Mathematisches Institut LMU München
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ADVANCED ANALYSIS
Exercise sheet 0 – 25.10.2022

We denote by (Ω, Σ, μ) a measure space.

Ex.1.1 (Properties of measures) Prove that

$$\mu(A) \leq \mu(B) \qquad \text{if } A \subset B,$$

$$\lim_{j\to\infty} \mu(A_j) = \mu\left(\bigcup_{j=1}^{\infty} A_j\right) \qquad \text{if } A_1 \subset A_2 \subset A_3 \subset \dots,$$

$$\lim_{j\to\infty} \mu(A_j) = \mu\left(\bigcap_{j=1}^{\infty} A_j\right) \qquad \text{if } A_1 \supset A_2 \supset A_3 \supset \dots \quad \text{and } \mu(A_1) < \infty.$$

Ex.1.2 (Measurability of lim sup and lim inf) Let (f_n) be a sequence of measurable functions. Prove that

$$\limsup_{n\to\infty} f_n(x) := \inf_n \{\sup_{k\geq n}\{f_k(x)\}\}, \qquad \text{and} \qquad \liminf_{n\to\infty} f_n(x) := \sup_n\{\inf_{k\geq n}\{f_k(x)\}\}$$

are measurable. In particular, prove that if (f_n) converges almost surely towards some function f, then f is measurable.

Ex.1.3 (Monotone Convergence Theorem) Let f_n be an increasing sequence of summable functions. Let $f(x)$ be defined as

$$f(x) := \lim_{n\to+\infty} f_n(x).$$

Prove that f is measurable and

$$\lim_{n\to+\infty} \int_\Omega f_n(x)\, d\mu(x) = \int_\Omega f(x) d\mu(x).$$

NB: The proof of the MCT has been given in the lecture (it's also in the book) but it remains to prove the statement about the convergence of the Riemann sum.

Ex.1.4 (Application of MCT) Find a simple condition on $f_n(x)$ so that

$$\sum_{n=0}^{\infty} \int_\Omega f_n(x)\, \mathrm{d}\mu(x) = \int_\Omega \left\{\sum_{n=0}^{\infty} f_n(x)\right\} \mathrm{d}\mu(x).$$

Ex.1.5 (Counter examples) We consider the measure space $(\mathbb{R}, \mathcal{B}, \mathrm{d}x)$, where $\mathrm{d}x$ denotes the Lebesgue measure.

1. Find a sequence of non-negative functions (f_n) such that
   (a) $\lim_{n\to\infty} \int_{\mathbb{R}} f_n$ doesn't exist
   (b) $f_n(x) \to f(x)$ for almost every x, where f is some measurable function, and $\lim_{n\to\infty} \int_{\mathbb{R}} f_n$ exists but
$$\lim_{n\to\infty} \int_{\mathbb{R}} f_n \neq \int_{\mathbb{R}} f.$$
2. Find a sequence of functions (f_n) such that
   (a) $f_n \geq 0$, $\int f_n < \infty$ for all n and
$$\int_{\mathbb{R}} \liminf_{n\to\infty} f_n(x)\mathrm{d}x < \liminf_{n\to\infty} \int_{\mathbb{R}} f_n$$
   (b) $\int |f_n| < \infty$ for all n and
$$\int_{\mathbb{R}} \liminf_{n\to\infty} f_n(x)\mathrm{d}x > \liminf_{n\to\infty} \int_{\mathbb{R}} f_n$$
3. Find a sequence of functions (f_n) such that $f_n(x) \to f(x)$ for almost every x, $\int |f_n| < \infty$ for all n but
$$\lim_{n\to\infty} \int_{\mathbb{R}} \Big| |f_n(x)| - |f_n(x) - f(x)| - |f(x)| \Big| \mathrm{d}x \neq 0$$

Ex.1.6 (An example of non-measurable set) Consider the quotient $\mathbb{R}/\mathbb{Q}$, that is the set of equivalence classes of real numbers modulo a rational number ($x \sim y$ iff $x - y \in \mathbb{Q}$). For any $a \in \mathbb{R}/\mathbb{Q}$ *choose* $x_a \in a \cap [0,1]$ and define
$$F := \{x_a, a \in \mathbb{R}/\mathbb{Q}\} \subset [0,1].$$
Prove that F is non-measurable.

NB: I guess this is a famous example, I took it from the lecture notes of Legall. Another famous is example is of course given by the Banach-Tarski theorem / paradox.