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Functional Analysis II

Holiday Extra

Problem A. *(Measurable functional calculus for commuting self-adjoint operators)*

For $n \in \mathbb{N}$ let $T_1, \dots, T_n$ be pairwise commuting, bounded, self-adjoint operators on a separable Hilbert space $\mathcal{H}$, and let $\mathcal{S} := \sigma(T_1) \times \dots \times \sigma(T_n) \subset \mathbb{R}^n$. The goal of this exercise is to prove that there exists a unique map $\Phi : \mathcal{M}_b(\mathcal{S}) \to \mathcal{B}(\mathcal{H})$ such that

(*i*) $\Phi(1) = \mathbb{I}$, and $\Phi(\pi_k) = T_k$, where $\pi_k(t_1, \dots, t_n) := t_k$ for all $k = 1, \dots, n$.

(*ii*) Φ is a linear multiplicative involution.

(*iii*) $\|\Phi(f)\| \leqslant \|f\|_\infty$ for all $f \in M_b(\mathcal{S})$, i.e. Φ is continuous.

(*iv*) If $(f_k)_k \subset \mathcal{M}_b(\mathcal{S})$ converges pointwise to some $f \in \mathcal{M}_b(\mathcal{S})$ and $\sup_n \|f_n\|_\infty < \infty$, then $\Phi(f_n)x \to \Phi(f)x$ for each $x \in \mathcal{H}$.

(*v*) $\Phi(f)S = S\Phi(f)$ for any $S \in \mathcal{B}(\mathcal{H})$ that commutes with $T_1, \dots, T_n$.

To do so, proceed as follows:

(a) Prove that for all Borel sets $A_1, \dots, A_n \subset \mathbb{R}$, the operators $\chi_{A_1}(T_1), \dots, \chi_{A_n}(T_n)$ commute, where χ_{A_i} denotes the characteristic function of A_i for $i = 1, \dots, n$.

(b) Define Φ_0 on step functions f over rectangles, i.e. for functions of the form

$$f = \sum_{i=1}^{N} c_i \, \chi_{A_1^{(i)} \times \dots \times A_n^{(i)}} , \quad \text{where } A_k^{(i)} \cap A_k^{(j)} = \emptyset \text{ if } i \neq j \ \forall k = 1, \dots, n$$

for some $N \in \mathbb{N}$, $c_i \in \mathbb{C}$ and Borel sets $A_k^{(i)} \subset \mathbb{R}$ for all $i = 1, \dots, N$, $k = 1, \dots, n$, by

$$\Phi_0(f) = f(T_1, \dots, T_n) := \sum_{i=1}^{N} c_i \, \chi_{A_1^{(i)}}(T_1) \cdots \chi_{A_n^{(i)}}(T_n) .$$

Prove that Φ_0 satisfies the properties (*i*), (*ii*), and (*iii*) above.

(c) Construct the continuous functional calculus on $\mathcal{S}$.

[*Hint:* Use uniform continuity and part (*b*).]

(d) Construct the measurable functional calculus defined by properties (*i*) – (*v*) above.

(e) Prove that there exists a spectral measure E on $\mathbb{R}^n$ (to be defined!) such that

$$\Phi(f) = \int_{\mathcal{S}} f(\lambda_1, \dots, \lambda_n) \, dE_{(\lambda_1, \dots, \lambda_n)}$$

for all $f \in \mathcal{M}_b(\mathcal{S})$, in particular, $T_k = \int_{\mathcal{S}} \lambda_k \, dE_{(\lambda_1, \dots, \lambda_n)}$ for all $k = 1, \dots, n$.

(f) Prove that there exists a finite measure space (M, Σ, μ), an isometric isomorphism $U : \mathcal{H} \to L^2(M)$, and bounded measurable functions $F_1, \dots, F_n$ on M, such that

$$(UT_kU^{-1}\varphi)(\xi) = F_k(\xi)\,\varphi(\xi)$$

for μ-almost all $\xi \in M$ and all $k = 1, \dots, n$.

Problem B. *(Measurable functional calculus for normal operators)*

Let T be a normal operator on a separable Hilbert space $\mathcal{H}$.

(a) Prove that there exists a unique map $\Phi : \mathcal{M}_b(\sigma(T)) \to \mathcal{B}(\mathcal{H})$ such that

  (i) $\Phi(1) = \mathbb{I}$, and $\Phi(z) = T$, where $z : \sigma(T) \to \mathbb{C}$, $z \mapsto z$.

  (ii) Φ is a linear multiplicative involution.

  (iii) $\|\Phi(f)\| \leqslant \|f\|_\infty$ for all $f \in M_b(\sigma(T))$, i.e. Φ is continuous.

  (iv) If $(f_k)_k \subset \mathcal{M}_b(\sigma(T))$ converges pointwise to $f \in \mathcal{M}_b(\mathcal{S})$ and $\sup_n \|f_n\|_\infty < \infty$, then $\Phi(f_n)x \to \Phi(f)x$ for each $x \in \mathcal{H}$.

  (v) $\Phi(f)S = S\Phi(f)$ for any $S \in \mathcal{B}(\mathcal{H})$ that commutes with T.

  [*Hint:* Problem 25.]

(b) There exists a spectral measure G on $\mathbb{C}$ (to be defined) such that

$$\Phi(f) = \int_{\sigma(T)} f(z)\, dG_z$$

for all $f \in \mathcal{M}_b(\sigma(T))$, in particular, $T = \int_{\sigma(T)} z\, dG_z$.

(c) There exists a finite measure space (M, Σ, μ), an isometric isomorphism $U : \mathcal{H} \to L^2(M)$, and a bounded measurable function F on M, such that

$$(UTU^{-1}\varphi)(\xi) = F(\xi)\,\varphi(\xi)$$

for μ-almost all $\xi \in M$.

This sheet will not be discussed in the coming exercise class. However, you are invited to ask questions about your solution attempts any time. Merry Christmas!
For more details please visit http://www.math.lmu.de/~tkoenig/16FA2exercises.php