Ludwig-Maximilians-Universität München
Prof. Dr. Thomas Vogel

Seminar Summer 2018

Characteristic classes

SEMINAR ON CHARACTERISTIC CLASSES

TOPIC

Characteristic classes are invariants associated to vector bundles (or principal bundles) over topological spaces. An important example of a vector bundle is the tangent bundle of a smooth manifold. Characteristic classes play an important role in differential geometry, complex geometry, and elsewhere.

We will discuss various constructions of vector bundles, in particular we will see that all vector bundles over a paracompact space can be obtained from one *universal* vector bundle over specific CW-complexes, so-called *classifying spaces*, depending on the nature of the vector bundles under consideration (real, complex, oriented).

From topological invariants of the classifying space BG, namely generators of $H^*(BG)$, one can then obtain the characteristic classes.

Another construction, the Chern-Weyl construction, of certain characteristic classes is useful in the study of the relationship between topological and geometeric properties of a Riemannian manifold. Another famous instance where the Chern-Weyl construction is used is the heat equation proof of the Atiyah-Singer index theorem.

Organizational meeting: Monday, March 9th, 4:16 pm in B251

TALKS

(1) **Vector bundles**
- Date:
- Speaker: T.V.
- Literature: [MS], p. 3–24
- Definition, Examples: Tangent bundle of a smooth manifold, normal bundles, Möbius band, Sections.

(2) **Constructions of vector bundles**
- Date:
- Speaker: T.V
- Literature: [MS], p.25–36
- Whitney sum, induced bundles (pull back), tensor product, dual bundle

(3) **Universal bundles**
- Date: 24.4.
- Speaker: M.R.
- Literature: [MS], p.55–68
- Tautological bundles over Grassmann manifolds, paracompact spaces, infinite Grassmannians and universal bundles

(4) **Cell structure for Grassmannians**
- Date: 8.5.
- Speaker: M.G.
- Literature: [MS], p.68–70, 73–81
- Construction of a CW-structure on Grassmannians

(5) **Stiefel-Whitney classes,** $H^*(G_n; \mathbb{Z}_2)$
- Date: 15.5.
- Speaker: D.A.
- Literature: [MS], p. 83–87, p. 37–47
- G_n is the infinite Grassmannian of n-planes in $\mathbb{R}^\infty$ (with the weak topology), assuming that Stiefel Whitney classes exist one shows that $H^*(G_n, \mathbb{Z}_2)$ is a polynomial algebra over $\mathbb{Z}_2$ freely generated by Stiefel-Whitney classes.

(6) **Oriented bundles, Euler class**
- Date: 29.5.
- Speaker: J.V.
- Literature: [MS], p. 95–103
- Oriented vector bundles, Euler class defined using the Thom isomorphism from the next talk, properties of the Euler class.

(7) **Thom isomorphism**
- Date: 5.6.
- Speaker: D.R.

- Literature: [MS], p. 105–114
- Definition of the Thom class/Thom isomorphism, the talk requires more algebraic topology than the others.

(8) **Smooth manifolds**
- Date: 12.6.
- Speaker: L.?.
- Literature: [MS], p. 115–130
- Euler class of smoooth manifolds, requires algebraic topology.

(9) **Complex vector bundles, Chern classes I**
- Date:
- Speaker:
- Literature: [MS], p. 149–159
- Definitions, conjugation, complex Grassmannian $H^*(G_n(\mathbb{C}^\infty), \mathbb{Z})$

(10) **Chern classes II**
- Date:
- Speaker:
- Literature: [MS], p. 159–173
- continuation of the previous talk, $H^*(G_n(\mathbb{C}^\infty), \mathbb{Z})$ is generated by Chern classes

(11) **Chern-Weyl construction 1**
- Date: 26.6.
- Speaker: A.S.
- Literature: [MS], p. 289–303
- connections, curvature, invariant polynomials

(12) **Chern-Weyl construction 2**
- Date: 3.7.
- Speaker: A.S.
- Literature: [MS], p. 303–314
- Gauß-Bonnet, Chern classes, if time permits generalize Gauß-Bonnet

REFERENCES

[BT] R. Bott, L. Tu, *Differential forms in algebraic topology*, Grad. Texts in Math. 82, Springer 1982.

[MS] J. Milnor, J. Stasheff, *Characteristic classes*, Annals of Math. Studies, Vol. 76, Princeton Univ. Press 1974.