Mathematisches Institut LMU München
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ADVANCED ANALYSIS
Exercise sheet 5 – 1.12.2022

Ex.1.1 (Elementary of properties of the Fourier transform)
Let f be a function in $L^1(\mathbb{R}^n)$ and denote by $\widehat{f}$ its Fourier transform. Prove that

1. the map $f \to \hat{f}$ is linear in f,
2. $\widehat{\tau_h f}(k) = e^{-2\pi i(k,h)}\widehat{f}(k)$, $h \in \mathbb{R}^n$
3. $\widehat{\delta_\lambda f}(k) = \lambda^n \widehat{f}(\lambda k)$ $\lambda > 0$,

where τ_h is the translation operator, i.e., $(\tau_h f)(x) = f(x-h)$, and δ_λ is the scaling operator such that $(\delta_\lambda f)(x) = f(x/\lambda)$.

Ex.1.2 (Fourier transforms of L^1 functions vanish at infinity)
Let $f \in L^1(\mathbb{R}^n)$ and let $\widehat{f}$ be its Fourier transform. Prove that $\widehat{f}(k) \to 0$ as $|k| \to \infty$.

Ex.1.3(The Hausdorff-Young inequality)
Let $1 \leq p, q \leq \infty$ and $C_{p,q} > 0$ such that for any $f \in L^1 \cap L^p$ we have

$$\|\widehat{f}\|_q \leq C_{p,q}\|f\|_p.$$

Show that

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Ex.1.4(Fourier transform and convolutions)
Let $1 \leq p, q, r \leq 2$ such that $1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. Show that for any $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$, we have

$$\widehat{f * g} = \widehat{f}\widehat{g}.$$

Hint: One essentially needs to make sure that all quantities make sense.

Ex.1.5(Fourier transform of $|x|^{\alpha-n}$)
Prove that there is some $C_{\alpha,n} > 0$ such that for any $f \in C_c^\infty(\mathbb{R}^n)$ and $0 < \alpha < n$, we have

$$\left(\frac{1}{|k|^\alpha}\widehat{f}\right)^\vee(x) = C_{\alpha,n}\int_{\mathbb{R}^n}\frac{1}{|x-y|^{n-\alpha}}f(y)\mathrm{d}y$$

Hint: One can use that there is some $C_\alpha > 0$ such that for any $k \in \mathbb{R}^n$

$$\frac{1}{|k|^\alpha} = C_\alpha > 0 \int_0^\infty e^{-\pi|k|^2\lambda} \lambda^{\alpha/2-1} \mathrm{d}\lambda.$$

Ex.1.6(Derivatives of distributions are distributions)

Let $T \in D'(\Omega)$, recall that for $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$, $D^\alpha T(\phi)$ is defined by

$$D^\alpha T(\phi) = (-1)^{|\alpha|} T(D^\alpha \phi)$$

for all $\phi \in D(\Omega)$, where $|\alpha| = \sum_{k=1}^n \alpha_k$.

1. Show that $D^\alpha T \in D'(\Omega)$.
2. Show that if $T = T_f$ for some $f \in C^\infty$, we have

$$D^\alpha T_f = T_{D^\alpha f}.$$