Mathematisches Institut LMU München
Arnaud Triay WiSe 22/23

ADVANCED ANALYSIS
Exercise sheet 4 – 24.11.2022

Ex.1.1 (Properties of f^*)

Let $f : \mathbb{R}^n \to \mathbb{C}$, measurable, vanishing at infinity. Let f^* be its symmetric-decreasing rearrangement. We recall that

$$f^*(x) := \int_0^\infty \mathbb{1}^*_{\{|f|>t\}} dt.$$

1. Show that f^* is radially symmetric, i.e. if $|x| = |y|$ then $f^*(x) = f^*(y)$.
2. Assume that for all $t > 0$

$$\{x, f^*(x) > t\} = \{x, |f(x)| > t\}^*$$

   and deduce from it that f^* is lower semicontinuous, that is that

$$\liminf_{x\to x_0} f^*(x) \geq f^*(x_0).$$

   Hint: Recall that by definition A^* is the <u>open</u> ball of volume $|A|$ centered at the origin.

Ex.1.2 (An "obvious" property)

Let $f : \mathbb{R}^n \to \mathbb{C}$, measurable, vanishing at infinity. Let f^* be its symmetric-decreasing rearrangement, we want to show that for all $t > 0$

$$\{x, f^*(x) > t\} = \{x, |f(x)| > t\}^*.$$

1. Prove that for all $0 < s < t$, we have

$$\{x, |f(x)| > t\}^* \subset \{x, |f(x)| > s\}^*.$$

   Hint: Is it true without the * already ?

2. Let $t > 0$ and $y \in \{|f| > t\}^*$, show that

$$f^*(y) > t$$

   and deduce from it that

$$\{x, f^*(x) > t\} \supset \{x, |f(x)| > t\}^*.$$

   Hint: use the layer cake representation (definition of f^*)

3. Let $y \notin \{|f| > t\}^*$, show that

$$\sup\{s, y \in \{|f| > s\}^*\} \leq t.$$

   Hint: by contradiction.

4. Show that, for $y \notin \{|f| > t\}^*$,

$$f^*(y) \leq t,$$

and deduce from it that

$$\{x, f^*(x) > t\} \subset \{x, |f(x)| > t\}^*.$$

Ex.1.3(L^p norms and rearrangements)

Let f be measurable and vanishing at infinity. For $1 \leq p \leq \infty$, show that $f \in L^p$ if and only if $f^* \in L^p$ and that

$$\|f\|_p = \|f^*\|_p.$$

Hint: one could use that $|f|^p = \int_0^\infty p\lambda^{p-1} \mathbb{1}_{\{|f|>\lambda\}} d\lambda$ *and the result of the previous exercise.*