Mathematisches Institut LMU München
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Introduction to the Calculus of Variations
Exercise sheet 1: Homework, due date 02.05

Ex. 1 (Implicit function theorem)

Let $f : \mathbb{R}^2 \to \mathbb{R}$, C^1 and such that $f(0,0) = 0$ and $\partial_y f(0,0) \neq 0$. We want to prove that there exist $\varepsilon > 0$ and a C^1 function $\varphi :]-\varepsilon, \varepsilon[\to]-\varepsilon, \varepsilon[$ such that for all $x, y \in]-\varepsilon, \varepsilon[$

$$f(x,y) = 0 \iff y = \varphi(x).$$

Without loss of generality, we assume $\partial_y f(0,0) > 0$.

1. Show that there exists $\varepsilon > 0$, such that for all $x \in [-\varepsilon, \varepsilon]$, $[-\varepsilon, \varepsilon] \ni y \mapsto f(x,y)$ is strictly increasing.
2. Deduce that for all $x \in [-\varepsilon, \varepsilon]$, there exists a unique $\varphi(x) \in [-\varepsilon, \varepsilon]$ such that $f(x,y) = 0$ if and only if $y = \varphi(x)$.
3. Using a Taylor expansion of f around $(0,0)$, show that φ is differentiable at 0.
4. Show that it is in fact differentiable on $]-\varepsilon, \varepsilon[$ and C^1 on this set.
5. (bonus) Generalize the theorem for $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$, for $n, m \geq 0$. Explain briefly the proof and the differences with the case $n = m = 1$.

Ex.2 (Weierstrass example)

Let $f(x, \xi) = x\xi^2$ for $x, \xi \in \mathbb{R}$ and consider for $\varepsilon \in [0,1)$

$$(P_\varepsilon) \qquad m_\varepsilon = \inf_{u \in X} \left\{ I(u) := \int_\varepsilon^1 f(x, u'(x)) \mathrm{d}x \right\},$$
$$X = \left\{ u \in C^1([0,1]), \quad u(\varepsilon) = 1, u(1) = 0 \right\}.$$

1. Show that for $\varepsilon \in (0,1)$ there is a unique minimizer of (P) in C^2.
2. Show that for $\varepsilon = 0$ there is no minimizer of (P) in $X \cap C^2$.
3. Find a sequence $\{u_n\} \subset C_p^1$ (piecewise C^1) such that $u_n(0) = 1, u_n(1) = 0$ and $I(u_n) \to 0$ as $n \to \infty$.
4. Show that $m_0 = 0$ and that there is therefore no minimizer of (P) in X.