrow (X, \{x_0\}) \\ (t_1, \ldots, t_k) &\longmapsto \begin{cases} \alpha(2t_1, t_2, \ldots, t_k) & \text{if } t_1 \in [0, 1/2] \\ \beta(2t_1 - 1, t_2, \ldots . t_k) & \text{if } t_1 \in [1/2, 1]. \end{cases} \end{aligned} \tag{4}$$
- **Theorem:** This is determines a group structure on $\pi_k(X, x_0)$.
- **Theorem:** $\pi_k(S^n) = 0$ for $k < n$ (by cellular approximation).
- **Theorem:** $\pi_k(X)$ is Abelian when $k \geq 2$.
- **Remark:** Since $(I^k, \partial I^k) \longrightarrow (S^k, *)$ is a relative homeomorphism, one can view maps $(I^k, \partial I^k) \longrightarrow (X^k, *)$ as maps $(S^k, *) \longrightarrow (X, *)$. This motivates
- **Definition:** $\pi_0(X, x_0) = \{(S^0, 1) \longrightarrow (X, x_0)\}/\sim$ where $\sim$ is again homotopy relative to the base point. (Recall $S^0 = \partial D^1 = \{1, -1\}$.)
- **Remark:** In general, $\pi_0(X, x_0)$ does not have a group structure. It is a pointed set since it has a distinguished element represented by $S^0 \longrightarrow \{x_0\} \subset X$. By definition, elements of $\pi_0(X, x_0)$ are in one-to-one correspondence with path-connected components of $X$.

## 13. Lecture on Nov. 27 - Role of the base point

- **Theorem:** $\pi_k(S^1, 1) = \{0\}$ for all $k \geq 2$.
- **Proof:** Let $F : (I^k, \partial I^k) \longrightarrow (S^1, 1)$ be continuous. This can be viewed a $I^{k-1}$ parametric family of maps $f_\tau : (I, 0) \longrightarrow (S^1, 1), \tau \in I^{k-1}$, for which we defined an auxiliary function $\varphi_\tau$ such that $f_\tau(t_1) = 1 \cdot E(\varphi_\tau(t_1))$. As shown on p. 9 in the case $k-1 = 1$, these maps $\varphi_\tau$ can be assembled to a map $\Phi : I^k \longrightarrow \mathbb{R}$ such that $E(\phi(t_1, t_2 \ldots, t_k)) = F(t_1, \ldots, t_k)$ with $\tau = (t_2, \ldots, t_k)$. If $\tau \in \partial I^{k-1}$, then $f_\tau$ is constant and so is $F|_{\partial I^k}$. Hence $\phi|_{\partial I^k} \equiv 0$ and one obtains a homotopy from $F$ to the constant map by convex interpolation of $\phi$ and the vanishing map (as in (2)).
- **Notation:** It is common to omit the base point from $\pi_k(X, x_0)$ or to write $*$ for some point.
- **Definition:** Let $[\gamma] \in \pi_1(X, x_0)$ and $[A] \in \pi_k(X, x_0), k \geq 1$, then let
$$\begin{aligned} \gamma \cdot A : (I^k \times \{1\}) \cup (\partial I^k \times J) &\longrightarrow (X, x_0) \\ (x, t) &\longmapsto \begin{cases} A(x) & \text{if } t = 1 \\ \gamma(t) & \text{if } x \in \partial I^k. \end{cases} \end{aligned}$$
There is a homeomorphism
$$\psi : (I^k, \partial I^k) = (I^k \times \{0\}, \partial I^k \times \{0\}) \longrightarrow \big((I^k \times \{1\}) \cup (\partial I^k \times J), \partial I^k \times \{0\}\big)$$
which is the identity/inclusion on $\partial I^k \times \{0\}$ (c.f. Figure 2). Then $[(\gamma \cdot A) \circ \psi]$ represents a well defined homotopy class $[\gamma] \cdot [A]$ in $\pi_k(X, x_0)$.
- **Remark:** For $k = 1$ this is the action of $\pi_1$ on itself by conjugation.

- **Theorem:** The groups $\pi_i(X, x_0)$ are $\pi_1(X, x_0)$-modules (more precisely, a $\mathbb{Z}[\pi_1(X, x_0)]$ module) with the module structure defined in the previous Definition.
- **Remark:** The module structure is independent from the choice of $\psi$ (Alexander trick, but beware of orientation reversing homeomorphisms...).
- **Remark:** $\pi_1$ is *not* Abelian in general, but there are more tools available to compute this group than for $\pi_k, k \geq 2$. Even the groups $\pi_k(S^n)$ are not known. A table with $\pi_k(S^n)$ for $k \leq 12$ and $n \leq 8$ can be found on p. 339 in [Ha].
- **Remark:** The choice of the base point matters. For example, if $x_0, x_1$ lie in different connected components of $X$, then there is no relationship between $\pi_k(X, x_0)$ and $\pi_k(X, x_1)$. If $\gamma$ is path from $x_0$ to $x_1$, then

$$\begin{aligned} \varphi_\gamma : \pi_1(X, x_1) &\longrightarrow \pi_1(X, x_0) \\ [\alpha] &\longmapsto [\gamma * \alpha * \gamma^{-1}] \end{aligned} \tag{5}$$

is a group homomorphism. Here $\gamma^{-1}(t) = \gamma(1-t)$ and $*$ denotes the concatenation of paths. The homomorphism depends on the homotopy class of $\gamma$ relative to the endpoints. If $\hat{\gamma} : [0,1] \longrightarrow X$ is a path with $\hat{\gamma}(0) = x_1, \hat{\gamma}(1) = x_0$ we obtain a homomorphism $\varphi_{\hat{\gamma}} : \pi_1(X, x_0) \longrightarrow \pi_1(X, x_1)$ as in (5). Then

$$\begin{aligned} \varphi_{\hat{\gamma}} \circ \varphi_\gamma : \pi_1(X, x_1) &\longrightarrow \pi_1(X, x_1) \\ \alpha &\longmapsto (\hat{\gamma} * \gamma) * \alpha * (\hat{\gamma} * \gamma)^{-1} \end{aligned}$$

is the conjugation of $\alpha$ with $(\hat{\gamma} * \gamma) \in \pi_1(X, x_1)$, so $\varphi_\gamma, \varphi_{\hat{\gamma}}$ are isomorphisms.
- **Remark:** $\alpha : (S^k, p) \longrightarrow (X, x_0)$ represents the trivial element iff $\alpha$ extends to a map $D^{k+1} \longrightarrow X$. In this case it is called *nullhomotopic*.
- **Definition:** $\alpha, \beta : (S^1, 1) \longrightarrow (X, x_0)$ are *freely homotopic* if they are homotopic but this homotopy does not have to respect base points. This notion makes sense for homotopy classes.
- **Proposition:** $\alpha, \beta \in \pi_1(X, x_0)$ are freely homotopic iff they are conjugate in $\pi_1(X, x_0)$.
- **Proof:** If $\alpha, \beta : (S^1, 1) \longrightarrow X$ are (freely) homotopic via $H : S^1 \times [0,1] \longrightarrow X$, then let $\gamma(t) = H(1, s)$. Then $\gamma * \alpha * \gamma^{-1}$ is homotopic to $\beta$ since $H$ provides an extension of $\gamma * \alpha * \gamma^{-1} * \beta^{-1} : S^1 \longrightarrow X$ to the disc (see Figure 5).

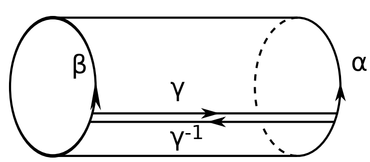


FIGURE 5. $\gamma * \alpha * \gamma^{-1} * \beta^{-1}$ is null homotopic

Conversely, we have to show that $\gamma * \alpha * \gamma^{-1}$ and $\alpha$ are freely homotopic. We view $\gamma * \alpha * \gamma^{-1}$ as a map $([0,1], \{0,1\}) \longrightarrow (X, x_0)$ with $(\gamma * \alpha * \gamma^{-1})(t) = \gamma(3t)$ for $t \in [0, 1/3]$ and $\gamma * \alpha * \gamma^{-1}(t) = \gamma^{-1}(3t-2)$ for $t \in [2/3, 1]$. Since we consider *free* homotopies, the following homotopy is admissible (we identify $[0,1]/\{0,1\}$

with $S^1$)

$$\begin{aligned} H : S^1 \times J &\longrightarrow X \\ (t,s) &\longmapsto \begin{cases} \gamma * \alpha * \gamma^{-1}(t+s/3) & \text{if } t+s/3 \le 1 \\ \gamma * \alpha * \gamma^{-1}(t+s/3-1) & \text{if } t+s/3 \ge 1. \end{cases} \end{aligned}$$

It shows that $\gamma * \alpha * \gamma^{-1}$ and $\alpha * \gamma^{-1} * \gamma$ are freely homotopic.

- Using the $\mathbb{Z}[\pi_1]$-module structure one obtains a similar relationship between $\pi_k(X, x_0)$ and $\pi_k(X, x_1)$ for $k \ge 2$. Note that a pointed map $S^k \longrightarrow X$ which freely null homotopic is also null homotopic via pointed maps.
- **Theorem:** Let $(X, x_0)$ be a CW-pair. Then $\pi_k(X, x_0)$ depends only on the $k+1$-skeleton of $X$.
- **Proof:** Cellular approximation.
- **Definition:** A pointed space is *$k$-connected* if $\pi_l(X, x_0)$ is trivial for $l \le k$. Instead of 0-connected one says *path-connected*, and instead of 1-connected *simply connected*.
- **Examples:** $\mathbb{CP}^n$ is simply connected for all $n$. The sphere $S^{k+1}$ is $k$-connected for all $k \ge 0$.
- **Definition:** Let $f : (X, x_0) \longrightarrow (Y, y_0)$ be a continuous map of pointed spaces. The *induced map* on homotopy groups is

$$\begin{aligned} \pi_k(f) : \pi_k(X, x_0) &\longrightarrow \pi_k(Y, y_0) \\ [\alpha] &\longmapsto [f \circ \alpha]. \end{aligned}$$

- **Theorem:** $\pi_k(f)$ is a well defined group homomorphism. It depends only on the homotopy class of $f$ as pointed map. Moreover, $\pi_k(g \circ f) = \pi_k(g) \circ \pi_k(f)$ if $g : (Y, y_0) \longrightarrow (Z, z_0)$ is continuous.

## 14. Lecture on Nov. 30 – Induced maps, Seifert van Kampen

- **Consequence:** If $f : X \longrightarrow Y$ is a homeomorphism of connected spaces and $x_0$ a base point, then $f_* : \pi_k(X, x_0) \longrightarrow \pi_k(Y, f(x_0))$ is an isomorphism with inverse $(f^{-1})_*$. This is also obvious if $f : (X, x_0) \longrightarrow (Y, f(x_0))$ is a homotopy equivalence relative to base points, the inverse is induced by the pointed homotopy inverse.
- **Notation:** It is common to write $\pi_k(f), f_\#, f_*$.
- **Example:** $f_n : (S^1, 1) \longrightarrow (S^1, 1), z \longmapsto z^n$ induces the multiplication by $n$ on $\pi_1(S^1, 1) \simeq \mathbb{Z}$. Recall that this isomorphism is given by the degree and note that if $\deg(\alpha) = \varphi(1)$ for a function $\varphi$ as in (1), then $\varphi_n$, the function associated to $f_n \circ \alpha$, is $\varphi_n(\cdot) = n\varphi(\cdot)$.
- **Theorem:** Homotopy groups are homotopy invariants, i.e. homotopy equivalent spaces have isomorphic homotopy groups.
- **Proof:** Let $X, Y$ be path connected. Assume that $f : X \longrightarrow Y$ is a homotopy equivalence with homotopy inverse $g : Y \longrightarrow X$. Then $g \circ f$ is homotopic to the identity of $X$ via $H : X \times [0,1] \longrightarrow X$ with $H(\cdot, 0) = \mathrm{id}$. Let $x_0 \in X$ be a base point. Then choose $y_0 = f(x_0)$, $x_1 = g(y_0)$ and $y_1 = f(x_1)$. Then

$$f_* : \pi_k(X, x_0) \longrightarrow \pi_k(Y, y_0) \qquad g_* : \pi_k(Y, y_0) \longrightarrow \pi_k(X, x_1)$$

are well defined. Let $\gamma : [0,1] \longrightarrow X, \gamma(t) = H(x_0, 1-t)$. This is a path from $x_1$ to $x_0$. Then $g \circ f \circ \alpha$ is homotopic to $\gamma \cdot \alpha$ for all $\alpha$ representing elements of

$\pi_k(X, x_0)$. In particular, $g_*$ is surjective and $f_*$ is injective. Now consider

$$f_{**} : \pi_k(X, x_1) \longrightarrow \pi_k(Y, y_1)$$

and apply the same argument to $f \circ g : (Y, y_0) \longrightarrow (Y, y_1)$ to show that $f_{**} \circ g_*$ is an isomorphism. Hence $g_* : \pi_k(Y, y_0) \longrightarrow \pi_k(X, x_1)$ is an isomorphism.

- **Remark:** If one considers pointed homotopy types (i.e. pointed, path connected spaces up to pointed homotopy equivalence), then one obtains much neater proofs and nicer statements because the pointed homotopy inverse $g$ to the pointed homotopy equivalence $f$ induces the inverse to $f_*$.
- **Theorem:** The map

$$\begin{aligned} \pi_k(X, x_0) \times \pi_k(Y, y_0) &\longrightarrow (X \times Y, (x_0, y_0)) \\ ([\alpha], [\beta]) &\longmapsto [(\alpha(\cdot), \beta(\cdot))] \end{aligned} \tag{6}$$

is an isomorphism.

- **Proof:** This map is injective: If $(\alpha, \beta) : (I^k, \partial I^k) \longrightarrow (X \times Y, *)$ is nullhomotopic, then the composition

$$\begin{aligned} \mathrm{pr}_X \circ (\alpha, \beta) : (I^k, \partial I^k) \longrightarrow (X \times Y, (x_0, y_0)) &\longrightarrow (X, x_0) \\ x \longmapsto (\alpha(x), \beta(x)) &\longmapsto \alpha(x) \end{aligned}$$

is nullhomotopic (with the projection $\mathrm{pr}_X : (X \times Y, (x_0, y_0)) \longrightarrow (X, x_0)$). Similarly for $\beta$ and $Y$.

This map is surjective: If $\gamma : (I^k, \partial I^k) \longrightarrow (X \times Y, (x_0, y_0))$ represents a homotopy class, then $\gamma = (\mathrm{pr}_X \circ \gamma, \mathrm{pr}_Y \circ \gamma)$.

This map is a group homomorphism: Direct check.

- **Remark:** The following theorem is the result of the application of the Seifert-van Kampen Theorem to CW-complexes. Depending on how we progress, we might prove this Theorem (and the following consequence), or not.
- **Set up:** Let $(X, x_0)$ be a connected CW-complex with $x_0 \in X^0$ and $\Gamma_0$ a maximal tree in $X^1$ with $x_0 \in \Gamma$. Orient all 1-cells which are not in $\Gamma$ and let $\gamma_i, i \in I$, be the collection of 1-cells which are not contained in $\Gamma$ and consider the free group $F_I$ which is generated by $\gamma_i, i \in I$.
- **For your information:** Let $S$ be a set. The free group $F_S$ generated by $S$ comes with an inclusion $\iota : S \hookrightarrow F_S$ and satisfies the following universal property:

For all groups $G$ and maps $f : S \longrightarrow G$ of sets there is a unique group homomorphism $\varphi$ such that $\varphi \circ \iota = f$.

$$\begin{tikzcd} & F_S \arrow[d, dashed, "\varphi"] \\ S \arrow[ur, "\iota"] \arrow[r, "f"'] & G \end{tikzcd}$$

- **Theorem, part 1:** $\pi_1(X^1, x_0) \simeq F_I$.
- **Example:** The fundamental group of the figure 8 (this is a one-point union of two circles) is isomorphic to $\mathbb{Z} * \mathbb{Z}$, i.e. the free group on two generators.
- **Set up:** We write $i : (X^1, x_0) \longrightarrow (X, x_0)$ for the inclusion map. For each 2-cell $e$ of $X$ choose a characteristic map $F_e : D^2 \longrightarrow X^2$. Fix a base point $e_0$ in $\partial D^2$. After a homotopy of $F_e$ we may assume $F_e : (S^1 = \partial D^2, e_0) \longrightarrow (X^1, x_0)$. Then

let $N$ be the smallest normal subgroup containing all elements of the form

$$(F_e)_*\left(\underbrace{id : S^1 \longrightarrow D^2}_{\text{generator of } \pi_1(\partial D^2, e_0)}\right) \in \pi_1(X^1, x_0). \tag{7}$$

- **Theorem, part 2:** The map $i_* : \pi_1(X^1, x_0) \longrightarrow \pi_1(X, x_0)$ is surjective and has kernel $N$. Hence $\pi_1(X, x_0) \simeq \pi_1(X^1, x_0)/N$.
- **Remark:**
  1. surjective follows from cellular approximation, it is equally clear that $N \subset \ker(i)$. The most difficult step is to show that $\ker(i) \subset N$. For a proof see [Ha], p. 43–54 or [StZ], Section 5.3.
  2. We homotope $F_e$ to arrange that $F_e(e_0) = x_0$. In general, there are many ways to do that, so the element of $\pi_1(X^1, x_0)$ described in (7) is well defined only up to conjugation (by the base point discussion last time). Since $N$ is the smallest *normal* subgroup containing these elements, this ambiguity does not matter.
- **Example:** $\mathbb{RP}^2$ has a CW-structure $X$ with exactly one cell in dimension $0, 1, 2$ such that $X^1 \simeq S^1$ and $\mathbb{RP}^2 = X^2 = X^1 \cup_f \overline{D}^2$ where the map $f : \partial\overline{D}^2 = S^1 \longrightarrow S^1$ has degree $\pm 2$. Hence $\pi_1(X^1, x_0) = \mathbb{Z}$. Since $f$ has degree $\pm 2$, the image of $\pi_1(\partial D^2, *)$ in $\pi_1(X^1, x_0) \simeq \mathbb{Z}$ consists of maps of even degree. Hence $\pi_1(\mathbb{RP}^2, x_0) \simeq \mathbb{Z}/2\mathbb{Z}$. Because $\mathbb{RP}^n, n \geq 2$, has a CW-structure such that the 2-skeleton is $\mathbb{RP}^2$, it follows that

$$\pi_1(\mathbb{RP}^n, *) = \mathbb{Z}/2\mathbb{Z} \text{ for } n \geq 2.$$

- **Application (Sketch):** Let $G$ be a group. There is a pointed CW-complex $(X, x_0)$ such that $\pi_1(X, x_0) \simeq G$. To construct $X$, consider $X^0 = \{x_0\}$. For each element $g \in G$ attach a 1-cell $\gamma_g$ to $X^0$, oriented in a way. Then for all $g, h \in G$ consider the path $\gamma_g * \gamma_h * \gamma_{gh}^{-1}$ in $X^1$ and attach a 2-disc to $X^1$ along this path. There is a well defined group homomorphism

$$\begin{aligned} \psi : G &\longrightarrow \pi_1(X, x_0) \\ g &\longmapsto \gamma_g. \end{aligned}$$

To find an inverse consider $[\gamma] \in \pi_1(X)$. After a homotopy, we may assume that $\gamma$ is cellular and the way $\gamma$ travels through $X^1$ determines a finite word $\gamma_{g_1} * \gamma_{g_2} * \ldots \gamma_{g_k}$. Then

$$\begin{aligned} \pi_1(X, x_0) &\longrightarrow G \\ [\gamma] &\longmapsto g_1 \cdot g_2 \cdot \ldots g_k. \end{aligned}$$

This is a well defined inverse of $\psi$.

## 15. Lecture on Dec. 4 – Coverings, Path lifting

- **Reference:** Chapter 9 in [Jä], Chapter 6 in [StZ],
- **Definition:** Let $\mathrm{pr} : Y \longrightarrow X$ be continuous. This map is a *covering (map)* if for all $x \in X$ there is an open neighborhood $U \subset X$, a discrete topological

space $\Lambda$, and a homeomorphism $\varphi : \mathrm{pr}^{-1}(U) \longrightarrow U \times \Lambda$ such that

$$\begin{tikzcd} \mathrm{pr}^{-1}(U) \arrow[rr, "\varphi"] \arrow[dr, "\mathrm{pr}"'] & & U \times \Lambda \arrow[dl, "\mathrm{pr}_1"] \\ & U & \end{tikzcd} \tag{8}$$

commutes. Here $\mathrm{pr}_1$ is the projection on the first factor. A neighborhood $U$ with this property is called *trivializing*. When using such neighborhoods we implicitly fix $\Lambda$ and $\varphi$ as above.

- **Remark:** If pr is a covering, and $x \in X$, then the fiber $Y_x := \mathrm{pr}^{-1}(x) = \varphi^{-1}\big(\{x\} \times \Lambda\big)$ is discrete and homeomorphic to $\Lambda$. The function $x \longrightarrow |Y_x|$ is locally constant, and constant if $X$ is connected. If this function is constant and finite, then pr is said to be *$n$-sheeted* with $n = |Y(x)|$. If it is constant and infinite, then it has *infinitely many sheets*.
- **Examples:**
  - $\mathrm{id} : X \longrightarrow X$ is a covering, it is 1-sheeted.
  - $E : \mathbb{R} \longrightarrow S^1, E(x) = e^{2\pi i t}$ is a covering. It has infinitely many sheets. Same for $\exp : \mathbb{C} \longrightarrow \mathbb{C} \setminus \{0\}$.
  - For $n \in \{1, 2, \ldots\}$, the map $f_n : S^1 \subset \mathbb{C} \longrightarrow S^1, z \longmapsto z^n$ is a covering and $n$-sheeted. Same when $f_n$ is viewed as map $\mathbb{C} \setminus \{0\} \longrightarrow \mathbb{C} \setminus \{0\}$.
  - The quotient map $S^n \longrightarrow \mathbb{RP}^n$ is a covering, it is 2-sheeted. For $[x] \in \mathbb{RP}^2$,
  $$\Lambda_{[x]} = \left\{ \frac{x}{\|x\|}, -\frac{x}{\|x\|} \right\}.$$
  For $x \in \mathbb{RP}^2$ let $U = \{[y] \mid \langle y, x\rangle \neq 0\}$. Then
  $$\begin{aligned} \mathrm{pr}^{-1}(U) &= \{y \in S^2 \mid \langle y, x\rangle \neq 0\} \\ &= \underbrace{\{y \in S^2 \mid \langle y, x\rangle > 0\}}_{U_+} \cup \underbrace{\{y \in S^2 \mid \langle y, x\rangle < 0\}}_{U_-} \end{aligned}$$
  where the union is disjoint. Moreover, $\mathrm{pr} : U\pm \longrightarrow U$ is a homeomorphism. Then
  $$u \in U_+ \longmapsto ([u], +1)$$
  $$u \in U_- \longmapsto ([u], -1)$$
  $$\begin{tikzcd} \mathrm{pr}^{-1}(U) \arrow[r, equal] \arrow[drr, "\mathrm{pr}"'] & U_+ \cup U_- \arrow[r, "\varphi"'] & U \times \{\pm 1\} \arrow[d, "\mathrm{pr}_1"] \\ & & U \end{tikzcd}$$
  - If $\Lambda \neq \emptyset$ is discrete, then $\mathrm{pr} : X \times \Lambda \longrightarrow X$ is a covering.
  - The inclusion $(-\infty, 0) \longrightarrow \mathbb{R}$ is not a covering although there is a diagram like (8) for all points in the image.
  - The groups $\mathrm{Spin}(n)$ which appear in the definition of Spin-structures and Dirac operators are 2-sheeted coverings of $\mathrm{SO}(n)$, for the definition of $\mathrm{Spin}(n)$ see [BrtD] p. 54–63.

- **Definition:** Two coverings $\mathrm{pr}_0 : Y_0 \longrightarrow X$ and $\mathrm{pr}_1 : Y_1 \longrightarrow X$ are *isomorphic* if there is a homeomorphism $\varphi : Y_0 \longrightarrow Y_1$ such that

(9)
$$\begin{tikzcd} Y_0 \arrow[r, "\varphi"] \arrow[dr, "\mathrm{pr}_0"'] & Y_1 \arrow[d, "\mathrm{pr}_1"] \\ & X \end{tikzcd}$$

commutes. The definition is analogous for pointed coverings.
- **Definition:** Let $f : Y \longrightarrow X$ be continuous. $f$ is a *local homeomorphism* if every $y \in Y$ has a neighborhood $U$ such that $f|_U$ is a homeomorphism onto its image and $f(U)$ is a neighborhood of $f(y)$.
- **Fact:** Coverings are local homeomorphisms, so is the inclusion in the list above. $E|_{(0,2)} : (0,2) \longrightarrow S^1$ is a surjective local homeomorphism, albeit not a covering.
- **Fact:** Local homeomporphism are open, i.e. they map open sets to open sets. Let $f : Y \longrightarrow X$ be a local homeomorphism and $U \subset Y$ open. For each $y \in Y$ let $U_y$ be an open neighborhood in $U$ of $y$ as above. Then
$$f(U) = \bigcup_{y \in Y} f(\underbrace{U \cap U_y}_{\text{open in } X})$$
is a union of open sets in $X$.
- **Consequence:** 1-sheeted coverings are homeomorphisms.
- **Warning:** Local homeomorphisms are not closed, in general. Neither are covering maps. However, this is the case when $X$ is Hausdorff and $Y$ is compact.
- **Fact:** If $f : Y \longrightarrow X$ is a finite sheeted covering of a compact space $X$, then $Y$ is compact.
- The following notion is fundamental in the study of covering spaces.
- **Definition:** Let $f : Y \longrightarrow X$ be continuous, $f(y_0) = x_0$ and $\alpha : [a,b] \longrightarrow X$ a path. A path $\widetilde{\alpha} : [a,b] \longrightarrow Y$ with starting point $y_0$ is a *lift of* $\alpha$ *with initial point* $y_0$ if $f \circ \widetilde{\alpha} = \alpha$. A map $f$ has the *path lifting property* if for every $\alpha : [0,1] \longrightarrow X$ and $y_0$ with $f(y_0) = \alpha(0) = x_0$ there is a unique lift $\widetilde{\alpha}$ with initial point $y_0$.
- **Example:** Look at the first Lemma in the lecture on Nov. 6.
- **Lemma:** Covering maps have the path lifting property.
- **Proof:** Let $\mathrm{pr} : Y \longrightarrow X$ be a covering and
$$T := \left\{ t \in [0,1] \;\middle|\; \begin{array}{l} \text{there is a unique } \widetilde{\alpha} : [0,t] \longrightarrow Y \text{ such that} \\ \widetilde{\alpha}(0) = y_0 \text{ and } \mathrm{pr}(\widetilde{\alpha}(s)) = \alpha(s) \text{ for } s \in [0,t] \end{array} \right\}.$$
This set contains 0. Let $\tau = \sup T$. If $\tau = 1$ we are done. If not, let $\widetilde{\alpha}$ be the unique lift of $\alpha|_{[0,\tau)}$, fix a trivializing neighborhood $U$ of $\alpha(\tau)$ and a homeomorphism $\varphi : \mathrm{pr}^{-1}(U) \longrightarrow U \times \Lambda_{\alpha(\tau)}$. We denote $\varphi(\widetilde{\alpha}(\tau)) = (\alpha(\tau), \lambda_0)$. Then for $\varepsilon > 0$ small enough, $\alpha((\tau - \varepsilon, \tau + \varepsilon)) \subset U$ and
$$\begin{aligned} \widehat{\alpha} : [0, \tau + \varepsilon) &\longrightarrow Y \\ t &\longmapsto \begin{cases} \widetilde{\alpha}(t) & \text{if } t < \tau \\ \varphi^{-1}(\alpha(t), \lambda_0) & \text{if } t \geq \tau - \varepsilon \end{cases} \end{aligned}$$
is continuous and a lift of $\alpha$ with starting point $y_0$. This lift is unique on $[0, \tau)$ by assumption and on $(\tau - \varepsilon, \tau + \varepsilon)$ a lift is uniquely determined by $\lambda_0 = \mathrm{pr}_2(\varphi(\alpha(\tau - \varepsilon/2))$ and $\alpha|_{(\tau-\varepsilon,\tau+\varepsilon)}$ because $\Lambda$ is discrete. Hence $\tau + \varepsilon/2 \in T$ contradicting to the choice of $\tau$. Hence $\tau = 1$ and $T = [0,1]$.

## 16. Lecture on Dec. 7 – Homotopy lifting, characteristic subgroups

- **Remark:** The following Lemma is a parametric version of the previous one.
- **Lemma (Homotopy lifting for coverings):** Let $\mathrm{pr} : Y \longrightarrow X$ be a covering, $h : Z \times [0,1] \longrightarrow X$ a homotopy and $\widetilde{h}_0 : Z = Z \times \{0\} \longrightarrow Y$ a lift of $h(\cdot, 0)$, i.e. $\mathrm{pr}(\widetilde{h}_0(z)) = h(z)$ for all $z \in Z$.

  Then there exists a unique continuous map $\widetilde{h} : Z \times [0,1] \longrightarrow Y$ such that $\mathrm{pr}(\widetilde{h}(z,t)) = h(z,t)$ and $\widetilde{h}_0(z) = \widetilde{h}(z,0)$:

$$\begin{tikzcd} Z \times \{0\} \arrow[r, "\widetilde{h}_0"] \arrow[d] & Y \arrow[d, "\mathrm{pr}"] \\ Z \times I \arrow[r, "h"] \arrow[ur, dashed, "\widetilde{h}"] & X. \end{tikzcd}$$

- **Proof:** Uniqueness follows for the previous lemma since the restriction of a lift $\widetilde{h}$ to $\{z\} \times [0,1]$ is a lift of the path $h|_{\{z\}\times\, I}$ with initial point $\widetilde{h}_0(z)$. This also means that we can define $\widetilde{h}$ as a family of path lifts with initial point determined by $\widetilde{h}_0$. We call the result $\widetilde{h}$ but we still have to show that this is continuous. Let $z \in Z$. We consider

$$T := \left\{ t \in [0,1] \;\middle|\; \begin{array}{l} \text{there is a neighborhood } V \text{ of } z \in Z \text{ and } \varepsilon > 0 \text{ such that} \\ \widetilde{h}(V \times (t-\varepsilon, t+\varepsilon)) \text{ is contained in } \varphi^{-1}(U \times \{\lambda\}) \\ \text{for a trivializing neighborhood } U \text{ of } h(z,t) \text{ and } \lambda \in \Lambda \end{array} \right\}.$$

  If $t \in T$, then $\widetilde{h}|_V$ is continuous on a neighborhood of $(z,t)$. Now $T$ is open by definition, $T$ contains 0 since $\widetilde{h}_0$ and $h$ are continuous. Let $t_0 \in \overline{T}$. Then there is a trivializing neighborhood $U$ of $h(z,t_0)$, $\varepsilon > 0$, and $V$ a neighborhood of $z$ such that $h(V \times (t-\varepsilon, t+\varepsilon)) \subset U$. Since $t_0 \in \overline{T}$ there is $t_1$ with $t_1 \in T$ and $|t_1 - t_0| < \varepsilon$. Then there is a neighborhood $V_1$ of $z$ in $Z$ such that $\widetilde{h}((V_1 \times \{t_1\}) \subset \varphi^{-1}(U \times \{\lambda\})$. But then $\widetilde{h}(z', t) \subset \varphi^{-1}(U \times \{\lambda\})$ for $t \in (t_0 - \varepsilon, t_0 + \varepsilon)$ and $z' \in V_1$ since $\Lambda$ is discrete. Hence $t_0 \in T$ and $T = [0,1]$ since $[0,1]$ is connected.
- **Consequence:** The map $\mathrm{pr}_* : \pi_1(Y, y_0) \longrightarrow \pi_1(X, x_0)$ is injective.
- **Consequence:** The map $\mathrm{pr}_* : \pi_k(Y, y_0) \longrightarrow \pi_k(X, x_0)$ is an isomorphism
- **Definition:** A space $X$ is *locally path connected* if for all $x \in X$ every neighbourhood $V$ of $x$ contains a path connected neighbourhood $V'$ of $x$.
- **Examples:**
  - CW-complexes are locally path connected.
  - For $M = \{0\} \cup \{1, 1/2, 1/3, 1/4, \ldots\}$ consider the cone $C_M = (M \times I)/(M \times \{1\})$ of $M$. Then $C_M$ is path connected (even contractible) but not locally path connected.
- **Theorem:** Let $Y$ be locally path connected and path connected. Let $\mathrm{pr} : \widetilde{X} \longrightarrow X$ be a covering and $f : Y \longrightarrow X$ a continuous map. Fix $y_0 \in Y$, $x_0 \in X$ and $\widetilde{x}_0 \in \widetilde{X}$ such that $f(y_0) = x_0 = \mathrm{pr}(\widetilde{x}_0)$. Then the following are equivalent:
  1. There is a unique map $\widetilde{f} : (Y, y_0) \longrightarrow (\widetilde{X}, \widetilde{x}_0)$ such that $\mathrm{pr} \circ \widetilde{f} = f$.
  2. $f_*(\pi_1(Y, y_0)) \subset \mathrm{pr}_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$

$$\begin{tikzcd} & \widetilde{X} \arrow[d, "\mathrm{pr}"] \\ Y \arrow[ur, dashed, "\widetilde{f}"] \arrow[r, "f"'] & X \end{tikzcd}$$

- **Proof:** (1)⇒(2): Assume that the lift $\widetilde{f}$ exists. Then

$$f_*(\pi_1(Y,y_0)) = \mathrm{pr}_*\left(\widetilde{f}_*(\pi_1(Y,y_0))\right) \subset \mathrm{pr}_*(\pi_1(\widetilde{X},\widetilde{x}_0)).$$

(2)⇒(1): Let $y \in Y$ and fix a path $\gamma_y : [0,1] \longrightarrow Y$ such that $\gamma_y(0) = y_0$ and $\gamma_y(1) = y$. We define

$$\widetilde{f}(y) = \text{ end point of the lift of } f \circ \gamma_y \text{ to } \widetilde{X} \text{ with starting point } \widetilde{x}_0.$$

This is well defined: Let $\gamma_y'$ be another path in $Y$ from $y_0$ to $y$. By assumption, there is $[\widetilde{\alpha}] \in \pi_1(\widetilde{X},\widetilde{x}_0)$ such that $\mathrm{pr} \circ \widetilde{\alpha}$ is homotopic to $f \circ \left(\gamma_y * \gamma_y'^{-1}\right)$ relative to $x_0$. By the homotopy lifting lemma, $[\widetilde{\alpha}]$ has a representative, we call it $\widetilde{\alpha}'$, which is a lift of $f \circ \left(\gamma_y * \gamma_y'^{-1}\right)$. Since all homotopies are relative to basepoints, $\mathrm{pr}^{-1}(x_0)$ is discrete, and $\widetilde{\alpha}$ is a closed loop, the same is true for $\widetilde{\alpha}'$. Hence the lifts of $f \circ \gamma_y$ and $f \circ \gamma_y'$ have the same endpoints. Thus we have a well defined map $\widetilde{f} : Y \longrightarrow \widetilde{X}$ such that $\mathrm{pr} \circ \widetilde{f} = f$.

$\widetilde{f}$ is continuous: Let $\widetilde{x} \in \widetilde{X}$ and $\mathrm{pr}(\widetilde{x}) = x$ and assume that $\widetilde{x} = \widetilde{f}(y)$. Then there is a path $\gamma_y$ from $y_0$ to $y$ such that the lift of $f \circ \gamma_y$ to $\widetilde{X}$ with starting point $\widetilde{x}_0$ ends at $\widetilde{x}$. Let $U \subset X$ a trivializing open neighborhood of $x$. There is $\lambda \in \Lambda$ such that $\mathrm{pr}^{-1}(U) \simeq U \times \Lambda$ and $\varphi^{-1}(U \times \{\lambda\})$ is an open neighborhood of $\widetilde{x} = (x,\lambda)$. Let $\widetilde{W} \subset U \times \{\lambda\}$ be neighborhood of $\widetilde{x}$. Then $f^{-1}(\mathrm{pr}(\varphi^{-1}(\widetilde{W})))$ contains a path connected neighbourhood $V$ of $y$

Every point $v \in V$ can be reached from $y_0$ by a path of the form $\gamma_y * \sigma_v$ where $\sigma_v$ is a path in $V$ from $y$ to $v$. Lifts of $f \circ (\gamma_y * \sigma_v)$ end in $\varphi{-}1(\widetilde{W})$. Since $\widetilde{W}$ can be chosen arbitrarily small, this implies that $\widetilde{f}$ is continuous.

Uniqueness: follows from the uniqueness for the lift of paths.

- **Definition:** Let $\mathrm{pr} : (Y,y_0) \longrightarrow (X,x_0)$ be a path connected covering which preserves base points. The *characteristic subgroup* of pr is $C(Y,y_0) = \mathrm{pr}_*(\pi_1(Y,y_0))$.
- **Lemma:** $[\gamma] \in \mathrm{pr}_*(\pi_1(Y,y_0))$ iff the lift of $\gamma$ to $Y$ with starting point $y_0$ is closed (i.e. the endpoint of the lift is also $y_0$).
- **Proof:** Let $[\widetilde{\gamma}] \in \pi_1(Y,y_0)$ be a loop, then $\mathrm{pr} \circ \widetilde{\gamma}$ has a closed lift (namely $\widetilde{\gamma}$) and by homotopy lifting the same is true for all closed loops based at $x_0$ which are homotopic to $\mathrm{pr} \circ \widetilde{\gamma}$ (homotopy relative to $x_0$).

  The converse is obvious: Let $\gamma \in \pi^1(X,x_0)$ such that the lift $\widetilde{\gamma}$ to $Y$ with starting point $y_0$ is closed. Then $\mathrm{pr}_*([\widetilde{\gamma}]) = [\gamma]$.
- **Corollary:** Let $(Y_0,y_0)$ and $(Y_1,y_1)$ be two coverings of $(X,x)$ such that $C(Y_0,y_0) = C(Y_1,y_1)$. Then the coverings are isomorphic as pointed covering spaces.

- **Proof:** Let $\varphi_0$ be the lift of $\mathrm{pr}_0$ and $\varphi_1$ be the lift of $\mathrm{pr}_1$. The following diagram commutes since lifts are unique, hence $\varphi_1 \circ \varphi_0 = \mathrm{id} : Y_0 \longrightarrow Y_0$.

$$\begin{tikzcd}
(Y_0, y_0) \arrow[r, "\varphi_0"'] \arrow[rr, bend left, "\mathrm{id}"] \arrow[dr, "\mathrm{pr}_0"'] & (Y_1, y_1) \arrow[r, "\varphi_1"'] \arrow[d, "\mathrm{pr}_1"] & (Y_0, y_0) \arrow[dl, "\mathrm{pr}_0"] \\
& (X, x). &
\end{tikzcd}$$

- **Observation:** Assume that $\mathrm{pr} : Y \longrightarrow X$ is a path connected covering and $y_0, y_0' \in \mathrm{pr}^{-1}(x_0)$. Then the characteristic subgroups of

$$\mathrm{pr} : (Y, y_0) \longrightarrow (X, x_0) \qquad\qquad \mathrm{pr}' : (Y, y_0') \longrightarrow (X, x_0)$$

are conjugate: Let $\gamma$ be a path in $Y$ from $y_0$ to $y_0'$. Then $\mathrm{pr} \circ \gamma$ is a closed loop based in $x_0$. Since conjugation with $\gamma$ induces isomorphisms of fundamental groups, there is a loop $\alpha'$ based at $y_0'$ such that $\gamma * \alpha' * \gamma^{-1}$ is homotopic to $\alpha$. Then $\mathrm{pr} \circ \alpha = \mathrm{pr}(\gamma * \alpha' * \gamma^{-1}) = (\mathrm{pr} \circ \gamma) * (\mathrm{pr}' \circ \alpha') * (\mathrm{pr} \circ \gamma^{-1})$. Hence.

$$\mathrm{pr}_*(\pi_1(Y, y_0)) = [\mathrm{pr} \circ \gamma] \cdot \mathrm{pr}'_* (\pi_1(Y, y_0')) \cdot [\mathrm{pr} \circ \gamma]^{-1}.$$

- **Consequence:** If $\mathrm{pr} : Y \longrightarrow X$ is a path connected covering, then the conjugacy class of the characteristic subgroup of $\pi_1(X, x_0)$ is independent of the choice of base point in $Y$.
- **Theorem:** Let $\mathrm{pr} : (\widetilde{X}, \widetilde{x}_0) \longrightarrow (X, x_0)$ a locally path connected and path connected covering and $f : (Y, y_0) \longrightarrow (X, x_0)$ continuous. The following are equivalent:
  1. There is a lift $\widetilde{f} : Y \longrightarrow \widetilde{X}$ of $f$, i.e. $\mathrm{pr} \circ \widetilde{f} = f$.
  2. $f_*(\pi_1(Y, y_0))$ is conjugate to a subgroup of the characteristic subgroup of $\mathrm{pr} : (\widetilde{X}, \widetilde{x}_0) \longrightarrow (X, x_0)$.
- **Proof:** (1)$\Rightarrow$(2): Let $\widetilde{f}$ be a lift and $\widetilde{x}_0' = \widetilde{f}(y_0)$. We write $\mathrm{pr}' : (\widetilde{X}, \widetilde{x}_0') \longrightarrow \pi_1(X, x_0)$. Let $\gamma$ is a path in $\widetilde{X}$ from $\widetilde{x}_0$ to $\widetilde{x}_0' = \widetilde{f}(y_0)$. Then

$$\begin{aligned}
f_*(\pi_1(Y, y_0)) &= (\mathrm{pr}' \circ \widetilde{f})_*(\pi_1(Y, y_0)) \\
&\subset \mathrm{pr}'_* \left(\pi_1(\widetilde{X}, \widetilde{x}_0')\right) \\
&= [\mathrm{pr} \circ \gamma]^{-1} \cdot \mathrm{pr}_* \left(\pi_1(\widetilde{X}, \widetilde{x}_0)\right) \cdot [\mathrm{pr} \circ \gamma].
\end{aligned}$$

(2)$\Rightarrow$(1): Assume that $f_*(\pi_1(Y, y_0)) \subset [\beta] \cdot \mathrm{pr}_*(\pi_1(\widetilde{X}, \widetilde{x}_0)) \cdot [\beta]^{-1}$ for $\beta \in \pi_1(X, x_0)$. Let $\widetilde{\beta}^{-1}$ be a lift of $\beta^{-1}$ to $\widetilde{X}$ with initial point $\widetilde{x}_0$. Then

$$\begin{aligned}
f_*(\pi_1(Y, y_0)) &\subset [\beta] \cdot \mathrm{pr}_*(\pi_1(\widetilde{X}, \widetilde{x}_0)) \cdot [\beta]^{-1} \\
&= [\mathrm{pr} \circ \widetilde{\beta}] \cdot \mathrm{pr}_*(\pi_1(\widetilde{X}, \widetilde{x}_0)) \cdot [\mathrm{pr} \circ \widetilde{\beta}]^{-1} \\
&= \mathrm{pr}_* \left(\widetilde{\beta} * \pi_1(\widetilde{X}, \widetilde{x}_0) * \widetilde{\beta}^{-1}\right) \\
&= \mathrm{pr}_* \left(\pi_1(\widetilde{X}, \widetilde{\beta}(0))\right)
\end{aligned}$$

According to the Lifting lemma from above there is a (unique) lift $\widetilde{f} : (Y, y_0) \longrightarrow \left(\widetilde{X}, \widetilde{\beta}(0)\right)$ of $f$ to $\widetilde{X}$ which maps $y_0$ to $\widetilde{\beta}(0)$.

- **Corollary:** Let $Y_0$ and $Y_1$ be two covering spaces of $X$ (both path connected and locally path connected). Then $Y_0, Y_1$ are isomorphic iff the characteristic subgroups are conjugate.
- **Reminder:** The index of a subgroup $H \subset G$ is the cardinality of $G/H$, this equals the cardinality of $H\backslash G$.
- **Remark:** Let $\mathrm{pr} : (Y, y_0) \longrightarrow (X, x_0)$ be a path connected covering, $Y$ path connected, and $\Lambda_0 = \mathrm{pr}^{-1}(x_0)$. For each $y_i \in \Lambda$ pick a path $\widetilde{\gamma}_i, y_i \in \Lambda$ in $Y$ from $y_0$ to $y_i$. Then for every path $\alpha$ representing an element of $C(Y, y_o) \cdot [\mathrm{pr} \circ \widetilde{\gamma}_i]$ the lift $\widetilde{\alpha}$ to $Y$ with initial point $y_0$ ends in $y_i$. This establishes a bijective map

$$\underbrace{C(Y, y_0)\Big\backslash\pi_1(X, x_0)}_{\text{right cosets}} \longrightarrow \Lambda_0.$$

- **Corollary:** If $\mathrm{pr} : Y \longrightarrow X$ is a $n$-sheeted covering, then $C(Y, y_0)$ has index $n$ in $\pi_1(X, x_0)$ (for all $y_0 \in \mathrm{pr}^{-1}(x_0)$).
- **Definition:** Let $X$ be a space and $G$ a group. A *(left) group action* of $G$ on $X$ is a homomorphism $\rho : G \longrightarrow \mathrm{Homeo}(X)$ such that $\rho(gh) = \rho(g) \circ \rho(h)$. A *right group action* satisfies $\rho(gh) = \rho(h) \circ \rho(g)$.
- **Definition:** Let $\mathrm{pr} : Y \longrightarrow X$ be a covering. A *decktransformation* is a homeomorphism $f : Y \longrightarrow Y$ such that $\mathrm{pr} \circ f = \mathrm{pr}$. They form the group $\mathrm{Deck}(\mathrm{pr})$ of decktransformations and acts on $Y$ from the left.
- **Remark:** Let $\mathrm{pr} : Y \longrightarrow X$ be a path connected and locally path connected covering and $f, g$ two decktransformations. If $f(y_0) = g(y_0)$ for some $y_0 \in Y$, then $f \equiv g$ by unique lifting.
- **Examples:**
  - $\mathrm{pr} = E : R \longrightarrow S^1, E(t) = e^{2\pi i t}$. Then $f_n : \mathbb{R} \longrightarrow \mathbb{R}, f_n(t) = t + n$ is a decktransformation for $n \in \mathbb{Z}$. The deckgroup is $\mathbb{Z}$.
  - $\mathrm{pr}_n : S^1 \longrightarrow S^1, \mathrm{pr}_n(z) = z^n$. Let $\zeta$ be a $n$-th root of unity. Then $f_m : S^1 \longrightarrow S^1, f_n(z) = \zeta^m z$ is a decktransformation for $n \in \mathbb{Z}$. The deckgroup is $\mathbb{Z}_n$.
  - $\mathrm{pr} : S^2 \longrightarrow \mathbb{RP}^2$. Then $f = -\mathrm{id}$ is a decktransformation, the deckgroup is $\mathbb{Z}_2$.
- **Definition:** Let $G$ be a group and $H$ a subgroup. The *normaliser* of $h$ in $G$ is the largest subgroup $N(H) \supset H$ such that $H$ is normal in $H$.
- **Remark:** The normaliser is well defined since any two subgroups of $G$ in which $H$ is normal generate a subgroup of $G$ with the same property.
- **Warning:** There is notation clash here: I will frequently write $H\backslash N(H)$ for the quotient group (represented by right cosets, which coincide with left cosets since $H$ is normal in $N(H)$). This is to be distinguished from the complement $H \setminus N(H)$ of $N(H)$ in $H$ by the reader (the latter possibility would always denote the empty set).
- **Theorem:** Let $\mathrm{pr} : (Y, y_0) \longrightarrow (X, x_0)$ be a path connected and locally path connected covering with base points and $C = C(Y, y_0) = \mathrm{pr}_*(\pi_1(Y, y_0))$ its characteristic subgroup. Then

$$\begin{aligned} \Gamma : C\Big\backslash N(C) &\longrightarrow \mathrm{Deck}(\mathrm{pr}) \\ [\gamma] &\longmapsto \begin{array}{l} \text{lift of pr to } (Y, \widetilde{\gamma}(1)) \text{ where } \widetilde{\gamma} \text{ is a lift of} \\ \gamma \text{ to } Y \text{ with initial point } y_0 \end{array} \end{aligned} \tag{10}$$

and

$$\begin{aligned} \mathrm{Deck}(\mathrm{pr}) &\longrightarrow C\backslash N(C) \\ f &\longmapsto [\mathrm{pr}\circ\gamma] \text{ where } \gamma \text{ is a path in } Y \text{ from } y_0 \text{ to } f(y_0) \end{aligned}$$

are mutually inverse homomorphisms.

- **Remark:** Deck(pr) is defined without reference to a base point. For $g \in \pi_1(X)$, the groups $C\backslash N(C)$ and $(gCg^{-1})\backslash N(gCg^{-1})$ are canonically isomorphic.
- **Proof:** well defined maps: first consider $\Gamma : N(C) \longrightarrow \mathrm{Deck}(\mathrm{pr})$. Let $\gamma \in N(C(Y, y_0)) \subset \pi_1(X, x_0)$ and $\widetilde{\gamma}$ the lift of $\gamma$ to $Y$ with initial point $y_0$. Then

$$\begin{aligned} C(Y, \widetilde{\gamma}(1)) &= [\mathrm{pr}\circ\widetilde{\gamma}]^{-1} \cdot C(Y, y_0) \cdot [\mathrm{pr}\circ\widetilde{\gamma}] \\ &= [\gamma]^{-1} C(Y, y_0)[\gamma] = C(Y, y_0) \end{aligned}$$

because $\gamma$ lies in the normaliser of $C(Y, y_0)$. According to the lifting theorem above, there is a unique lift

$$\begin{tikzcd} & (Y, \widetilde{\gamma}(1)) \arrow[d, "\mathrm{pr}"] \\ (Y, y_0) \arrow[ur, dashed, "\Gamma(\gamma)"] \arrow[r, "\mathrm{pr}"'] & (X, x_0) \end{tikzcd}$$

This map is well defined on right cosets since lifts of loops in $C(Y, y_0)$ are closed. Thus $\Gamma$ as in (10) is well defined. and we have also showed that $\Gamma$ is injective

Let $f \in \mathrm{Deck}(\mathrm{pr})$ and $\widetilde{\gamma}$ a path from $y_0$ to $f(y_0)$. Then, since $\mathrm{pr}\circ f = \mathrm{pr}$, $f$ can be viewed as a lift of pr to a map $f : (Y, y_0) \longrightarrow (Y, f(y_0))$. Hence $\mathrm{pr}_*(\pi_1(Y, y_0)) = C(Y, y_0) \subset C(Y, f(1))$. The same argument applied to $f^{-1}$ shows $C(Y, f(1)) \subset C(Y, y_0)$.

Now let $\widetilde{\gamma}$ be a path from $y_0$ to $f(1)$. Then $\pi_1(Y, f(1)) = [\mathrm{pr}\circ\widetilde{\gamma}]^{-1}\pi_1(Y, y_0)[\mathrm{pr}\circ\widetilde{\gamma}]$. Projecting this to $(X, x_0)$ we obtain $C(Y, f(1)) = [\gamma]^{-1} \cdot C(Y, y_0) \cdot [\gamma] = C(Y, y_0)$. Therefore $\gamma \in N(C)$. This shows that the second map is well defined.

$\Gamma$ is a group homomorphism: Let $\alpha, \beta \in N(C)$ and $\widetilde{\alpha}$, respectively $\widetilde{\beta}$, be the lift of $\alpha$ respectively $\beta$, with initial point $y_0$ and $\widehat{\beta}$ be the lift of $\beta$ with initial point $\widetilde{\alpha}(1)$. Now $\Gamma(\alpha)$ is a deck transformation which moves $y_0$ to $\widetilde{\alpha}(1)$, so $\Gamma(\alpha)\circ\widetilde{\beta} = \widehat{\beta}$. Therefore

$$\begin{aligned} \Gamma(\alpha\cdot\beta)(y_0) &= \widetilde{\alpha * \beta}(1) = \widetilde{\alpha} * \widehat{\beta}(1) = \widehat{\beta}(1) \\ &= \Gamma(\alpha)\left(\widetilde{\beta}(1)\right) = \Gamma(\alpha)\big(\Gamma(\beta)(y_0)\big). \end{aligned}$$

Group homomorphism, second map: Let $f, g$ be two decktransformations, $\widetilde{\alpha}$ a path from $y_0$ to $f(y_0)$ and $\widetilde{\beta}$ a path from $y_0$ to $g(y_0)$. Then $\widetilde{\alpha} * (f\circ\widetilde{\beta})$ is a path from $y_0$ to $(f\circ g)(y_0)$. Because $f$ is a deck transformation

$$\begin{aligned} \left[\mathrm{pr}\circ\left(\widetilde{\alpha} * (f\circ\widetilde{\beta})\right)\right] &= [\mathrm{pr}\circ\widetilde{\alpha}] \cdot [\mathrm{pr}\circ f\circ\widetilde{\beta}] \\ &= [\mathrm{pr}\circ\widetilde{\alpha}] \cdot [\mathrm{pr}\circ\widetilde{\beta}]. \end{aligned}$$

This shows that that the second map is a group homomorphism.

The maps are mutally inverse: This can be read of from the definitions.

- **Consequence:** $C\backslash N(C)$ acts on $Y$ from the left.

- **Definition:** A path connected covering $Y \longrightarrow X$ is called *universal* if the characteristic subgroup is $\{1\}$. It is *regular* or *normal* if its characteristic subgroup is normal in $\pi_1(X)$.
- **Corollary:** If $\mathrm{pr} : Y \longrightarrow X$ is a normal covering, then $\mathrm{Deck}(\mathrm{pr})$ acts transitively on fibers, i.e. for all $x_0$ and $y_0, y_0' \in \mathrm{pr}^{-1}(x_0)$ there is $f \in \mathrm{Deck}(\mathrm{pr})$ such that $f(y_0) = y_0'$.
- So far we have studied coverings and their isomorphisms/automorphisms. In particular, there is at most one (up to isomorphism) covering of $X$ (path connected, locally path connected space) whose characteristic subgroup is conjugate to a given subgroup $C \subset \pi_1(X)$. We want to study under which conditions such a covering exists. This needs some vocabulary.
- **Definition:** Let $G$ be a group. A $G$-action on a space $X$ is *properly discontinuous* if for all $x \in X$ there is an open neighborhood $U$ such that $gU \cap U = \emptyset$ for all $g \neq 1$. The same for right $G$-actions.
- **Remark:** If $G \curvearrowright X$ is properly discontinuous, then $g \cdot x = x$ implies $g = 1$, i.e. properly discontinuous actions are free.
- **Warning:** There is another property of $G \curvearrowright X$ which is also called *properly discontinous*: $X$ has to be locally compact and for all compact $K \subset X$ the set $\{g \mid K \cap gK \neq \emptyset\}$ is finite.
- **Example:** The deck group action on a covering is properly discontinuous (look at preimages of trivializing neighbourhoods)
- **Example:** Let $\mathbb{Z} \curvearrowright S^1$ be defined by $n \cdot z = e^{2\pi i n a} z$ for $a \in \mathbb{R} \setminus \mathbb{Q}$. This group action is free, but not properly discontinuous.
- **Theorem:** Let $\mathrm{pr} : (\widetilde{X}, \widetilde{x}_0) \longrightarrow (X, x_0)$ be the universal covering of the path connected and locally path connected space $X$ and $H \subset \pi_1(X, x_0)$ a subgroup. Then $X$ has a covering with characteristic subgroup $H$.
- **Proof:** Consider the homomorphism $\Gamma$ and $\Gamma(H) \subset \mathrm{Deck}(\widetilde{X}, \widetilde{x}_0)$ from (10) and let

$$\begin{aligned} \mathrm{pr}_H : (\widehat{X}, \widehat{x}_0) = (\Gamma(H) \backslash \widetilde{X}, [\widetilde{x}_0]) &\longrightarrow (X, x_0) \\ [\widetilde{y}] &\longmapsto \mathrm{pr}(\widetilde{y}). \end{aligned}$$

Let $[y] \in X$ and $U$ an open neighbourhood of $y$ such that $U \cap g \cdot U = \emptyset$ for $g \neq 1$. Then $g \cdot U, g \in G$, is a pairwise disjoint collection of open sets such that $\cup_g (g \cdot U) = \mathrm{pr}^{-1}(\mathrm{pr}(U))$. In particular, $\mathrm{pr}(U) \subset X$ is open, $\mathrm{pr}|_U : U \longrightarrow \mathrm{pr}(U)$ is a homeomorphism, and

$$\begin{tikzcd} (\mathrm{pr}|_U(u), \Gamma(h)) \arrow[rr, mapsto] & & \Gamma(h)(u) \\ \mathrm{pr}|_U(U) \times \Gamma(H) \arrow[rr, "\varphi"'] \arrow[dr, "\mathrm{pr}_1"'] & & \mathrm{pr}^{-1}(\mathrm{pr}(U)) \arrow[dl, "\mathrm{pr}"] \\ & \mathrm{pr}(U) & \end{tikzcd}$$

establishes that $\mathrm{pr}_H$ is a covering.

Let $[\alpha] \in \pi_1(X, x_0)$ and consider the lift $\widehat{\alpha}$ of $\alpha$ to $\widehat{X}$ with initial point $\widehat{x}_0$. This is closed if and only if the lift of $\widetilde{\alpha}$ has endpoint in $\Gamma(H)$, i.e. $[\alpha] \in H$.
- **Consequence:** Let $(X, x_0)$ be connected, locally path connected space with base point $x_0$ and $H \subset \pi_1(X, x_0)$ a subgroup. In order to show that there is

a covering with characteristic group $H$ it is enough to construct the universal cover.

- **Definition:** A topological space $X$ is *semi locally simply connected* if for every $x \in X$ and every neighborhood $U$ of $x$ there is a neighbouhood $U' \subset U$ such that every loop in $U'$ is null homotopic in $X$.
- **Example:** cone over the Hawaiian earrings.
- **Terminology:** A space is *sufficiently connected* if it is path connected, locally path connected, and semi locally simply connected.
- **Theorem:** Let $X$ be a sufficiently connected space. Then there is there is a universal covering $\mathrm{pr} : Y \longrightarrow X$.
- **Proof Idea/Stroke of genius:** We need a space with the path lifting property. Let $\gamma : ([0, 1], 0) \longrightarrow (X, x_0)$ be a path in $X$. Then

$$\begin{aligned} [0,1] &\longrightarrow \mathcal{P}(X, x_0) = \{\text{Paths } \gamma : [0,1] \longrightarrow X \text{ starting at } x_0\} \\ s &\longmapsto \left( \begin{array}{rl} \gamma_s : [0,1] & \longrightarrow X \\ t & \longmapsto \gamma(st) \end{array} \right) \end{aligned} \tag{11}$$

is a lift of $\gamma$ to the set $\mathcal{P}(X, x_0)$ with respect to the end point map

$$\begin{aligned} \mathrm{pr} : (\mathcal{P}(X, x_0), c_0) &\longrightarrow (X, x_0) \\ \gamma &\longmapsto \gamma(1) \end{aligned}$$

where $c_0$ is the constant path sitting at $x_0$. This only a set theoretic cover, and if $\gamma$ is homotopic of $\gamma'$ are homotopic relative there end points, then the endpoints (namely the paths $\gamma$ and $\gamma'$) are not equal. This suggests we should look at spaces of paths with the equivalence relation *homotopy relative endpoints.* It turns out that this can be carried out.
- **Proof:** We consider

$$\widetilde{X} = \{ \text{ paths in } X \text{ starting in } x_0\} / \sim$$

with $\sim$ being homotopy relative to the endpoints. As base point in $\widetilde{X}$ we take the (homotopy class of) the constant path $c_0(t) \equiv x_0$. The projection is

$$\begin{aligned} \mathrm{pr} : (\widetilde{X}, c_0) &\longrightarrow (X, x_0) \\ [\gamma] &\longmapsto \gamma(1). \end{aligned}$$

This is well defined since we use homotopy relative to end points as equivalence relation. We need to define a topology on $\widetilde{X}$ with all kinds of properties.

Let $\gamma \in \widetilde{X}$ and $U$ a path connected neighbourhood of $\gamma(1)$. Then let

$$V(\gamma, U) = (\gamma * \{\text{paths in } V \text{ startign in } \gamma(1)\}) \,/\text{homotopy relative endpoints.}$$

**Definition of the topology on $\widetilde{X}$:** A set $V \subset \widetilde{X}$ is open if and only of for every point $[\gamma]$ in $V$ there is $\gamma \in [\gamma]$ and $\gamma(1) \subset U \subset X$ such that $V(\gamma, U) \subset V$.

**Topology:** direct check

pr **is continuous:** Let $U \subset X$ open and $[\gamma] \in \mathrm{pr}^{-1}(U)$. Because $U$ is open and $X$ is locally pah connected, there is $U' \subset U$ around $\gamma(1)$. Then $V(\gamma, U') \subset U$. This shows that pr is continuous.

$\widetilde{X}$ **is path connected:** Let $\gamma$ represent $[\gamma]$. Then $s \longmapsto \gamma_s$ is a path (as in (11)) starting at $c_0$ and ending at $\gamma$ once we show it is continuous. Let $s_0 \in [0, 1]$. There is a path connected neighborhood $U$ of $\gamma_s(1) = \gamma(s)$ and because $\gamma$ is continuous it spends some time in $U$, i.e. there is $\varepsilon > 0$ such that $\gamma((s_0 - \varepsilon, s_0 + \varepsilon)) \subset U$. Then $\gamma_s \in V(\gamma_{s_0}, U)$ for $s \in (s_0 - \varepsilon, s_0 + \varepsilon)$.

pr **has discrete fibers:** Let $x \in X$ and $\gamma$ a path from $x_0$ to $x$ (representing a point in $\widetilde{X}$). Because $X$ is semi locally simply connected, there is a path connected $U \subset X$ such that $i_* : \pi_1(U, x) \longrightarrow \pi_1(X, x)$ is trivial. Then $V(\gamma, U)$ is open and contains no other point of $\mathrm{pr}^{-1}(x)$.

pr **is locally trivial:** very similar to the last step.

$\widetilde{X}$ **is simply connected:** Let $\gamma_s : [0,1] \longrightarrow X$ be a loop of paths such that $s \longmapsto [\gamma_s]$ is continuous (in the topology on $\widetilde{X}$, not necessarily in any other natural topology on the space of paths). Then $\sigma : tau \longmapsto \gamma_\tau(1)$ is a path such that $s \longmapsto \sigma(s\cdot)$ is homotopic (relative endpoints) to $\gamma_s$. Since $[\gamma_1]$ is homotopic relative endpoints to the constant path. The resulting homotopy can used to show that the original loop of paths is null-homotopic.

## 17. Lecture on Jan. 8 – Definition of singular homology, points and $H_0$

- **Definition:** The *standard $n$-simplex* is the subspace
$$\Delta^n = \{(x_0, \ldots, x_n) \mid \sum_i x_i = 1, x_i \geq 0\}.$$
- **Definition:** A *singular $n$-simplex* in $X$ is a continuous map
$$\sigma : \Delta^n \longrightarrow X.$$
The space of *$n$-chains* $C_n(X)$ is the free Abelian group generated by the singular $n$-simplices in $X$. If $n < 0$, then $C_n(X) = \{0\}$.
- 0-simplices in $X$ are maps of points, 1-simplices are maps of intervals, etc.
- **Notation:** An $n$-simplex is the convex hull of its vertices. The vertices of the standard $n$-simplex are $v_i = (0, \ldots, x_i = 1, \ldots, 0)$, $i = 0, \ldots, n$. We write $[w_{i_0}, \ldots, w_{i_k}]$ for the the smallest convex subspace of $\Delta^n$ which contains the vertices $w_{i_0}, \ldots, w_{i_k}$ of $\Delta^n$. We order the vertices so that $i_0 < i_2 < \ldots < i_k$. There is a unique affine homeomorphism
$$\Delta^k \longrightarrow [w_{i_0}, \ldots, w_{i_k}] \tag{12}$$
which maps the standard vertex $v_j \in \Delta^k$ to $w_{i_j} \in \Delta^n$.
- **Definition:** Let $\sigma$ be a $n$-simplex in $X$ for $n > 0$. Then
$$\partial_n \sigma = \sum_{i=0}^{n} (-1)^i \sigma|[v_0, \ldots, \widehat{v_i}, \ldots v_n] \in C_{n-1}(X) \tag{13}$$
where $\widehat{v_i}$ denotes the omission of the $i$-th vertex, we use the inverse of (12) to really obtain an element of $C_{n-1}(X)$. (13) defines a unique homomorphism
$$\partial_n : C_n(X) \longrightarrow C_{n-1}(X).$$
This is the *boundary operator*. For the case $n \leq 0$, $\partial_n$ is defined to be zero.
- **Fundamental Lemma:** $\partial_{n-1} \circ \partial_n = 0$.

- **Proof:** Let $\sigma$ be a singular $n$-simplex in $X$. Then

$$\begin{aligned}(\partial_{n-1}\circ\partial_n)\sigma &= \partial_{n-1}\left(\sum_{i=0}^{n}(-1)^i\sigma|[v_0,\dots,\widehat{v_i},\dots v_n]\right)\\ &= \sum_{i=0}^{n}(-1)^i\left(\partial_{n-1}\sigma|[v_0,\dots,\widehat{v_i},\dots v_n]\right)\\ &= \sum_{i=0}^{n}(-1)^i\left(\sum_{j=0}^{i-1}(-1)^j\sigma|[v_0,\dots,\widehat{v_j},\dots,\widehat{v_i},\dots,v_n]\right.\\ &\qquad\left.+\sum_{j=i}^{n-1}(-1)^{j-1}\sigma|[v_0,\dots,\widehat{v_i},\dots,\widehat{v_j},\dots,v_n]\right)\\ &= 0\end{aligned}$$

because $\sigma|[v_0,\dots,\widehat{v_i},\dots,\widehat{v_j},\dots,v_n]$ appears twice for fixed $i,j$ but with different signs.
- This implies $\mathrm{im}(\partial_n)\subset\ker(\partial_{n-1})$.
- **Definition:** The *singular complex of* $X$ is

$$\dots\xrightarrow{\partial_{n+1}}C_n(X)\xrightarrow{\partial_n}C_{n-1}(X)\xrightarrow{\partial_{n-1}}\dots$$

More generally, a chain complex is a sequence $C_n$ of Abelian groups with linear maps $\partial_n : C_n\longrightarrow C_{n-1}$ such that $\partial_n\circ\partial_{n+1}=0$ for all $n$.
- **Definition:** The *$n$-th singular homology group of* $X$ is

$$H_n(X) := H_*((C_*(X),\partial)) = \frac{\ker(\partial_n)}{\mathrm{im}(\partial_{n+1})}.$$

The definitions works for all chain complexes.
- **Terminology:** Elements of $H_i(X)$ are called *homology classes*, two singular chains $a,a'$ representing the same homology class are *homologous* (then $a-a'=\partial\tau$ for some $\tau$), a chain representing the trivial class is *null-homologous*. We denote

$$C_*(X)=\bigoplus_n C_n(X) \qquad\qquad H_*(X)=\bigoplus_n H_n(X).$$

A singular chain $\sigma$ with $\partial\sigma=0$ is called a *cycle*, a singular chain $\sigma$ is a boundary if there is a singular chain $\tau$ such that $\partial\tau=\sigma$. Because of $\partial^2=0$, boundaries are always cycles.

  Analogous terminology is used for general chain complexes.
- **Definition:** Let $f : X\longrightarrow Y$ be continuous. Then

$$\begin{aligned}f_* : C_n(X)&\longrightarrow C_n(Y)\\ \sum_i n_i\sigma_i&\longmapsto\sum_i n_i f\circ\sigma_i\end{aligned}$$

is the induced map on chains. Other common notations include $C_n(f), f_n$.
- **Definition:** A family of homomorphisms $f_n : C_n(X)\longrightarrow C_n(Y)$ forms a chain map if $\partial_n^Y\circ f_n = f_{n-1}\circ\partial_n^X$ for all $n$.

  The definition for general chain complexes is analogous.
- **Lemma:**

  (0) Chain maps induce homomorphisms on homology groups.

(i) For $f : X \longrightarrow Y$ continuous, the induced map $f_*$ is a chain map.
(ii) $(g \circ f)_* = g_* \circ f_*$.

- **Remark:** There are other types of homology for particular classes of spaces. For example: Simplicial homology of simplicial complexes, cellular homology for CW-complexes.
- **Disadvantage of singular homology:** $C_n(X)$ is a huge Abelian group, usually of infinite rank, and it is nearly impossible to compute anything (a notable exception to this is the one-point space) from the definition compared with simplicial/cellular homology.
- **Advantage of singular homology:** It is obviously a topological invariant, i.e. homeomorphic spaces have isomorphic chain groups/homology groups. This is not true/obvious for simplicial or cellular homology.
- **Theorem (homology of a point):** Let $X = \{*\}$ be the one-point space. Then

$$H_n(X) = \begin{cases} \mathbb{Z} & \text{if } n = 0 \\ 0 & \text{otherwise} \end{cases}$$

- **Proof:** For all $n$ there is a unique singular $n$-simplex $\sigma_n$ in $X$. Let $n > 0$. Then $\partial_n \sigma_n$ has $n+1$-summands with alternating signs. Hence $\partial_n \sigma_n = \sigma_{n-1}$ if $n$ is even and $\partial_n \sigma_n = 0$ if $n$ is odd. Hence

$$\ker(\partial_n) = \begin{cases} 0 & \text{odd } n \\ \mathbb{Z} & \text{even } n \end{cases}$$

which implies the claim for odd $n$. If $n$ is even, then every element in $C_n(X)$ is in the image of $\partial_{n+1} C_{n+1}(X)$. This implies the claim for positive, even $n$. The case $n = 0$ is different because we defined $C_{-1} = \{0\}$ despite of the usual convention that there **is** a map from the empty set into $X$ (exactly one).

- **Lemma (disjoint union):** Let $X = \bigcup X_i$ where $X_i$ are path connected components of $X$. Then

$$C_n(X) = \bigoplus_i C_n(X_i)$$
$$H_n(X) = \bigoplus_i H_n(X_i).$$

- **Proof:** This is immediate since the $n$-simplex is path connected and the boundary operator turns $n$-chains in $X_i$ into $n-1$-chains in $X_i$.
- **Proposition:** If $X$ is path connected, then

$$\varepsilon : H_0(X) \longrightarrow \mathbb{Z}$$
$$\left[\sum_i n_i p_i\right] \longmapsto \sum_i n_i$$

is an isomorphism.

- **Proof:** $\ker(\partial_0) = C_0(X)$ is generated by points, elements of $C_0$ are finite sums of points with multiplicities (positive or negative). Let $\sigma : \Delta^1 \longrightarrow X$ be a singular 1-simplex. Then $\partial_1(\sigma) = \sigma(v_1) - \sigma(v_0)$. These elements generate (proof by induction on the minimal number of different summands) the kernel of

$$\begin{aligned} \varepsilon : C_0(X) &\longrightarrow \mathbb{Z} \\ \sum_{i=1}^{m} n_i p_i &\longmapsto \sum_{i=1}^{m} n_i. \end{aligned}$$

## 18. Lecture on Jan. 11 – Definition of reduced homology, homotopy invariance

- **Definition:** The *reduced singular chain complex of $X$* is

$$\ldots \longrightarrow C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

  and the reduced homology groups $\widetilde{H}_n(X)$ are the homology groups of the reduced singular chain complex.
- **Remark:** This makes sense since $\varepsilon \circ \partial_1 = 0$. It also makes sense because it drops our convention not to count maps from the empty set into a space. Clearly, $H_n(X) = \widetilde{H}_n(X)$ for $n > 0$ and $H_0(X) \simeq \widetilde{H}_0(X) \oplus \mathbb{Z}$ if $X \neq \emptyset$. This last isomorphism depends on the choice of a path connected component of $X$.
- **Remark:** The empty set set $H_*(\emptyset) = 0$ and $H_{-1}(\emptyset) = \mathbb{Z}$, the reduced homology vanishes in all other degrees.
- **Notation:** From now on we omit indices in boundary operators (both referring to the space in question and the degree of a simplex).
- **Theorem (Homotopy invariance):** Let $f, g : X \longrightarrow Y$ be continuous maps which are homotopic. Then $f_* = g_* : H_*(X) \longrightarrow H_*(Y)$.
- **Proof overview:** First, we give a decomposition of $\Delta^n \times I$ into $n+1$-simplices. This and the homotopy $H : X \times I \longrightarrow Y$ with $H(\cdot, 0) = f$ and $H(\cdot, 1) = g$ will then be used to define an operator $P : C_*(X) \longrightarrow C_{*+1}(Y)$, called prism operator. It will turn out that $P$ has the interesting property

$$\partial \circ P + P \circ \partial = g_* - f_*. \tag{14}$$

  Operators with this property are called *chain homotopies*. One can check easily that this implies that $g_* = f_*$ on $H_*$.
  - **Step 1:** Let $\sigma$ be a $n$-simplex, $\sigma_0 = [v_0, \ldots, v_n] \times \{0\} \subset \sigma \times I \subset \mathbb{R}^{n+1} \times \mathbb{R}$, and $\sigma_1 = [w_0, \ldots, w_n] \times \{1\} \subset \sigma \times I$. Then $[v_0, \ldots, v_n, w_n] \subset \mathbb{R}^{n+1} \times \mathbb{R}$ is the convex hull of $\sigma_0$ and $w_n$, it is a $n+1$-simplex. Also, $[v_0, \ldots, v_{n-1}, w_{n-1}, w_n] \subset \mathbb{R}^{n+1} \times \mathbb{R}$ is the convex hull of its vertices and the two simplices intersect in the convex hull of $[v_0, \ldots, v_{n-1}, w_n]$.
    This continues: $[v_0, \ldots, v_k, w_k, \ldots, w_n], k = 0, \ldots, n$ is a collection of $n+1$-simplices which are lying one above the other (with respect to the coordinate on the second factor of $\mathbb{R}^{n+1} \times \mathbb{R}$). This is illustrated in Figure 6.
  - **Step 2:** Let $H : X \times I \longrightarrow Y$ be a homotopy from $f = H(\cdot, 0)$ to $g$, and $\sigma : \Delta^n \longrightarrow X$ a singular $n$-simplex in $X$. Then we define

$$P(\sigma) = \sum_i (-1)^i H \circ \underbrace{(\sigma \times \mathrm{id}_I)|[v_0, \ldots, v_i, w_i, \ldots, w_n]}_{\text{sing. } n+1\text{-simplex in } X \times I} \tag{15}$$

    and extend linearly to define $P : C_n(X) \longrightarrow C_{n+1}(Y)$.

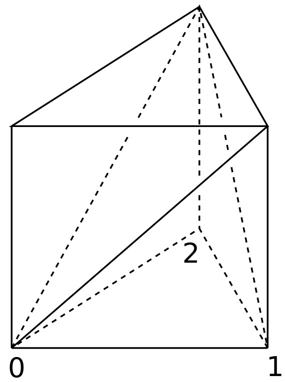


FIGURE 6. Decomposition of )2-simplex)$\times I$ into 3-simplices.

– **Step 3:** We want to check (14). It suffices to do that for singular $n$-simplices in $X$, the case for general singular chains follows from linearity. One computes:

$$
\begin{aligned}
\partial P(\sigma) &= \partial\left(\sum_i (-1)^i H\circ(\sigma\times \mathrm{id}_I)|[v_0,\dots,v_i,w_i,\dots,w_n]\right)\\
&= \sum_i\left(\sum_{j=0}^{i}(-1)^{i+j}H\circ(\sigma\times\mathrm{id}_I)|[v_0,\dots\widehat{v_j},\dots,v_i,w_i,\dots,w_n]\right.\\
&\quad\left.+\sum_{j=i}^{n}(-1)^{i+1+j}H\circ(\sigma\times\mathrm{id}_I)|[v_0,\dots,v_i,w_i,\dots,\widehat{w_j},\dots,w_n]\right).
\end{aligned}
$$

$$
\begin{aligned}
P\partial(\sigma) &= P\left(\sum_i(-1)^i\sigma|[v_0,\widehat{v_i},\dots,v_n]\right)\\
&= \sum_i\left(\sum_{j=0}^{i-1}(-1)^{i+j}H\circ(\sigma\times\mathrm{id}_I)\,|[v_0,\dots,v_j,w_j,\dots,\widehat{w_i},\dots,w_n]\right.\\
&\quad\left.+\sum_{j=i+1}^{n}(-1)^{i+j-1}H\circ(\sigma\times\mathrm{id}_I)\,|[v_0,\dots,\widehat{v_i},\dots,v_j,w_j,\dots,w_n]\right)
\end{aligned}
$$

Adding these two equalities, almost everything cancels. The only terms that survive are those where $i=j$, i.e.

$$
\begin{aligned}
\partial\circ P(\sigma)+P\circ\partial(\sigma) = \sum_{i=0}^{n}\big(&(-1)^{2i}H\circ(\sigma\times\mathrm{id}_I)|[v_0,\dots,v_{i-1},w_i,\dots,w_n]\\
&+(-1)^{2i+1}H\circ(\sigma\times\mathrm{id}_I)|[v_0,\dots,v_i,w_{i+1},\dots,w_n]\big).
\end{aligned}
$$

Expanding the sum, one has $2n+2$-summands, all cancel except the first and the last (if $i=0$, then $[v_0,\dots,v_{i-1},w_i,\dots,w_j]=[w_0,\dots,w_n]$ etc.). Hence,

$$
\begin{aligned}
\partial\circ P(\sigma)+P\circ\partial(\sigma) &= (-1)^{2\cdot 0}H\circ(\sigma\times\mathrm{id}_I)|[w_0,\dots,w_n]+(-1)^{2\cdot 0+1}H\circ(\sigma\times\mathrm{id}_I)|[v_0,\dots,v_n]\\
&= g\circ\sigma - f\circ\sigma.
\end{aligned}
$$

This proves (14).

- **Conclusion:** Assume that $\sigma$ is a cycle representing the homology class $[\sigma]$. Then
$$g_*\sigma - f_*\sigma = \partial P(\sigma) + P(\underbrace{\partial\sigma}_{=0}) = \partial(P(\sigma)).$$
Hence $g_*\sigma, f_*\sigma$ are homologous.

- **Corollary:** Let $f : X \longrightarrow Y$ be a homotopy equivalence. Then $f_* : H_*(X) \longrightarrow H_*(Y)$ is an isomorphism, the inverse is induced by any homotopy inverse of $f$.
- **Definition:** Let $(C_*, \partial)$ and $(C'_*, \partial')$ be chain complexes and $f_*, g_* : C_* \longrightarrow C'_*$ chain maps. Then $f_*$ and $g_*$ are *chain homotopic* if there is a group homomorphism $P : C_* \longrightarrow C'_{*+1}$ such that
$$P \circ \partial + \partial' \circ P = g_* - f_*.$$
The map $P$ is a *chain homotopy.*

## 19. Lecture on Jan. 15 – Definition of relative homology, long exact sequence in homology

- **Definition:** Let $X$ be a space and $A \subset X$ a subspace. Then the inclusion $i : A \longrightarrow X$ induces an inclusion of $C_*(A)$ into $C_*(X)$ which allows to view $C_*(A)$ as subgroup of $C_*(X)$. We define
$$C_*(X, A) = \frac{C_*(X)}{C_*(A)}.$$
Since $i_*$ is a chain map, the operator $\partial : C_*(X, A) \longrightarrow C_{*-1}(X, A)$ is well defined. The homology groups $H_*(X, A)$ of the chain complex $(C_*(X, A), \partial)$ are the *relative homology groups of* $(X, A)$.
- **Terminology:** A chain $\sigma \in C_*(X)$ represents a class in $H_*(X, A)$ if $\partial\sigma \in C_*(A)$, $\sigma$ is then called a *relative cycle*. A relative cycle $\sigma$ represents the trivial class in $H_*(X, A)$ if there are singular chains $\tau \in C_*(X)$ and $\gamma \in C_*(A)$ such that $\sigma = \gamma + \partial\tau$. In this case $\sigma$ is a *relative boundary.*
- **Terminology:** A sequence of homomorphisms of groups
$$\dots A_i \xrightarrow{f_i} A_{i+1} \xrightarrow{f_{i+1}} A_{i+2} \dots$$
is *exact* if $\ker(f_{i+1}) = \operatorname{image}(f_i)$ for all $i \in \mathbb{Z}$ (this implies $f_{i+1} \circ f_i = 0$ for all $i$). A *short exact sequence* is a collection of groups and group homomorphisms
$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$
which is exact. If this is the case then in particular $f$ is injective and $g$ is surjective.
- **Example:** If $f : A \longrightarrow B$ is injective, then
$$0 \longrightarrow A \longrightarrow B \longrightarrow B/(f(A)) \longrightarrow 0$$
is exact.
- **Notation:** One usually writes 0 for the trivial group (containing only the unit element) in the context of Abelian groups while denoting the same group with 1 in the context of groups which are not assumed to be Abelian.
- **Remark:** In order to define exact sequences one does not use the group structure really. It is sufficient to consider pointed sets, i.e. sets with a distinguished element.

- **Setup:** Let $(A_*, \partial), (B_*, \partial), (C_*, \partial)$ be chain complexes (the boundary maps of $A_*, B_*, C_*$ do not have to be related, although they are denoted by the same symbol) and $f_* : A_* \longrightarrow B_*$, $g_* : B_* \longrightarrow C_*$ chain maps such that

$$0 \longrightarrow A_i \longrightarrow B_i \longrightarrow C_i \longrightarrow 0 \tag{16}$$

is (short) exact. This is a short exact sequence of chain complexes. We want to understand how the homology groups are related.

- **Theorem (long exact sequence in homology):** Short exact sequences of chain complexes induce long exact sequences in homology

$$\dots \xrightarrow{\partial_*} H_i(A_*) \xrightarrow{f_*} H_i(B_*) \xrightarrow{g_*} H_i(C_*) \xrightarrow{\partial_*} H_{i-1}(A_*) \xrightarrow{f_*} \dots \tag{17}$$

$$[\gamma] \longmapsto [f_*^{-1}\partial\beta]$$

where $\beta$ is a chain in $B$ such that $g(\beta) = \gamma$ and $f^{-1}\partial^B\beta$ is a chain $\alpha$ in $A_{i-i}$ such that $f_*\alpha = \partial\beta$. Unlike in (16), the maps $f_*, g_*$ denote the induced maps on homology and $\partial^B$ is the boundary operator of $B_*$, the superscript is there only for emphasis.

- **Proof:** The following is an example of *diagram chasing*.
  - $\partial_*$ is well defined: Let $\gamma \in C_i$ be a cycle. $g_*$ is surjective, so there is $\beta \in B_i$ such that $g_*(\beta) = \gamma$. Since $g_*$ is a chain map, $g_*(\partial\beta) = \partial g_*(\beta) = \partial\gamma = 0$. Because $\mathrm{im}(f_*) = \ker(g_*)$, there is $\alpha \in A_{i-1}$ such that $f_*\alpha = \beta$. Finally, $\partial^A\alpha = \partial^A f_*^{-1}(\partial^B\beta) = f^{-1}(\partial^B\partial^B\beta) = 0$ since $f_*$ is an injective chain map and $(\partial^B)^2 = 0$. Hence $\alpha$ is a cycle and represents a homology class of $A_*$, but it might depend on choices.
  Assume that $\alpha', \alpha$ satisfy $f_*(\alpha) = f_*(\alpha') = \beta$. Then $f_*(\alpha - \alpha') = 0$, hence $\alpha = \alpha'$ by injectivity of $f_*$.
  Assume that $\beta', \beta$ satisfy $g_*(\beta') = \gamma = g_*(\beta)$. Then $g_*(\beta - \beta') = 0$ and there is a (unique) $\hat{\alpha} \in A_i$ such that $f_*(\hat{\alpha}) = \beta - \beta'$. Then $f_*^{-1}(\partial\beta) - f_*^{-1}(\partial\beta') = \partial\hat{\alpha}$, i.e. $[f_*^{-1}(\partial\beta)] = [f_*^{-1}(\partial\beta')]$ in homology.
  - $\mathrm{im}(f_*) \subset \ker(g_*)$: obvious.
  - $\mathrm{im}(g_*) \subset \ker(\partial_*)$: Assume $\gamma = g_*\beta$ and $\beta$ is a cycle in $B$. Then $\partial_*\gamma = 0$ by the definition of $\partial_*$.
  - $\mathrm{im}(\partial_*) \subset \ker(f_*)$: Let $\gamma$ represent a class in $H_i(C)$. Then

  $$f_*(\partial_*(\gamma)) = f_*(f_*^{-1}(\partial^B\beta)) = \partial^B\beta$$

  represents the trivial homology class in $H_{i-1}(B)$.
  - $\ker(g_*) \subset \mathrm{im}(f_*)$: Let $[\sigma] \in H_i(B)$ such that $g_*([\sigma]) = 0$, i.e. $g_*(\sigma) = \partial\tau$. Since $g_* : B_{i+1} \longrightarrow C_{i+1}$ is surjective, there is $\beta \in B_{i+1}$ such that $\sigma - \partial\beta \in \ker(g_*)$. Then $\sigma - \partial\beta \in \mathrm{im}(f_i)$. Let $f_*(\alpha) = \sigma - \partial\beta$. Then, since $f_*$ is a chain map and $f_*$ is injective on chains,

  $$\begin{aligned} f_*(\partial(\alpha)) &= \partial(\sigma - \partial\beta) \\ &= 0 \end{aligned}$$

  implies $\partial\alpha = 0$ and $f_*([\alpha]) = [\sigma - \partial\beta] = [\sigma]$.
  - $\ker(\partial_*) \subset \mathrm{im}(g_*)$: Let $\sigma \in H_i(C)$ be in $\ker(\partial_*)$ and $\beta \in C_i$ such that $g_*(\beta) = \sigma$. Then there is $\alpha \in A_i$ such that $\partial\alpha = f_*^{-1}\partial\beta = \partial_*\sigma$, i.e. $f_*(\partial\alpha) = \partial\beta$. Hence, $\beta - f_*\alpha$ is a cycle and $g_*([\beta - f_*\alpha]) = [g_*(\beta)] = [\sigma]$. Hence $[\sigma] \in \mathrm{im}(g_*)$.

– $\ker(f_*) \subset \operatorname{im}(\partial_*)$: Let $\sigma \in A_i$ such that $[\sigma] \in \ker(f_*)$, i.e. $\sigma$ is a cycle and $f_*\sigma = \partial\beta$ for some $\beta \in B_{i+1}$. Then by definition, $[\sigma] = \partial_*[g_*(\beta)]$.

- **Remark:** We will apply this usually to the short exact sequence of singular chain complexes of a pair of spaces, i.e. for $A \subset X$ we have a long exact sequence

$$\ldots H_i(A) \xrightarrow{i_*} H_i(X) \xrightarrow{\mathrm{pr}_*} H_i(X,A) \xrightarrow{\partial_*} H_{i-1}(A) \ldots \tag{18}$$

$$[\sigma] \longmapsto [\partial\sigma]$$

Recall that $\sigma \in H_i(X,A)$ implies that $\partial\sigma \in C_{i-1}(A)$ and it is clear that $\partial(\partial\sigma) = 0$. It is not clear that $\partial\sigma$ is a boundary in $C_*(A)$.

- **Remark:** Slightly more generally, if $A \subset B \subset X$, then the short exact sequence of singular chains

$$0 \longrightarrow C_*(B,A) \longrightarrow C_*(X,A) \longrightarrow C_*(X,B) \longrightarrow 0$$

induces a long exact sequence

$$\ldots H_i(B,A) \longrightarrow H_i(X,A) \longrightarrow H_i(X,B) \xrightarrow{\partial_*} H_{i-1}(B,A) \ldots$$

## 20. Lecture on Jan. 18 – Excision

- **Reference:** The following is essentially from [Ha].
- **Remark:** One can define *relative homotopy groups* $\pi_i(X,A)$ for pairs of spaces and obtain an exact sequence similar to (18). Homotopy invariance is built into their definition. A property that relative homology groups have, but homotopy groups don't, is the **excision theorem**.
- **Theorem (Excision/Ausschneidung):** Let $Z \subset A \subset X$ such that $\overline{Z} \subset \mathring{A}$. Then the inclusion of pairs

$$(X \setminus Z, A \setminus Z) \hookrightarrow (X,A)$$

induces an isomorphism $H_*(X \setminus Z, A \setminus Z) \simeq H_*(X,A)$.

- Let $\mathfrak{U} = (U_i)_{i \in I}$ be an open covering of $X$. Then we denote

$$C_*^{\mathfrak{U}}(X) = \text{subcomplex of } C_*(X) \text{ generated by singular simplices}$$
$$\text{completely contained in one of the sets } U_i.$$

This a subcomplex since the boundary of a simplex in $U_i$ is a sum of simplices in $U_i$.

- To prove the excision theorem one applies the following proposition (which is important on its own).
- **Proposition (open cover):** The inclusion $\iota : C_*^{\mathfrak{U}}(X) \longrightarrow C_*(X)$ is a chain homotopy equivalence.
- **Organization of Proof:** One first studies affine simplices (=convex hulls of vertices, with an ordering on the set of vertices) lying in $\Delta^n$. Often things will be defined using induction on $n$, and in order to avoid exceptions for $n = 0$, sometimes we will artificially introduce chains of degree $-1$ (as in the reduced chain complex).

  The definition of chain homotopy inverse $\rho$ of $\iota$ is then more algebraic.

- **Barycentric subdivision of $\Delta^n$:** Let $[v_0, \dots, v_n]$ be the convex hull of its vertices, we assume that the vertices $v_0, \dots, v_n$ are in general position.

  The **barycenter** of $[v_0, \dots, v_n]$ is
$$b = \frac{v_0 + \ldots + v_n}{n+1}.$$
  The barycentric subdivision of $\Delta^0$ is $\Delta^0$. Now assume that the barycentric subdivision of affine subsimplices of $\Delta^{n-1}$ is defined. Then the barycentric subdivision of a $n$-simplex $[v_0, \dots, v_n]$ is the collection of simplices $[b, w_0, \dots, w_{n-1}]$ where $[w_0, \dots, w_{n-1}]$ is a simplex appearing in the barycentric subdivision of $[v_0, \dots, \widehat{v_j}, \dots, v_n]$.
- **Geometric Fact:** Let $[v_0, \dots, v_n]$ be the convex hull of points in $\mathbb{R}^n$ in general position. Then the diameter of every simplex in the barycentric subdivision of $[v_0, \dots, v_n]$ is bounded from above by
$$\frac{n}{n+1}\mathrm{diam}([v_0, \dots, v_n]) = \frac{n}{n+1}\max\{|v_i - v_j|\}.$$
  What matters here is that the factor in front of $\mathrm{diam}([v_0, \dots, v_n])$ is smaller than 1 and independent of $[v_0, \dots, v_n]$.
  - $|v - \sum_i t_i v_i| = |\sum_i t_i(v - v_i)| \leq \sum_i t_i \max\{v - v_i\} = \max_i\{|v - v_i|\}$
  - $\mathrm{diam} = \max\{|v_i - v_j|\}$.
  - $b_i$ barycenter of $i$-th face (opposite to $v_i$). Then
$$b = \frac{n}{n+1}b_i + \frac{1}{n+1}v_i.$$
    Hence $b$ lies on $\overline{b_i v_i}$, the distance of $b$ from $v_i$ is $\leq \frac{n}{n+1}\mathrm{diam}([v_0, \dots, v_n])$.
- **Definition of $LC_*(Y)$:** Let $Y$ be a convex subset in some Euclidian space, for example standard simplices, and $LC_*(Y)$ the subgroup of affine chains (i.e. the maps $\sigma : \Delta^* \longrightarrow Y$ generating this group are affine). We keep the original boundary operator. Then $(LC_*(Y), \partial)$ is a chain complex.
- **Adding a vertex to simplices in $LC_*(Y)$:** and define $LC_{-1}(Y) = \mathbb{Z}$ generated by $[\emptyset]$. Let $b \in Y$ and
$$\begin{aligned} b : LC_n(Y) &\longrightarrow LC_{n+1}(Y) \\ [w_0, \dots, w_n] &\longmapsto [b, w_0, \dots, w_n]. \end{aligned}$$
  Here it is important that we are considering convex subsets of Euclidean space. Then $\partial(b[w_0, \dots, w_n]) = [w_0, \dots, w_n] - b(\partial[w_0, \dots, w_n])$. Hence
$$\partial b + b\partial = \mathrm{id} - 0, \tag{19}$$
  so $b$ can be viewed as chain homotopy on $LC_*$ connecting the identity to the zero map. (Hence $H_*((LC_*(Y), \partial)) = \{0\}$.)
- **Subdivision operator on affine chains:** Let $\lambda : \Delta^n \longrightarrow Y$ be a generator of $LC_*(Y)$ and $b_\lambda$ the barycenter of $\lambda$, i.e. $b_\lambda = \lambda(\text{barycenter of } \Delta^n)$. Then we set $S([\emptyset]) = [\emptyset]$ (defining $S$ on $LC_{-1}(Y)$) and
$$S\lambda = b_\lambda(S\partial\lambda).$$
- **$S$ is a chain map on $LC_*(Y)$:** This is true for $* = -1$ and $* = 0$ (by $S\lambda = b_\lambda S\partial\lambda = b_\lambda S[\emptyset] = b_\lambda[\emptyset]$. Then we use $\partial b_\lambda \lambda + b_\lambda \partial\lambda = \lambda$ and induction
$$\begin{aligned} \partial S\lambda &= \partial(b_\lambda(S\partial\lambda)) = S\partial\lambda - b_\lambda \partial S\partial\lambda \\ &= S\partial\lambda - b_\lambda S\partial\partial\lambda = S\partial\lambda. \end{aligned}$$

- **Example:** $S$ is the identity on $LC_0(Y)$ and $LC_{-1}(Y)$. If $\lambda : \Delta^1 \longrightarrow Y$ is an affine 1-simplex, then

$$\begin{aligned}\partial\lambda &= \lambda(v_1 \in \Delta^1) - \lambda(v_0)\\ S\lambda &= b_\lambda S\partial\lambda\\ &= \left[\frac{\lambda(v_1)+\lambda(v_0)}{2}, \lambda(v_1)\right] - \left[\frac{\lambda(v_1)+\lambda(v_0)}{2}, \lambda(v_0)\right].\end{aligned}$$

- $S$ **is chain homotopic to** id**:** We define an operator $T : LC_*(Y) \longrightarrow LC_{*+1}(Y)$ inductively by $T = 0$ on $LC_{-1}(Y)$ and $T\lambda = b_\lambda(\lambda - T\partial\lambda)$.

  On $LC_{-1}$ we have $S = \mathrm{id}$ and $T = 0$. Then $T\partial + \partial T = 0 = 1 - S$ on $LC_{-1}$. Now using (19) and induction (from second to third line)

$$\begin{aligned}\partial T\lambda = \partial(b_\lambda(\lambda - T\partial\lambda)) &= (\lambda - T\partial\lambda) - b_\lambda(\partial\lambda - \partial T\partial\lambda)\\ &= (\lambda - T\partial\lambda) - b_\lambda(\partial\lambda - (\partial\lambda - S\partial\lambda - T\partial\partial\lambda))\\ &= (\lambda - T\partial\lambda) - b_\lambda(S\partial\lambda),\end{aligned}$$

  hence $\partial T\lambda + T\partial\lambda = \lambda - S\lambda$ (by definition of $S$).
- **Example:** Let $\lambda : \Delta^0 \longrightarrow Y$ be a 0-simplex. Then $T\lambda = (\Delta^1 \longrightarrow \lambda(\Delta^0)) - 0$. Thus $T\lambda$ maps $\Delta^1$ onto the point $\lambda(\Delta^0)$.

  For $n = 1$: Let $\lambda : \Delta^1 \longrightarrow Y$ be a $LC_1$-simplex. Then $b_\lambda$ is the mid point of $\lambda(v_0)$ and $\lambda(v_1)$ and

$$\begin{aligned}T\sigma = &\big(\Delta^2 \longrightarrow Y, w_0 \longmapsto b_\sigma, w_1 \longmapsto \sigma(v_0), v_2 \longmapsto \sigma(v_1)\big)\\ &+ \big(\Delta^2 \longrightarrow Y, \text{one side goes to } \sigma(v_0), \text{the opposite vertex to } b_\lambda\big)\\ &- \big(\Delta^2 \longrightarrow Y, \text{one side goes to } \sigma(v_1), \text{the opposite vertex to } b_\lambda\big)\end{aligned}$$

- From now on we forget about the crutch $LC_{-1}(Y) = \mathbb{Z}\langle[\emptyset]\rangle$ and return to standard definition of $C_*(X)$. This is ok because $T : LC_{-1}(Y) \longrightarrow LC_0(Y)$ is the zero map, i.e. it does not really care about its domain and the relation $\partial T + T\partial = \mathrm{id} - S$ is still valid.
- **Subdivision of general chains:** We view $\Delta^n$ as a chain in $C_*(\Delta^n)$, then $\sigma = \sigma_*\Delta^n$ for every singular $n$-simplex in $X$. Then define

$$\begin{aligned}S : C_*(X) &\longrightarrow C_*(X)\\ \sigma &\longmapsto \sigma_*(S\Delta^n).\end{aligned}$$

  Now extend linearly to general singular chains.
- $S$ **is a chain map:** We use that maps induced by continuous maps are chain maps, and $S\partial = \partial S$ on affine simplices in convex subsets of Euclidean space.

$$\begin{aligned}\partial S\sigma = \partial\sigma_* S\Delta^n &= \sigma_*(\partial S\Delta^n) = \sigma_* S\partial\Delta^n\\ &= \partial\sigma_* S\left(\sum_j (-1)^j \underbrace{\Delta_j^n}_{\text{boundary faces}}\right)\\ &= \sum_j (-1)^j \sigma_* S\Delta_j^n\\ &= \sum_j (-1)^j S\sigma|_{\Delta_j^n} = S\partial\sigma.\end{aligned} \tag{20}$$

- $S$ **is chain homotopic to the identity:** Define $T$ on $C_*(X)$ in analogy with $S$

$$\begin{aligned} T : C_*(X) &\longrightarrow C_{*+1}(X) \\ \sigma &\longmapsto \sigma_*(T\Delta^n). \end{aligned}$$

  Using the last lines of (20) for $T$ instead of $S$ we get

$$\begin{aligned} \partial T\sigma &= \partial \sigma_* T\Delta^n = \sigma_* \partial T\Delta^n \\ &= \sigma_*(\Delta^n - S\Delta^n - T\partial\Delta^n) \\ &= \sigma - S\sigma - \sigma_* T\partial\Delta^n \\ &= \sigma - S\sigma - T\partial\sigma. \end{aligned}$$

- **Iteration of barycentric subdivision:** $T$ is a chain homotopy from id to $S$. Then a direct computation shows that

(21)
$$D_m = \sum_{i=0}^{m-1} TS^i$$

  is a chain homotopy from id to $S^m$ (on $C_*(X)$). For $m = 0$, the sum is empty, so $D_0 = \mathrm{id}$.
- **Definition of** $\rho : C_*(X) \longrightarrow C_*^{\mathfrak{U}}(X)$**, first attempt:** Let $U_i, i \in I$ be the open sets in $\mathfrak{U}$ and $\sigma : \Delta^n \longrightarrow X$ a singular $n$-simplex in $X$. Then $\sigma^{-1}(U_i), i \in I$, is an open cover of $\Delta^n$ which has a Lebesgue number $\delta$. Then, by the geometric fact on p. 41, there is $m \in \mathrm{N}$ such that all non-trivial summands of $S^m\Delta^n$ have diameter $< \delta$. This means that they are mapped to one of the sets $U_i$.

  Let $m(\sigma)$ be the smallest $m$ such that each simplex appearing in $S^m\Delta^n$ is mapped to one of the sets $U_i$ by $\sigma$ ($i$ depends on the simplex). Then $\partial\rho(\sigma)$ is in $C_*^{\mathfrak{U}}(X)$ but the subdivision might not be optimal. Thus it is not clear that $\rho$ is a chain map.
- **Definition of** $\rho : C_*(X) \longrightarrow C_*^{\mathfrak{U}}(X)$**, second attempt:** Let $\sigma$ and $m(\sigma)$ be as in the first attempt and recall the definition (21). Set

$$\begin{aligned} D : C_*(X) &\longrightarrow C_{*+1}(X) \\ \sigma &\longmapsto D_{m(\sigma)}\sigma. \end{aligned}$$

  We have already stated

$$\partial \underbrace{D_{m(\sigma)\sigma}}_{=D\sigma} + \underbrace{D_{m(\sigma)}\partial\sigma}_{\neq D\sigma} = \sigma - S^{m(\sigma)}\sigma.$$

  Reorganizing the summands

(22)
$$\partial D\sigma + D\partial\sigma = \sigma - \underbrace{\left(S^{m(\sigma)} + D_{m(\sigma)}\partial\sigma - D(\partial\sigma)\right)}_{=:\rho(\sigma)}.$$

  Therefore, we define

(23)
$$\begin{aligned} \rho : C_*(X) &\longrightarrow C_*(X) \\ \sigma &\longmapsto S^{m(\sigma)}\sigma + D_{m(\sigma)}\partial\sigma - D\partial\sigma \end{aligned}$$

- $\rho$ **is a chain map:** This follows from (22) (apply $\partial$ to both sides and compare with what you get when you apply the relation to $\partial\sigma$).
- $\rho$ **is chain homotopic to** id**:** by (22).

- **$\rho(\sigma)$ lies in $C_*^{\mathfrak{U}}(X)$:** This is of course our goal! The first summand in the definition (23) of $\rho(\sigma)$ is obviously contained in $C_*^{\mathfrak{U}}(X)$.

  Notice that $m(\text{any simplex in } \partial(\sigma)) \leq m(\sigma)$. Therefore the difference $D_{m(\sigma)}\partial\sigma - D\partial\sigma$ consists of $T \circ S^i(\partial\sigma)$ for $i > m(\partial\sigma)$. All of the summands appearing here are singular simplices mapping into open sets of the cover $\mathfrak{U}$.
- We have shown that $\iota \circ \rho : C_*(X) \longrightarrow C_*(X)$ is chain homotopic to the identity. By the definition of $\rho$, it follows that if $\sigma \in C_*^{\mathfrak{U}}(X)$, then $m(\sigma) = 0$. Hence, $S^{m(\sigma)}\sigma = \sigma$ and $\rho \circ \iota = \mathrm{id}_{C_*^{\mathfrak{U}}(X)}$.
- **This concludes the proof of the open covering proposition cf. p 40**.

## 21. Lecture on Jan. 22 – Applications/Harvest

- **Excision, AB version:** Let $A, B \subset X$ be sets whose interiors cover $X$. Then $(B, A \cap B) \longrightarrow (X, A)$ induces an isomorphism

$$H_*(B, A \cap B) \longrightarrow H_*(X, A).$$

- **Proof:** Let $\mathfrak{U} = \{\mathring{A}, \mathring{B}\}$. The maps $\rho, D$ arising in the proof of the open cover Proposition both turn singular chains in $A$, resp. $A \cap B$, into singular chains in $A$, resp. $A \cap B$. So

$$\frac{C_*^{\mathfrak{U}}(X)}{C_*(A)} \longrightarrow \frac{C_*(X)}{C_*(A)}$$

  is well defined and it induces an isomorphism on homology. Moreover, the inclusion

$$\frac{C_*(B)}{C_*(A \cap B)} \longrightarrow \frac{C_*(X)}{C_*(A)}$$

  is an isomorphism because the right hand side is (like the left) generated by chains supported in $B$.
- **Excision, standard version:** The statement is on p.40. Apply the previous version to $B = X \setminus Z$ and the $A$ that is already around.
- **The two versions are equivalent:** To prove AB-version from standard version, apply standard version to $Z = X \setminus B$.
- **Definition:** A pair of spaces $(X, A)$ is *good* if $A \subset X$ is closed and there is a neighborhood of $A$ in $X$ such that $A$ is a deformation retract of the neighborhood.
- **Example:** Subcomplexes of CW-complexes, boundaries inside manifolds with boundary.
- **Theorem:** Let $(X, A)$ be a good pair. Then there is an exact sequence

$$\ldots \longrightarrow \widetilde{H}_i(A) \xrightarrow{i_*} \widetilde{H}_i(X) \xrightarrow{j_*} \widetilde{H}_i(X/A) \xrightarrow{\partial} \widetilde{H}_{i-1}(A) \longrightarrow \ldots \tag{24}$$

  in reduced homology. Here $j : X \longrightarrow X/A$ is the quotient map.
- **Proof:** Note that $H_n(A, \{a\}) \simeq \widetilde{H}_n(A)$ when $a \in A$. From the long exact sequence of $\{a\} \subset A \subset X$ (last remark before Jan. 18) one gets

$$\ldots \longrightarrow \widetilde{H}_i(A) \xrightarrow{i_*} \widetilde{H}_i(X) \xrightarrow{j_*} H_i(X, A) \xrightarrow{\partial} \widetilde{H}_{i-1}(A) \longrightarrow \ldots$$

  so the map $\partial$ in the above diagram is still the one obtained in the long exact sequence. We are done once we identify $H_n(X, A)$ with $\widetilde{H}_n(X/A)$.
- **Lemma:** For good pairs the quotient map $X \longrightarrow X/A$ induces an isomorphism $H_n(X, A) \longrightarrow H_n(X/A, [A]) \simeq \widetilde{H}_n(X/A)$.

- **Proof:** Let $V$ be the neighborhood of $A$ as in the assumptions.

$$\begin{tikzcd}
H_n(X,A) \arrow[r, "\simeq"] \arrow[d] & H_n(X,V) \arrow[d] & H_n(X\setminus A, V\setminus A) \arrow[l] \arrow[d] \\
H_n(X/A,[A]) \arrow[r] & H_n(X/A,V/A) & H_n((X/A)\setminus [A], (V/A)\setminus [A]) \arrow[l]
\end{tikzcd}$$

The vertical maps are all induced by quotient maps, horizontal maps are induced by inclusions. Since $A \subset V$ is a deformation retract the arrows from left to right are isomorphisms. (We proved homotopy invariance of homology for single spaces, the proof works in the same way for pairs. This can also be seen from the long exact sequence. Note that $H_*(X) = H_*(X,\emptyset)$.)

The arrows from right to left are isomorphisms by excision.

Finally, the quotient map inducing the vertical map on the right is a homeomorphism! It therefore induces an isomorphism between homologies. Therefore all vertical maps in the diagram are isomorphisms.
- **Example:** If $A = \{0, 1, 1/2, 1/3, \ldots\} \subset X = [0,1]$ one can show that $\widetilde{H}_1(X/A)$ and $H_1(X,A)$ are not isomorphic (the first group is much bigger). This is fine since the pair $(X,A)$ is not good.
- **Theorem (Naturality):** Let $f : (X,A) \longrightarrow (Y,B)$ be continuous. Then the diagram

$$\begin{tikzcd}
\ldots \arrow[r] & H_n(A) \arrow[r] \arrow[d, "f_*"] & H_n(X) \arrow[r] \arrow[d, "f_*"] & H_n(X,A) \arrow[r, "\partial_*"] \arrow[d, "f_*"] & H_{n-1}(A) \arrow[r] \arrow[d, "f_*"] & \ldots \\
\ldots \arrow[r] & H_n(B) \arrow[r] & H_n(Y) \arrow[r] & H_n(Y,B) \arrow[r, "\partial_*"] & H_{n-1}(B) \arrow[r] & \ldots
\end{tikzcd}$$

commutes. The rows are the long exact sequences of the pairs. When both pairs are good, an analogous statement holds for the sequence in (24).
- **Proof:** Diagram chase.
- **Fundamental Example:** Let $S^0 = \partial[-1,1] \subset \mathbb{R}$. Recall that the reduced homology of a point vanishes and

$$\widetilde{H}_i(S^0) = \begin{cases} \mathbb{Z} & i = 0 \\ 0 & i \neq 0. \end{cases}$$

Recall that $S^n$ is homeomorphic to $D^n/S^{n-1}$. From (24) we get, since $D^n$ is homotopy equivalent to a point,

$$\ldots \longrightarrow \widetilde{H}_i(S^{n-1}) \xrightarrow{i_*} \widetilde{H}_i(D^n) = 0 \xrightarrow{j_*} \widetilde{H}_i(S^n) \xrightarrow{\partial_*} \widetilde{H}_{i-1}(S^{n-1}) \longrightarrow \ldots$$

The map $\partial_* : H_i(S^n) \longrightarrow H_{i-1}(S^{n-1})$ is an isomorphism by exactness. By induction,

$$\widetilde{H}_i(S^n) = \begin{cases} \mathbb{Z} & i = n \\ 0 & i \neq n. \end{cases}$$

This implies for $n \neq 0$

$$H_i(S^n) = \begin{cases} \mathbb{Z} & i = n, 0 \\ 0 & \text{otherwise.} \end{cases}$$

- **Corollary:** $S^{n-1}$ is not a retract of $D^n$.

- **Proof:** By contradiction, if there is a retraction $f$, then $f \circ i$ is the identitiy of $S^{n-1}$ where $i : S^{n-1} \longrightarrow D^n$ is the inclusion. So $i_*$ is injective on $H_{n-1}(S^{n-1})$. But $D^n$ is contractible, .
- **Brouwer fixed point theorem:** Every continuous map $f : D^n \longrightarrow D^n$ has a fixed point.
- **Proof:** As in the case $n = 2$ on p. 10.
- **Theorem:** Let $U \subset \mathbb{R}^m, V \subset \mathbb{R}^n$ be nonempty open sets which are homeomorphic. Then $m = n$.
- **Proof:** For all $u \in U$ we have $H_*(U, U \setminus x) = H_*(\mathbb{R}^m, \mathbb{R}^m \setminus x) = H_*(D^m, \partial D^m)$ by excision. Similarly for $V$. If there is a homeomorphism from $U$ to $V$, then these groups have to be isomorphic.

## 22. Lecture on Jan. 25 – Generators of $H_n(S^n)$, degrees of maps between spheres of the same dimension

- **Reference:** The following is essentially from [Ha], with a certain number of modifications.
- **Explicit generators:** Recall that $(D^n, \partial D^n)$ and $(\Delta^n, \partial\Delta^n)$ are homeomorphic. Then $i_n : \Delta^n \longrightarrow (\Delta^n, \partial\Delta^n)$ represents a generator of $H_n(\Delta^n, \partial\Delta^n) \simeq \mathbb{Z}$.
  - $\Delta^n$ is a cycle in $C_n(\Delta^n, \partial\Delta^n)$.
  - For $n = 0$, $D^n, \Delta^n$ are points and $\Delta^0$ is a generator by the computation of the homology of a point on p. 35.
  - Recall that $\Delta_0^n$ denotes the face in $\Delta^n$ opposite to $v_0$. Let $\Lambda = (\partial\Delta^n)\setminus\mathring{\Delta}_0^n$, note that $\Lambda$ is a deformation retract of $\Delta^n$, $\Lambda$ is homeomorphic to $\Delta_0^n \simeq \Delta^{n-1}$ and hence contractible. Apply the long exact sequence of the triple $(\Delta^n, \partial\Delta^n, \Lambda)$ to see that $\partial_*$ is an isomorphism and excision + homotopy invariance to see that the second arrow below is an isomorphism.

    $$H_n(\Delta^n, \partial\Delta^n) \xrightarrow{\partial_*} H_{n-1}(\partial\Delta_n, \Lambda) \longleftarrow H_{n-1}(\Delta_0^n, \partial\Delta_0^n)$$

    Then $\partial_* i_n$ and the image of $i_{n-1}$ under the inclusion into $(\partial\Delta_n, \Lambda)$ both map to generators, i.e. they coincide up to sign.

  The long exact sequence of $(\Delta^n, \partial\Delta^n \simeq S^{n-1})$ shows that $\partial_*([i_n])$ is a generator of $H_{n-1}(S^{n-1})$.
- **Slight improvement of the lecture:** There is another generator of $H_n(\partial\Delta^{n+1})$ which is useful. Note that $\partial\Delta^{n+1} = \Lambda \cup \Delta_0^{n+1}$. Both $\Lambda, \Delta_0^{n+1}$ are homeomorphic to $n$-simplices (which are identified by a projection orthogonal to $\Delta_0^{n+1}$). This suggests that the difference of $i_0 : \Delta^n \longrightarrow \Delta_0^{n+1}$ (preserving the order of vertices) and $i_\Lambda : \Delta^n \longrightarrow \Lambda \subset \partial\Delta^{n+1}$ (so that $v_j \in \Delta^n$ is mapped to $i_0(v_j)$) is a generator.
  - $i_0 - i_\Lambda$ is a cycle for all $n$.
  - This is true for $n = 0$.
  - If $n > 0$, then look at: $i_0 : \Delta^{n+1} \longrightarrow \Delta_0^{n+1}$ generates $H_n(\Delta_0^{n+1}, \partial\Delta_0^{n+1})$. By excision, the first arrow in the following diagram is an isomorphism. The second arrow is also induced by an inclusion, it is part of the long exact sequence of the pair $(\partial\Delta^{n+1}, \Lambda)$ and it is an isomorphism because

$n > 0$ and $\Lambda$ is contractible:

$$\begin{array}{ccccc} H_n(\Delta_0^{n+1}, \partial\Delta_0^{n+1}) & \longrightarrow & H_n(\partial\Delta^{n+1}, \Lambda) & \longleftarrow & H_n(\partial\Delta^{n+1}) \\ [i_0] & \longmapsto & [i_0] & \longleftarrow\!\shortmid & [i_0] - [i_\Lambda] \end{array}$$

The preimage of $i_0 - i_\Lambda$ under the composition of these isomorphisms is $i_0$, i.e. a generator. Therefore $i_0 - i_\Lambda$ is a generator of $\widetilde{H}_n(\partial\Delta^{n+1} \simeq S^n)$.

- **Definition:** Let $f : S^n \longrightarrow S^n$ be continuous. Then $f_* : H_n(S^n) \simeq \mathbb{Z} \longrightarrow H_n(S^n)$ is of the form $f_*([\sigma]) = d[\sigma]$ for a fixed $d \in \mathbb{Z}$. This is the *degree* of $f$, i.e. $\deg(f) = d$.
- **Warning:** In this definition, domain and image are the *same* sphere. If we consider maps between different spheres, we will need to identify the spheres before we can talk about the degree of such maps.
- **Properties:** $\deg(id) = 1$, homotopy invariance, multiplicative (i.e. $\deg(f \circ g) = \deg(f)\deg(g)$). In particular, self-homeomorphisms and self homotopy equivalences of $S^n$ have degree $\pm 1$.
- **More properties:** If $f : S^n \longrightarrow S^n$ is a reflection along a hyperplane in $\mathbb{R}^{n+1}$, then $\deg(f) = -1$. For this use the generator $i_0 - i_\Lambda$ such that $\Delta_0^n$ and $\Lambda$ cover opposite half spheres intersecting along the equator corresponding to the hyperplane in a symmetric fashion. Then $f_*(i_0 - i_\Lambda) = (i_\Lambda - i_0)$. Moreover, $\deg(-id) = (-1)^{n+1}$ since $-\mathrm{id}$ is the composition of $n+1$ reflections.
- **One more property:** For $n = 1$ the degree defined above coincides with the degree we already know (look at the standard generator of $H_1(S^1)$).
- **Lemma:** If $f : S^n \longrightarrow S^n$ has no fixed point, then $\deg(f) = \deg(-id)$.
- **Proof:** If $f$ has no fixed point, then there is a homotopy from $f$ to $-id$ (along unique minimal geodesics from $f(x)$ to $-x$).
- **Proposition:** Let $G$ be a group that acts on $S^n$ for $n = 2k$ such that $g(x) = x$ implies $g = 1_G$. Then $G = \mathbb{Z}_2$ (or $\{1\}$). $S^{2k}$ arises only as covering of $\mathbb{RP}^{2k}$.
- **Proof:** The degree defines a homomorphism $G \longrightarrow \{\pm 1\}$. Since all elements except the identity have no fixed points, the kernel of this map is trivial ($\deg(g) = \deg(-\mathrm{id}) = (-1)^{2n+1} = -1$ for $g \neq 1_G$).
- **Theorem:** Every vector field on a sphere of even dimension has a zero.
- **Proof:** A nowhere vanishing vector field on $S^n$ induces a homotopy from $id$ to $-id$. But in even dimension $\deg(id) = 1 \neq -1 = \deg(-id)$.
- **Example:** On $S^{2n-1} \subset \mathbb{R}^{2n}$ the vectorfield
$$X(x_1, y_1, \ldots, x_n, y_n) = (-y_1, x_1, \ldots, -y_n, x_n)$$
never vanishes.
- **Current goal:** Compute the homology of a CW-complex from CW structure. So let $X$ be a connected CW-complex, given in terms of a cell decomposition, attation maps.
- **Lemma:** $H_k(X^n, X^{n-1}) = 0$ if $k \neq n$. $H_n(X^n, X^{n-1})$ is free Abelian with one generator for each $n$-cell.
- **Proof:** $(X^n, X^{n-1})$ is a good pair and $X^n/X^{n-1}$ is a one point union of $n$-spheres, with one sphere per $n$-cell in $X^n$.
- **Lemma:** $H_k(X^n) = 0$ for $k > n \geq 0$.
- **Proof:** The long exact sequence of the pair $(X^n, X^{n-1})$ contains the piece

$$H_{k+1}(X^n, X^{n-1}) \longrightarrow H_k(X^{n-1}) \longrightarrow H_k(X^n) \longrightarrow H_k(X^n, X^{n-1}). \tag{25}$$

If $k > n$, the inclusion induced map $H_k(X^{n-1}) \longrightarrow H_k(X^n)$ is an isomorphism. The claim follows by induction and $H_k(X^0) = 0$ for $k > 0$.

## 23. Lecture on Jan. 29 – Cellular chain complex, cellular homology

- **Lemma:** The inclusion $X^n \longrightarrow X$ induces an isomorphism on $H_k$ if $k < n$.
- **Proof:** Let $[\sigma] \in H_k(X)$. Then the image of $\sigma$ is contained in a finite subcomplex, and hence $X^m$ for big enough $m$. By the previous two Lemmas (see (25)), $\sigma$ is homologous to a chain in $X^{m-1}$ when $m > k$. Thus there is a chain $\sigma_{m-1}$ which is homologous to $\sigma$ and whose image is contained in $X^{m-1}$. Iterating this procedure, one ends up with $\sigma_n \sim \sigma$ such that $\sigma_n \in C_k(X^n)$. Hence $i : X^n \longrightarrow X$ is surjective on $H_k$ when $n > k$.

  Assume a $k$-cycle $\sigma$ in $X^n$ bounds a $k+1$-chain $\tau$. This chain is contained in some $C_{k+1}(X^m), m \geq n$ and it represents a class in $H_{k+1}(X^m, X^n)$ which maps to $\sigma$ under $\partial_*$. Because of the naturality of the long exact sequences, the diagram

$$\begin{tikzcd}
H_{k+1}(X^{m-1}, X^n) \arrow[r, "\partial_*"] \arrow[d] & H_k(X^n) \arrow[d, "="] \\
H_{k+1}(X^m, X^n) \arrow[r, "\partial_*"] \arrow[d] & H_k(X^n) \\
H_{k+1}(X^m, X^{m-1}) &
\end{tikzcd}$$

  commutes. The vertical part on the left is part of the long exact sequence of the triple $(X^m, X^{m-1}, X^n)$. The map $H_{k+1}(X^{m-1}, X^n) \longrightarrow H_{k+1}(X^m, X^n)$ is surjective when $k+1 \neq m$. Thus one can reduce the dimension needed to carry a chain with the same properties as $\tau$ when $k+1 > m \geq n$. This shows that $i_* : H_k(X^n) \longrightarrow H_k(X)$ is injective.
- **The cellular chain complex:** Look at the diagram

(26)
$$\begin{tikzcd}
H_n(X^{n+1}, X^n) = 0 & & \\
H_n(X^{n+1}) = H_n(X) \arrow[u] & & H_{n-1}(X^{n-1}, X^{n-2}) \\
H_n(X^n) \arrow[u] \arrow[r, "j_n"] & H_n(X^n, X^{n-1}) \arrow[r, "\partial_*"] \arrow[ur, "d_n"] & H_{n-1}(X^{n-1}) \arrow[u, "j_{n-1}"'] \\
H_{n+1}(X^{n+1}, X^n) \arrow[u, "\partial_*"] \arrow[ur, "d_{n+1}"'] & & H_{n-1}(X^{n-2}) = 0 \arrow[u]
\end{tikzcd}$$

  The row is part of the long exact sequence of $(X^n, X^{n-1})$, the left column is part of the long exact sequence of $(X^{n+1}, X^n)$, the right column corresponds to $(X^{n-1}, X^{n-2})$. The arrows $d_{n+1}$ and $d_n$ are the compositions of other arrows. Because the horizontal sequence is exact, $d_n \circ d_{n+1} = 0$.

  Therefore $(H_n(X^n, X^{n-1}), d_n)$ is a chain complex. This is one incarnation of the *cellular chain complex* of $X$.
- **Notation:** The homology of this chain complex is $H_*^{CW}(X)$, this is called *cellular homology*.

- **Theorem:** $H_*^{CW}(X) \simeq H_*(X)$.
- **Proof:** We use three previous Lemmas and the diagram (26).
  - From the left vertical row: $H_n(X) \simeq H_n(X^n)/\mathrm{image}(\partial_* : H_{n+1}(X^{n+1}, X^n) \longrightarrow H_n(X^n))$.
  - Since $H_n(X^{n-1}) = 0$, the map $j_n$ is injective, i.e. image$(\partial_*)$ is mapped isomorphically of image$(j_n \circ \partial_*) = \mathrm{image}(d_{n+1})$.
  - $j_{n-1}$ is injective. Hence $\ker(\mathrm{d_n}) = \ker(\partial_*)$.

  Now one can read of $H_n(X) \simeq \frac{ker(d_n)}{\mathrm{image}(d_{n+1})}$
- **Corollary:** The homology of a finite $CW$-complex is finitely generated, if there are no $n$-cells, then $H_n(X) = 0$. If there are $k$ $n$-cells, then $H_n(X)$ can be generated by at most $k$ elements.
- **Corollary:** Assume $X^{2n+1} \simeq X^{2n}$ for all $n$. $H_{2n+1}(X) = 0$ and $H_{2n}(X)$ is free Abelian, the rank of $H_{2n}(X)$ coincides with the number of $2n$-cells.
- **Example:** This Corollary is tailored to suit the following example. There are other instances where it works.

$$H_k(\mathbb{CP}^n) \simeq \begin{cases} 0 & k \text{ odd or } k > 2n \text{ or } k < 0 \\ \mathbb{Z} & 0 \leq k \leq 2n \text{ even} \end{cases}$$

$$H_k(\mathbb{HP}^n) \simeq \begin{cases} 0 & k > 4n \text{ or } k < 0 \\ 0 & k \text{ not a multiple of } 4 \\ \mathbb{Z} & 0 \leq k \leq 4n \text{ divisible by } 4 \end{cases}$$

- **Description of $H_n(X^n, X^{n-1})$:** This is isomorphic to a Abelian group generated by the $n$-cells in $X$.
- **Description of $d_n$:** Let $e_\alpha^n$ be a $n$-cell, these cells represent a basis of $H_n(X^n, X^{n-1})$. In terms of this basis

$$d_n(e_\alpha^n) = \sum_{e_\beta^{n-1}\ (n-1)\text{-cell}} \deg\left(\varphi_\alpha : \partial D_\alpha^n \longrightarrow X^{n-1}/(X^{n-1} \setminus e_\beta^{n-1})\right) e_\beta^{n-1}. \tag{27}$$

  where $\varphi_\alpha$ is the attaching map of $e_\alpha^n$.
- **Proof:** $n = 0$: There are no $-1$-cells, so $d_0$ is zero.

  $n = 1$: Each one cell $e_\alpha^1$ corresponds to a generator of $H_1(X^1, X^0)$, it is mapped the difference of the end points.

  $n > 1$: Again, one looks at a diagram. $\phi_\alpha$ is the characteristic map of the cell $e_\alpha^n$ and $\varphi_\alpha$ its restriction to the boundary of $D_\alpha^n$. $q_\beta : X^{n-1}/X^{n-2} \longrightarrow S_\beta^{n-1}$ collapes all points in the complement of $e_\beta^{n-1}$ to a point and identifies the result with $D_\beta^{n-1}/\partial D_\beta^{n-1} = S_\beta^{n-1}$ via the (inverse of the) characteristic map of $e_\beta^{n-1}$. We use $n \geq 2$ for the bottom right isomorphism (the pair $(X^{n-1}, X^{n-2})$ is good).

$$\begin{tikzcd}
H_n(D_\alpha^n, \partial D_\alpha^n) \arrow[r, "\partial_*", "\simeq"'] \arrow[d, "\phi_\alpha"] & H_{n-1}(\partial D_\alpha^n) \arrow[r, "\Delta_{\alpha\beta}"] \arrow[d, "\varphi_\alpha"] & \widetilde{H}_{n-1}(S_\beta^{n-1}) \\
H_n(X^n, X^{n-1}) \arrow[r, "\partial_*"] \arrow[dr, "d_n"'] & H_{n-1}(X^{n-1}) \arrow[d, "j_{n-1}"] & \widetilde{H}_{n-1}(X^{n-1}/X^{n-2}) \arrow[u, "q_{\beta*}"'] \\
 & H_{n-1}(X^{n-1}, X^{n-2}) \arrow[ur, "\simeq"'] &
\end{tikzcd}$$

$\Delta_{\alpha\beta}$ is the map that makes everything commute. Since $n-1>0$, the reduced homologies in the right column coincide with the non-reduced versions. By definition of the degree,

$$\Delta_{\alpha\beta}[\partial D^n_\alpha] = \deg\left(\varphi_\alpha : \partial D^n_\alpha \longrightarrow X^{n-1}/(X^{n-1}\setminus e^{n-1}_\beta)\right)[S^{n-1}_\beta]$$

- **Remark:** The above two descriptions would allows us to define the cellular chain complex without reference to singular homology. The cellular chain complex is very amenable to computations. The non-trivial lesson from the above discussion is that homeomorphic CW-complexes have isomorphic cellular homology.
- **Computation of degrees from local data:** In order to use the cellular chain complex, one needs to be able to compute the degree effectively. Here is a method that works under very mild topological assumptions.

  Assume that $f : S^n \longrightarrow S^n, n \geq 2$, is continuous and $y \in S^n$ has finite $f^{-1}(y) = \{x_1, \dots, x_m\}$. Fix a neighborhood $V$ of $y$ and pairwise disjoint neighborhoods $U_i$ of $x_i$ such that $f(U_i) \subset V$ .

  There are obvious inclusions: $(U_i, U_i \setminus x_i) \longrightarrow (S^n, S^n \setminus x_i)$ induces an iso. on homology by excision. The same is true for $(V, V\setminus y) \longrightarrow (S^n, S^n \setminus y)$. Then

$$\begin{array}{ccc} H_n(U_i, U_i \setminus x_i) & \xrightarrow{f_*} & H_n(V, V\setminus y) \\ \downarrow \simeq & & \downarrow \simeq \\ H_n(S^n, S^n\setminus x_i) & \xrightarrow{f_*} & H_n(S^n, S^n \setminus y) \end{array}$$

The lower map is multiplication with an integer, the *local degree* of $f$ at $x_i$. This means that

$$f_*([U_i, U_i \setminus x_i]) = \deg(f, x_i)[V, V\setminus y])$$

where $[U_i, U_i\setminus x_i]$ respectively $[V, V\setminus y]$ is a generator of $H_n(U_i, U_i\setminus x_i)$ respectively $H_n(V, V\setminus y)$. These generators (there is a choice which would affect the sign of the local degree) are related via

$$H_n(S^n) \xrightarrow{j_{i*}} H_n(S^n, S^n\setminus x_i) \xleftarrow{\simeq} H_n(U_i, U_i \setminus x_i)$$
$$H_n(S^n) \xrightarrow{j_{y*}} H_n(S^n, S^n\setminus y) \xleftarrow{\simeq} H_n(V, V \setminus y)$$

where the arrows from right to left are excision isomorphisms (induced by inclusions), $j_i : S^n \longrightarrow (S^n, S^n \setminus x_i)$, and $i_y : S^n \longrightarrow (S^n, S^n\setminus y)$. For fixed $i$ let

$$k_i : (U_i, U_i \setminus x_i) \longrightarrow (S^n, S^n \setminus x_i)$$

be the inclusion. Then, by excision and disjoint union

$$H_n\left(\bigcup_i U_i, \bigcup_i (U_i \setminus x_i)\right) \simeq \bigoplus_i H_n(U_i, U_i\setminus x_i) \xrightarrow{\oplus_i k_{i*}} H_n(S^n, S^n \setminus f^{-1}(y))$$

is an isomorphism. Because of the long exact sequence of the pair $(S^n, S^n\setminus y)$, the inclusion $(S^n, \emptyset) \longrightarrow (S^n, S^n \setminus y)$ induces an isomorphism $H_n(S^n) \longrightarrow$

$H_n(S^n, S^n \setminus y)$. Consider the diagram

$$\begin{tikzcd}
\bigoplus_i H_n(U_i, U_i \setminus x_i) \arrow[r, "\text{induced by } f"] \arrow[d, "\oplus k_{i*}", "\simeq"'] & H_n(V, V-y) \arrow[d, "\simeq"] \\
H_n(S^n, S^n \setminus f^{-1}(y)) \arrow[r, "f_*"] & H_n(S^n, S^n \setminus y) \\
H_n(S^n) \arrow[r, "f_*"] \arrow[u, "j"] & H_n(S^n) \arrow[u, "\simeq"']
\end{tikzcd} \tag{28}$$

The lowest line is multiplication by the degree, each summand in the top line is multiplied with the local degree. In order to finish, we need to express $j([S^n])$ in terms of the direct sum decomposition of $H_n\left(\bigcup_i U_i, \bigcup_i (U_i \setminus x_i)\right)$.

Let $i \in \{1, \ldots, m\}$ be fixed. By the long exact sequence of the pair $(S^n, S^n \setminus x_i)$ the inclusion induces an isomorphism $j_i : H_n(S^n) \longrightarrow H_n(S^n, S^n \setminus x_i)$. The inclusion $(S^n, S^n \setminus f^{-1}(y)) \longrightarrow (S^n, S^n \setminus x_i)$ fits into

$$\begin{tikzcd}
 & (\cup_i U_i, \cup_i (U_i \setminus x_i)) \arrow[dl] \arrow[d] \\
(S^n, S^n \setminus x_i) & (S^n, S^n \setminus f^{-1}(y)) \arrow[l] \\
 & S^n \arrow[u] \arrow[ul]
\end{tikzcd}$$

On homology, the top left arrow induces a projection onto the $i$-th summand. Thus the image of a fixed generator $[S^n] \in H_n(S^n)$ is

$$\sum_i j_{i*}([S^n]).$$

We can now compute the degree in terms of local degrees as follows ($f_*$ refers to maps induced by $f$, these may differ from line to line, $i_y$ is the inclusion $S^n \longrightarrow (S^n, S^n \setminus y)$). From (28) we get

$$\begin{aligned}
i_{y*} f_*([S^n]) &= \deg(f) i_{y*}([S^n]) \\
&= f_*(\sum_i j_{i*}[S^n]) = \sum_i \deg(f, x_i) i_{y*}[S^n]) \\
&= \left( \sum_i \deg(f, x_i) \right) i_{y*}([S^n]).
\end{aligned}$$

Therefore

$$\deg(f) = \sum_i \deg(f, x_i) \tag{29}$$

- **Example:** We can finally compute $H_*(\mathbb{RP}^n)$. Recall that $X = \mathbb{RP}^n$ has a CW-structure such that
  - $X^k = \mathbb{RP}^k = \{[x_0 : \ldots : x_k : 0 \ldots : 0]\} \subset \mathbb{RP}^n$ for $0 \leq k \leq n$,
  - $X^{k+1} = X^k \cup \overline{e}^{k+1}$ when $0 \leq k+1 \leq n$,
  - the attaching map $\varphi_k : \partial \overline{D}^{k+1} = S^k \longrightarrow X^k = \mathbb{RP}^k$ is the two fold covering (universal for $k > 1$).

– In particular, $\varphi_k = \varphi_k \circ (-\mathrm{id})$, since $-\mathrm{id}$ is a deck transformation of the covering map.

We now compute the degree of the map

$$\varphi_{k+1} : \partial D^{k+1} = S^k \longrightarrow X^k / X^{k-1} = \overline{e}^k / \dot{e}^k.$$

$\overline{e}_k$ was already identified with $\overline{D}^k$ (using the characteristic map) as indicated in Figure 23.

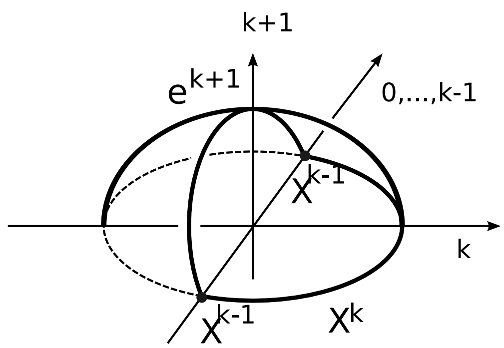


FIGURE 7. $X^{k+1}$ from $X^k \supset X^{k-1}$

We use the point $y = [0 : \ldots : 1] \in X^k$ and apply (29)

$$d_k(e^{k+1}) = (1 + (-1)^{k+1})e^k.$$

It turns out that $H_*(\mathbb{RP}^n)$ depends on the parity of $n$. If $n = 2m$ is even, the cellular chain complex is

$$0 \longrightarrow C_{2m} \simeq \mathbb{Z} \xrightarrow{\cdot 2} C_{2m-1} \simeq \mathbb{Z} \xrightarrow{\cdot 0} \ldots \xrightarrow{\cdot 0} C_2 \simeq \mathbb{Z} \xrightarrow{\cdot 2} C_1 \simeq \mathbb{Z} \xrightarrow{\cdot 0} C_0 = \mathbb{Z} \longrightarrow 0$$

The homology is therefore

$$H_k(\mathbb{RP}^{2m}) = \begin{cases} 0 & k \notin \{0, \ldots, 2m\} \\ \mathbb{Z} & k = 0 \\ 0 & k \text{ even} \\ \mathbb{Z}_2 & k \in \{1, 3, \ldots, 2m-1\}. \end{cases}$$

If $n = 2m+1$ is odd, the cellular chain complex of $\mathbb{RP}^n$ is

$$0 \longrightarrow C_{2m+1} \simeq \mathbb{Z} \xrightarrow{\cdot 0} C_{2m} \simeq \mathbb{Z} \xrightarrow{\cdot 2} C_{2m-1} \xrightarrow{\cdot 0} \ldots \xrightarrow{\cdot 0} C_2 \simeq \mathbb{Z} \xrightarrow{\cdot 2} C_1 \simeq \mathbb{Z} \xrightarrow{\cdot 0} C_0 = \mathbb{Z} \longrightarrow 0$$

The homology is therefore

$$H_k(\mathbb{RP}^{2m+1}) = \begin{cases} 0 & k \notin \{0, \ldots, 2m+1\} \\ \mathbb{Z} & k = 0 \text{ or } k = 2m+1 \\ 0 & k \text{ even} \\ \mathbb{Z}_2 & k \in \{1, 3, \ldots, 2m-1\}. \end{cases}$$

Finally, the homology of $\mathbb{RP}^\infty$ is

$$H_k(\mathbb{RP}^\infty) = \begin{cases} 0 & k < 0 \\ \mathbb{Z} & k = 0 \\ 0 & k \text{ even} \\ \mathbb{Z}_2 & k > 0 \text{ odd}. \end{cases}$$

- **Definition:** Let $X$ be a topological space. The *k-th Betti number* of $X$ is
$$b_k(X) = \mathrm{rank}(H_k(X)) \in \mathbb{N}_0 \cup \{\infty\}.$$
The rank of an Abelian group is the number of generators of a maximal free Abelian subgroup.
- **Definition:** Let $X$ be a finite (or, equivalently, compact) CW-complex with $c_i$ cells in dimension $i$. The *Euler characteristic of* $X$ is
$$\chi(X) = \sum_i (-1)^i c_i.$$
- **Theorem:** $\chi(X) = \sum_i (-1)^i b_i(X)$. In particular, $\chi(X)$ is a topological invariant and does not depend on the cell structure.
- **Proof:** Consider the cellular chain complex of $X$:
$$0 \longrightarrow C_n^{CW} \xrightarrow{d_n} C_{n-1}^{CW} \xrightarrow{d_{n-1}} \ldots \longrightarrow C_1^{CW} \xrightarrow{d_1} C_0^{CW} \longrightarrow 0.$$
For all $k$ there are short exact sequences
$$0 \longrightarrow B_k = \{k-\text{boundaries}\} \longrightarrow Z_k = \{k-\text{cycles}\} \longrightarrow H_k \longrightarrow 0$$
$$0 \longrightarrow Z_k \longrightarrow C_k \longrightarrow B_{k-1} \longrightarrow 0.$$
The second sequence implies $\mathrm{rank}(C_k) = \mathrm{rank}(Z_k) + \mathrm{rank}(B_{k-1})$. The first implies $b_k(X) = \mathrm{rank}(Z_k) - \mathrm{rank} B_k$. Therefore,
$$\begin{aligned}\chi(X) &= \sum_i (-1)^i c_i = \sum_i (-1)^i (\mathrm{rank}(Z_i) + \mathrm{rank}(B_{i-1})) \\ &= \sum_i (-1)^i (b_i(X) + \mathrm{rank}(B_i) + \mathrm{rank}(B_{i-1})) \\ &= \sum_i (-1)^i b_i(X).\end{aligned}$$
- **Remark:** Let $\widehat{X} \longrightarrow X$ be a $n$-sheeted cover of a compact CW-complex. Then $\widehat{X}$ has a CW-structure such that the restriction of the covering map to each cell of $\widehat{X}$ is a homeoomorphism onto a cell of $X$. Hence $\chi(\widehat{X}) = n\chi(X)$. There is no nice relationship between homology groups of a space and its covers.
- **Theorem (Mayer-Vietoris):** Let $A, B \subset X$ be open sets such that $A \cup B = X$. Then there is a long exact sequence
$$\begin{aligned}\ldots \longrightarrow H_i(A\cap B) \longrightarrow H_i(A) \oplus H_i(B) \longrightarrow H_i(X) &\xrightarrow{\partial_*} H_{i-1}(A\cap B) \longrightarrow \ldots \\ [\sigma] \longmapsto (i_{A\cap B\to A*}[\sigma], -i_{A\cap B\to B*}[\sigma]) & \\ ([\alpha],[\beta]) \longmapsto [i_{A*}\alpha + i_{B*}\beta] &\end{aligned} \tag{30}$$
- **Proof:** Consider the chain complex $C_*^{\mathfrak{U}}(X)$ for $\mathfrak{U} = \{A, B\}$. There is a short exact sequence
$$0 \longrightarrow C_n(A\cap B) \longrightarrow C_n(A) \oplus C_n(B) \longrightarrow C_n^{\mathfrak{U}}(X) \longrightarrow 0$$
for all $n$ and the maps involved are chain maps. The corresponding long exact sequence is of the form in 30, the only non-elementary fact used is that the inclusion $(C_*^{\mathfrak{U}}(X), \partial) \longrightarrow (C_*(X), \partial)$ is a chain homotopy equivalence (open cover on 40) . It induces an isomorphism on homology.
- **Remark:** This is frequently applied to sets $A, B$ covering $X$ which are not necessarily open when there are open sets $A' \supset A$ and $B' \supset B$ such the smaller set is a deformation retract of the larger one.

- **Remark:** A similar sequence exists in reduced homology, one uses

$$\begin{tikzcd}
0 \arrow[r] & C_0(A \cap B) \arrow[r] \arrow[d, "\varepsilon"] & C_0(A) \oplus C_0(B) \arrow[r] \arrow[d, "\varepsilon\oplus\varepsilon"] & C_*^{\mathfrak{U}}(X) \arrow[r] \arrow[d, "\varepsilon"] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r] & \mathbb{Z} \oplus \mathbb{Z} \arrow[r] & \mathbb{Z} \arrow[r] & 0 \\
 & n \arrow[r, mapsto] & (n, -n) & & \\
 & & (a,b) \arrow[r, mapsto] & (a+b) &
\end{tikzcd}$$

  to augment the chain complex.
- **Theorem (Jordan non-separation):** Let $h : \overline{D}^k \longrightarrow S^n$ be an embedding and $k \leq n$. Then
$$\widetilde{H}_i(S^n \setminus h(D^k)) = 0 \text{ for all } i$$
- **Proof:** By induction on $k$. The base case $k = 0$ is obvious. Assume that the statement is proved for $k-1$ and let $D^k = I^k = D^{k-1} \times I$. We write
$$A = S^n \setminus h(D^{k-1} \times [0, 1/2]) \qquad B = S^n \setminus h(D^{k-1} \times [1/2, 1]).$$
  Both sets are open. Then $A \cup B = S^n \setminus h(D^{k-1} \times \{1/2\})$ and $A \cap B = S^n \setminus h(D^k)$. By the Mayer-Vietoris sequence and the inductive hypothesis $\widetilde{H}_*(D^{k-1} \times \{1/2\}) = 0$
$$\widetilde{H}_i(A \cap B) \longrightarrow \widetilde{H}_i(A) \oplus \widetilde{H}_i(B)$$
  is an isomorphism. If there is $[\sigma] \neq 0 \in \widetilde{H}_i(A \cap B)$, then $\sigma \neq 0$ in $\widetilde{H}_i(A)$ or $\widetilde{H}_i(B)$ (up to sign the maps in the Mayer-Vietoris sequence are induced by inclusions).

  Assume $\sigma \neq 0$ in $\widetilde{H}_i(A = S^n \setminus h(D^{k-1} \times [0, 1/2])$. Iterating this process we find intervals $[a_i, b_i] \subset I$ such that $b_i - a_i = 2^{-i}$, $[a_i, b_i] \subset [a_{i-1}, b_{i-1}]$ such that $\sigma \neq 0 \in \widetilde{H}_i(S^n \setminus h(D^{k-1} \times [a_i, b_i])$ for all $i$.

  By induction, $\sigma = 0$ in $\widetilde{H}_i(S^n \setminus h(D^{k-1} \times a_\infty))$ with $a_\infty = \cap_i [a_i, b_i]$. Then there is a $i+1$-chain $\tau$ in $S^n \setminus h(D^{k-1} \times a_\infty)$ such that $\partial\tau = \sigma$. The space $h(D^{k-1} \times a_\infty)$ is compact, therefore $\tau$ is a chain in the complement of an open neighborhood of $h(D^{k-1} \times a_\infty)$. This complement is contained in the complement of one of the sets $h(D^{k-1} \times [a_i, b_i])$ where $\sigma$ is not zero. This is a contradiction.
- **Remark:** The complement of an embedded closed ball in $S^n$ has the homology of a ball. This does not mean that it is homeomorphic/homotopy equivalent to a ball. For $n = 2$ this is true, in higher dimension it is not.
- **Theorem:** Let $h : S^k \longrightarrow S^n$ be an embedding. Then
$$\widetilde{H}_i(S^n \setminus h(S^k)) = \begin{cases} \mathbb{Z} & i = n-k-1 \\ 0 & \text{otherwise.} \end{cases}$$
- **Proof:** By induction. For $k = 0$, $S^n \setminus h(S^0)$ is homeomorphic to $S^{n-1} \times \mathbb{R}$. The theorem is true in this case.

  Assume that $k > 0$ and let $S^k = D_+ \cup D_-$ where $D_\pm 0 = \{\pm x_0 > 0\}$ are discs. Then by the Mayer-Vietoris sequence applied to $A = S^n \setminus h(D_+)$ and

$B = S^n \setminus h(D_-)$

$$0 = \widetilde{H}_i(A) \oplus \widetilde{H}_i(B) \longrightarrow \widetilde{H}_i(A \cup B) \longrightarrow \widetilde{H}_{i-1}(A \cap B) \longrightarrow 0$$

as long as $i - 1 \geq 0$. The claim follows by induction.

- **Remark:** This implies the generalized Jordan curve theorem: The complement of codimension-1-sphere in $S^n$ consists of two open homology balls. The complement has two path connected components.
- **Theorem (Invariance of domain):** Let $U \subset \mathbb{R}^n$ be an open set and $f : U \longrightarrow \mathbb{R}^n$ continuous and injective (for example a homeomorphism). Then $h(U) \subset \mathbb{R}^n$ is open.
- **Proof:** $S^n \setminus \{pt\}$ is homeomorphic to $\mathbb{R}^n$. We will show that $h(U)$ is open in $S^n$. Let $x \in U$ and $D^n \subset U$ a close ball containing $x$ in its interior. Then $\partial D^n$ is a sphere and $h(\partial D^n)$ decomposes $D^n$ in two path components, one of them is $h(D^n \setminus \partial D^n)$ and homeomorphic to an open ball, the other is $S^n \setminus h(D^n)$. Their union is open since $h(\partial D^n)$ is compact.

  We show that $h(D^n \setminus \partial D^n)$ is open. Let $y \in h(D^n \setminus \partial D^n)$. There is an open ball $V$ around $y$ in $S^n \setminus h(\partial D^n)$. Then $V$ lies in the same path connected component of $S^n \setminus h(\partial D^n)$ as $y$.
- **Remark:** We defined the singular chain complex using a chain complex $(C_i)$ whose elements were of the form $\sum_j a_j\sigma_j \in C_i$ where $\sigma_j$ are singular $i$-simplices and $a_j \in \mathbb{Z}$. Instead of $\mathbb{Z}$, one can use any other Abelian group, for example $G = \mathbb{Z}_2$. One then writes $C_i(X; G)$. When one reinterprets the boundary operator by viewing $(-1)$ as element in $\mathrm{Aut}(G)$, then everything we have discussed so far goes through without changes. The homology of a point is then $H_0(pt; G) = G$, and the homology groups of the spheres change accordingly.
- **Definition:** The homology of the resulting chain complex is $H_*(X; G)$.
- The reduced chain complex is defined using the map $\varepsilon : C_0(X; G) \longrightarrow G$ defined by summing the coefficients. The homology of the reduced chain complex with coefficients in $G$ is $\widetilde{H}_*(X; G)$.
- **Lemma:** Let $\varphi : G_1 \longrightarrow G_2$ be a morphism of Abelian groups. The inclusion $\varphi : C_i(X, G_1) \longrightarrow C_i(X, G_2)$ is a chain map. For a continuous map $f : X \longrightarrow Y$,

$$\begin{array}{ccc} H_i(X; G_1) & \xrightarrow{\varphi} & H_i(X; G_2) \\ \big\downarrow{\scriptstyle f_*} & & \big\downarrow{\scriptstyle f_*} \\ H_i(Y; G_1) & \xrightarrow{\varphi} & H_i(Y; G_2). \end{array}$$

  commutes.
- **Consequence:** Let $f : S^n \longrightarrow S^n$ be a continuous map. Then $f : \widetilde{H}_n(S^n; G) \longrightarrow \widetilde{H}_n(S^n; G)$ is multiplication with an integer $m$ (viewed as endomorphism of $G$, i.e. $m \cdot g = g + g + \ldots + g$).
- **Proof:** For $g \in G$, there is an morphism $\varphi_g : \mathbb{Z} \longrightarrow G$ defined by $1 \to g$. By the previous naturality lemma, $f_*(g) = f_*(\varphi_g(1)) = \varphi_g(f_*(1)) = \varphi_g(m) = g \cdot m$. This works for all $g$ with the same $m$.
- **Remark:** Using the previous Lemma, one obtains a cellular chain complex whose homology is $H_*(X; G)$. The chain groups $C_i^{CW}(X, G)$ are $\oplus G$, with one copy of $G$ for each $i$-cell and the boundary operator is defined by (27).
- **Example:** The cellular chain complex of $\mathbb{RP}^n$ with coefficients in $\mathbb{Z}_2$: For the standard CW-structure $C_k(\mathbb{RP}^n; \mathbb{Z}_2) = \mathbb{Z}_2$ iff $0 \leq k \leq n$ and 0 otherwise. All

differentials are zero. Then

$$H_k(\mathbb{RP}^n;\mathbb{Z}_2) \simeq \begin{cases} \mathbb{Z}_2 & 0 \le k \le n \\ 0 & \text{otherwise} \end{cases}$$

- **Proposition:** Let $f : S^n \longrightarrow S^n$ be continuous such that $f(-x) = -f(x)$. Then the degree of $f$ is odd.
- **Preliminaries:** Let $\mathrm{pr} : \widehat{X} \longrightarrow X$ be a 2-sheeted covering. This is automatically normal,, we denote the non-trivial deck transformation by $\varphi$. Consider the short exact sequence

$$0 \longrightarrow C_n(X;\mathbb{Z}_2) \xrightarrow{\tau} C_n(\widehat{X};\mathbb{Z}_2) \xrightarrow{\mathrm{pr}_*} C_n(X;\mathbb{Z}_2) \longrightarrow 0$$

where $\tau$ is defined as follows. For a singular $n$-simplex in $X$ we choose a lift $\widehat{\sigma} : \Delta^n \longrightarrow \widehat{X}$ such that $\mathrm{pr} \circ \widehat{\sigma} = \sigma$. Then

$$\tau(\sigma) = \widehat{\sigma} + \varphi_*(\widehat{\sigma}).$$

This determines $\tau$ on general chains by linearity and $\tau$ is a chain map. Thus there is an associated long exact sequence, the map induced by $\tau$ is a *transfer map*, the associated long exact sequence is a *transfer sequence*.
- **Proof of the Proposition:** Consider the covering $\mathrm{pr} : S^n \longrightarrow \mathbb{RP}^n$. Since $f(-x) = -f(x)$, $f$ induces a map $\widehat{f} : \mathbb{RP}^n \longrightarrow \mathbb{RP}^n$. We need to understand how $\widehat{f}, f$ interact with the short exact sequence: Using the definition of $\tau$, one can check that the diagram

$$\begin{tikzcd}
0 \arrow[r] & C_i(\mathbb{RP}^n;\mathbb{Z}_2) \arrow[r, "\tau"] \arrow[d, "\widehat{f}_*"] & C_i(S^n;\mathbb{Z}_2) \arrow[r, "\mathrm{pr}_*"] \arrow[d, "f_*"] & C_i(\mathbb{RP}^n;\mathbb{Z}_2) \arrow[r] \arrow[d, "\widehat{f}_*"] & 0 \\
0 \arrow[r] & C_i(\mathbb{RP}^n;\mathbb{Z}_2) \arrow[r, "\tau"] & C_i(S^n;\mathbb{Z}_2) \arrow[r, "\mathrm{pr}_*"] & C_i(\mathbb{RP}^n;\mathbb{Z}_2) \arrow[r] & 0
\end{tikzcd}$$

commutes. By naturality of the long exact sequence there is map from the transfer sequence to itself:

$$\begin{tikzcd}
H_1(S^n;\mathbb{Z}_2) \arrow[r] \arrow[d, "f_*"] & H_1(\mathbb{RP}^n;\mathbb{Z}_2) \arrow[r] \arrow[d, "\widehat{f}_*"] & H_0(\mathbb{RP}^n;\mathbb{Z}_2) \arrow[r, "\tau_*"] \arrow[d, "\widehat{f}_*"] & H_0(S^n;\mathbb{Z}_2) \arrow[r, "\mathrm{pr}_*"] \arrow[d, "f_*"] & H_0(\mathbb{RP}^n;\mathbb{Z}_2) \arrow[d, "\widehat{f}_*"] \\
H_1(S^n;\mathbb{Z}_2) \arrow[r] & H_1(\mathbb{RP}^n;\mathbb{Z}_2) \arrow[r] & H_0(\mathbb{RP}^n;\mathbb{Z}_2) \arrow[r, "\tau_*"] & H_0(S^n;\mathbb{Z}_2) \arrow[r, "\mathrm{pr}_*"] & H_0(\mathbb{RP}^n;\mathbb{Z}_2)
\end{tikzcd}$$

The right-most map $\mathrm{pr}_*$ is an isomorphism, so the map preceding it must be zero. Therefore, the connecting morphism $H_1(\mathbb{RP}^n;\mathbb{Z}_2) \longrightarrow H_0(\mathbb{RP}^n;\mathbb{Z}_2)$ is sujective, hence it is an isomorphism. Moreover, the map $\widehat{f}_* : H_0(\mathbb{RP}^n;\mathbb{Z}_2) \longrightarrow H_0(\mathbb{RP}^n;\mathbb{Z}_2)$ is an isomorphism. Hence $\widehat{f}_* : H_1(\mathbb{RP}^n;\mathbb{Z}_2) \longrightarrow H_1(\mathbb{RP}^n;\mathbb{Z}_2)$ is an isomorphism.

  Inductively, one obtains that $\widehat{f}_* : H_n(\mathbb{RP}^n;\mathbb{Z}_2) \longrightarrow H_n(\mathbb{RP}^n;\mathbb{Z}_2)$ is an isomorphism. Finally, look at

$$\begin{tikzcd}
H_{n+1}(\mathbb{RP}^n;\mathbb{Z}_2) = 0 \arrow[r] \arrow[d] & H_n(\mathbb{RP}^n;\mathbb{Z}_2) \arrow[r] \arrow[d, "\widehat{f}_*"] & H_n(S^n;\mathbb{Z}_2) = \mathbb{Z}_2 \arrow[r] \arrow[d, "f_*"] & H_n(\mathbb{RP}^n;\mathbb{Z}_2) \arrow[d, "\widehat{f}_*"] \\
H_{n+1}(\mathbb{RP}^n;\mathbb{Z}_2) = 0 \arrow[r] & H_n(\mathbb{RP}^n;\mathbb{Z}_2) \arrow[r] & \mathbb{H}_n(S^n;\mathbb{Z}_2) = \mathbb{Z}_2 \arrow[r] & H_n(\mathbb{RP}^n;\mathbb{Z}_2)
\end{tikzcd}$$

The horizontal maps in the middle are injective, hence they are isomorphisms. Hence, $f_*$ is an isomorphism. We have shown that it is also the multiplication with an integer, which must be odd.

Using the morphism $\varphi : \mathbb{Z} \longrightarrow \mathbb{Z}_2$ of coefficient groups we conclude, that the degree of $f$ is odd.

- **Corollary (Borsuk-Ulam):** Let $g : S^n \longrightarrow \mathbb{R}^n$ be a continuous map. Then there is a point $x$ such that $f(x) = f(-x)$.
- **Proof:** Assume not and consider
$$\begin{aligned} f : S^n &\longrightarrow S^{n-1} \\ x &\longmapsto \frac{g(x) - g(-x)}{\|g(x) - g(-x)\|}. \end{aligned}$$
The restriction of this map to the equator has the symmetry property needed for the previous proposition. Therefore, it has odd degree. However, this restriction is null-homotopic (shrink $S^{n-1}$ in one of the hemispheres), so the degree would be zero. This is a contradiction.
- **Remark:** There is a strong relationship between $H_*(X)$ and $H_*(X; A)$, the corresponding theorem is the universal coefficient theorem. This is for next semester. Just like the study of $C^* = Hom(C_*, \mathbb{Z})$ and a ring structure on the resulting cohomology $H^*(X)$.

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[3] there are slight differences between the printed and the online versions