Mathematisches Institut LMU München
Arnaud Triay SoSe 23

Introduction to the Calculus of Variations
Exercise sheet 4

CONTENTS

I. WARM UP

Ex. 4.1 (Countability)

We say a set I is (at most) countable if there exists an injective function $\varphi : I \to \mathbb{N}$.

1. Show that if I_1 and I_2 are countable, then $I_1 \cup I_2$ is also countable. Deduce that $\mathbb{Z}$ is countable.

2. Show that if I_1 and I_2 are countable, then $I_1 \times I_2$ is also countable. Deduce that $\mathbb{Q}$ is countable. *Hint: The decomposition $2^{n_1}3^{n_3}$ is unique for $n_1, n_2 \in \mathbb{N}$.*

3. Show that if I_k, $k \geq 1$ are countable then $\cup_{k\geq 1} I_k$ is countable.

4. Show that $[0,1]$ is not countable. *Hint: By contradiction, if it were so, one could write $[0,1] = \{x_n\}_{n\in\mathbb{N}}$ and one can use the decimal decomposition $x_n = 0, x_n^{(1)} x_n^{(2)} ...$ where $x_n^{(k)} \in [|0,9|]$. Then one could construct $x \in [0,1]$ that does not belong to $\{x_n\}_{n\in\mathbb{N}}$ by choosing $x^{(n)}$ different from $x_n^{(n)}$.*

5. Show that if I_k, $k \geq 1$ are countable then $\prod_{k\geq 1} I_k$ may not be countable.

Ex. 4.2 (Approximation by step functions)

Denote

$$\mathcal{E}(\mathbb{R}^n) = \left\{ \sum_{k=1}^{N} \alpha_k \mathbb{1}_{A_k}, A_k = \{a_i^{(k)} \leq x_i \leq b_i^{(k)}\}, a_i^{(k)} < b_i^{(k)} \in \mathbb{R}, \alpha_k \in \mathbb{C}, N \geq 1 \right\}.$$

We assume to know that $\mathcal{E}(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$. Show that for $1 \leq p < \infty$, the following assertions are equivalent

- $u_N \rightharpoonup u$ weakly in $L^p(\mathbb{R}^n)$
- $\{u_N\}_N$ is bounded in L^p and $\int_A u_N \underset{N\to\infty}{\longrightarrow} \int_A u, \forall A = \{a_i \leq x_i \leq b_i\}, a_i < b_i \in \mathbb{R}$.

Ex. 4.3 (Approximation by C_c^∞ functions)

Let $f \in L^p(\mathbb{R}^n)$, $1 \leq p < \infty$ and let $\chi \in C_c^\infty(\mathbb{R}^3)$. For $\varepsilon > 0$, define $\chi_\varepsilon(x) = \varepsilon^{-d}\chi(x/\varepsilon)$. We assume to know that $\mathcal{E}(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$.

1. Show that $f_N = \mathbb{1}_{|x|\leq N}\mathbb{1}_{|f|\leq N} f$ converges to f in $L^p(\mathbb{R}^n)$ as $N \to \infty$.
2. Show that $f_N * \chi_\varepsilon$ is in $C_c^\infty(\mathbb{R}^n)$.
3. Show that $f_N * \chi_\varepsilon$ converges to f_N as $\varepsilon \to 0$. *Hint: One could approximate f_N by a a step function.*
4. Conclude.

Ex. 4.4 (Lack of compactness)

Let $\varphi \in L^p(\mathbb{R}^n)$, $1 < p < \infty$, $k \in \mathbb{R}^n$ and define

$$\varphi_N^{(1)}(x) = N^{-d/p}\varphi(N^{-1}x), \qquad \varphi_N^{(2)}(x) = N^{d/p}\varphi(N^1 x)$$
$$\varphi_N^{(3)}(x) = \varphi(x - kN), \qquad \varphi_N^{(4)}(x) = e^{ik\cdot xN}\varphi(x).$$

Show that these functions converge weakly to 0 in $L^p(\mathbb{R}^n)$.

Ex. 4.5 (L^1_{loc} is the least regular)

Let $\Omega \subset \mathbb{R}^n$ be measurable.

1. Show that for all $1 \leq p \leq \infty$,
$$L^p(\Omega) \subset L^1_{\text{loc}}(\Omega) = \{f : \Omega \to \mathbb{C}, \int_{K\cap\Omega} |f| < \infty, \forall K \subset \Omega \text{ compact }\}.$$
2. Show that $L^p(\mathbb{R}^n) \not\subseteq L^p_{\text{loc}}(\mathbb{R}^n) \not\subseteq L^1_{\text{loc}}(\mathbb{R}^n)$ for all $p > 1$.

II. EXERCISES

Ex. 4.5 (Sobolev inequality for gradients $n = 2$)

We want to prove that for all $2 \leq q < \infty$ there is a constant $C_{Sob,q} > 0$ such that for all $f \in H^1(\mathbb{R}^2)$ we have

$$\|f\|_{L^q(\mathbb{R}^2)} \leq C_{Sob,q}\|f\|_{H^1(\mathbb{R}^2)} \tag{1}$$

where $2^* = 2n/(n-2)$.

We assume to know that

- For all $1 \leq p \leq 2$ there exists some $C > 0$ such that
$$\|\widehat{f}\|_{L^{p'}} \leq C\|f\|_{L^p}$$
where $1/p' + 1/p = 1$ and
$$\widehat{f}(k) = \frac{1}{(2\pi)}\int_{\mathbb{R}^3} f(x)e^{-ik\cdot x}\mathrm{d}x.$$

1. Justify (in detail) that it is enough to show (1) for $f \in C_c^\infty$.

2. For $f \in C_c^\infty(\mathbb{R}^2)$, compute $\widehat{\nabla f}(k)$ in terms of $\widehat{f}(k)$. Deduce from this a formula for $\|f\|_{H^1}$ in terms of $\widehat{f}$.

3. Let $q > 2$ and $p = q'$, writing

$$\widehat{f}(k) = \widehat{f}(k)(1 + |k|^2)^{1/2} \times (1 + |k|^2)^{-1/2}$$

show that

$$\|\widehat{f}\|_p \leq C\|f\|_{H^1}$$

for some $C > 0$ independent on f.

4. Conclude.

Ex. 4.6 (Sobolev inequality for gradients $n \geq 3$)

Let $n \geq 3$, we want to prove that there is a constant $C_{Sob} > 0$ such that for all $f \in L^1_{\text{loc}}$ with $\nabla f \in L^2(\mathbb{R}^n)$ (in the sense of distributions) we have

$$\|f\|_{L^{2^*}(\mathbb{R}^n)} \leq C_{Sob}\|\nabla f\|_{L^2(\mathbb{R}^n)}$$

where $2^* = 2n/(n-2)$.

We assume to know two things:

- The Hardy-Littlewood-Sobolev inequality: for all $p, r > 1$ and $0 < \lambda < n$ such that $\frac{1}{p} + \frac{\lambda}{n} + \frac{1}{r} = 2$, we have

$$\left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x) \frac{1}{|x-y|^\lambda} g(y) \mathrm{d}x \mathrm{d}y \right| \leq C\|f\|_p \|g\|_r, \tag{2}$$

  for all measurable functions f, g.

- The Plancherel formula, for any $f, g \in L^2$

$$\int_{\mathbb{R}^n} \overline{f} g = \int_{\mathbb{R}^n} \overline{\widehat{f}}\, \widehat{g}$$

1. Justify (in detail) that it is enough to show (2) for $f \in C_c^\infty$.

2. Prove that there is come $C_\alpha > 0$, such that for all $k \in \mathbb{R}^n$

$$\frac{1}{|k|^\alpha} = C_\alpha \int_0^\infty e^{-\pi |k|^2 \lambda} \lambda^{\alpha/2 - 1} \mathrm{d}\lambda.$$

3. Prove that there is some $C_{\alpha,n} > 0$ such that for any $f \in C_c^\infty(\mathbb{R}^n)$ and $0 < \alpha < n$, we have

$$\left(\frac{1}{|k|^\alpha} \widehat{f} \right)^\vee (x) = C_{\alpha,n} \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-\alpha}} f(y) \mathrm{d}y.$$

4. Let $f, g \in C_c^\infty(\mathbb{R}^n)$, show that

$$\left|\int_{\mathbb{R}^n} fg\right| \leq \left(\int_{\mathbb{R}^{\ltimes}} |k|^2|\widehat{f}(k)|^2\right)^{1/2} \left(\int_{\mathbb{R}^{\ltimes}} |k|^{-2}|\widehat{g}(k)|^2\right)^{1/2}$$
$$\leq C\|\nabla f\|_{L^2(\mathbb{R}^n)}\|g\|_{L^p(\mathbb{R}^n)}$$

for an appropriate p to be computed.

5. Justify that for any $1 \leq p < \infty$

$$\sup_{\|g\|_{L^p}=1} \left|\int_{\mathbb{R}^n} fg\right| = \|f\|_{L^{p'}}.$$

6. Conclude.

Ex. 4.7 (Bounded sequences in L^p have weak limits)
Let $1 < p < \infty$, $\Omega \subset \mathbb{R}^d$ and we assume to know that $L^p(\Omega)$. Let $\{f_n\} \subset L^p(\Omega)$, bounded, we want to show that there exists a subsequence and $f \in L^p(\Omega)$, such that

$$f_{n'} \underset{n\to\infty}{\rightharpoonup} f \tag{3}$$

weakly in L^p

1. Let $\{\phi_k\} \subset L^p(\Omega)$ be a dense sequence. By a diagonal argument, show that there exists a subsequence $f_{n'}$, such that for all $k \geq 1$, the sequence $\{\langle f_{n'}, \varphi_k\rangle\}_{n'}$ converges, and denote by ℓ_k its limit.
2. Let $g \in L^{p'}$ show that $\{\langle f_{n'}, g\rangle\}_{n'}$ is convergent (where $\{f_{n'}\}$ is the subsequence from 1.)
3. Define, for all $g \in L^{p'}$, $F(g) = \lim\limits_{n\to\infty} \langle f_{n'}, g\rangle$ and show that there exists $f \in L^p$ such that $F(g) = \int fg$.
4. Conclude. What properties of L^p did we use ?