Mathematisches Institut LMU München
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ADVANCED ANALYSIS
Exercise sheet 7 – 15.12.2022

Let $\Omega \subset \mathbb{R}^n$ be an open set.

For $n \geq 1$ and $y \in \mathbb{R}^n$, define

$$G_y(x) = \begin{cases} -\frac{1}{|\mathbb{S}^1|} \ln(|x-y|), & n = 2, \\ \frac{1}{|\mathbb{S}^{n-1}|(n-2)} \frac{1}{|x-y|^{n-2}}, & n \neq 2. \end{cases}$$

Ex.1.1 (Distributional Laplacian of Green functions)

1. Justify that $G_y \in L^1_{\text{loc}}(\mathbb{R}^n)$ for all $y \in \mathbb{R}^n$.
2. Take $y = 0$, we want to show that
$$-\Delta G_0 = \delta_0,$$
in $D'(\mathbb{R}^n)$, where $\delta_0(\phi) = \phi(0)$ for all $\phi \in D(\mathbb{R}^n)$. Pick $\phi \in D(\mathbb{R}^n)$ and write for $r > 0$,
$$\int_{\mathbb{R}^n} G_0 \Delta\phi = \int_{|x|\leq r} G_0 \Delta\phi + \int_{|x|>r} G_0\Delta\phi = I(r) + J(r).$$
   (a) Prove that $I(r) \to 0$ as $r \to 0$.
   (b) Using an integration by part, rewrite $J(r)$ and find its limit for $r \to 0$.
   (c) Conclude.

Ex.1.2 (Inverting the Laplacian – Poisson's equation)
Let $f \in L^1_{\text{loc}}(\mathbb{R}^n)$. We want to find a solution $u \in L^1_{\text{loc}}$ to the equation

$$-\Delta u = f$$

in $D'(\Omega)$.

1. Define
$$\widetilde{G}(y) = \begin{cases} \frac{1}{(1+|y|)^{n-2}}, & n \geq 3, \\ \ln(1+|y|), & n = 2, \\ |y|, & n = 1. \end{cases}$$
Show that the followings are equivalent.
   (a) $G_x f \in L^1(\mathbb{R}^n)$ for a.e. $x \in \mathbb{R}^n$,
   (b) $\widetilde{G} f \in L^1(\mathbb{R}^n)$.

2. Assume $\widetilde{G}f \in L^1(\mathbb{R}^n)$ and define

$$u(x) = \int_{\mathbb{R}^n} G_x(y) f(y) \mathrm{d}y.$$

   (a) Show that $u \in L^1_{\text{loc}}(\mathbb{R}^n)$.
   Hint: It is enough to show it belongs to $L^1(B(0,R))$ for any $R > 0$. One can consider the function

$$H_R(y) = \int_{B(0,R)} |G_y(x)| \mathrm{d}x$$

   and show that there is some constant $C_R > 0$ such that $H_R(y) \leq C_R \widetilde{G}(y)$ for all $y \in \mathbb{R}^n$ and some $C_R > 0$.

   (b) Verify that

$$-\Delta u = f$$

   in $D'(\mathbb{R}^n)$.

3. Show that $u \in W^{1,1}_{\text{loc}}(\mathbb{R}^n)$ and compute

$$\partial_{x_i} u$$

   for all $1 \leq i \leq n$.

Ex.1.3(The dual of $W^{1,p}(\Omega)$)
Let $1 \leq p < \infty$ and $g_0, g_1, \ldots, g_n \in L^{p'}(\Omega)$ where we recall that $\frac{1}{p} + \frac{1}{p'} = 1$.

1. Justify that

$$T = g_0 + \sum_{i=1}^{n} \partial_{x_i} g_i \in D'(\Omega). \tag{1}$$

2. Show that for all $\phi \in D(\Omega)$,

$$|T(\phi)| \leq C \|\varphi\|_{W^{1,p}(\Omega)},$$

   for some constant $C > 0$, where we recall that

$$\|\varphi\|_{W^{1,p}(\Omega)} = \|\varphi\|_{L^p(\Omega)} + \sum_{i=1}^{n} \|\partial_{x_i} \varphi\|_{L^p(\Omega)}.$$

3. Show that T can be extended to a continuous function on $W^{1,p}(\Omega)$ and conclude that $T \in (W^{1,p}(\Omega))'$, where $(W^{1,p}(\Omega))'$ is the dual of $W^{1,p}(\Omega)$. *Hint:* Is $W^{1,p}(\Omega)$ complete?

In fact, any element of the dual of $W^{1,p}(\Omega)$ is given by an expression like (1), see Theorem 6.14.