LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN
MATHEMATISCHES INSTITUT

Prof. T. Ø. Sørensen PhD
T. König

Winter term 2016/17
February 2, 2017

Functional Analysis II
Assignment 14

Problem 53. Let A be a symmetric operator on a separable Hilbert space $\mathcal{H}$ such that its domain $\mathcal{D}(A)$ contains an orthonormal basis of $\mathcal{H}$ consisting of eigenvectors of A. Prove:

(a) A is essentially self-adjoint.

(b) $\sigma(\overline{A})$ is the closure of the set of the eigenvalues of A.

Problem 54. *(Dirichlet and Neumann Laplacian in one dimension)*

On the Hilbert space $L^2([0,1])$ consider the densely defined operators A_D and A_N given by

$$\mathcal{D}(A_D) = \{\psi \in C^2([0,1]) \,|\, \psi(0) = 0 = \psi(1)\}, \qquad A_D\psi = -\psi'',$$
$$\mathcal{D}(A_N) = \{\psi \in C^2([0,1]) \,|\, \psi'(0) = 0 = \psi'(1)\}, \qquad A_N\psi = -\psi''.$$

(a) Prove that A_D and A_N are symmetric.

(b) Prove that A_D is essentially self-adjoint and find $\sigma(\overline{A_D})$.

(c) Prove that A_N is essentially self-adjoint and find $\sigma(\overline{A_N})$.

(d) Let $A_{D,N}$ be given by $\mathcal{D}(A_{D,N}) = \mathcal{D}(A_D) \cap \mathcal{D}(A_N)$ and $A_{D,N}\psi = -\psi''$. Prove that $A_{D,N}$ has at least two distinct self-adjoint extensions.

Problem 55. *(Self-adjoint operators generate unitary groups)*

Let $\mathcal{H}$ be a Hilbert space. A family $\{U(t)\}_{t\in\mathbb{R}} \subset \mathcal{B}(\mathcal{H})$ is called a *(one-parameter) unitary group of operators* if $U(t)$ is unitary for all $t \in \mathbb{R}$ and if $U(t)U(s) = U(t+s)$ for all $s,t \in \mathbb{R}$. Moreover, $\{U(t)\}_{t\in\mathbb{R}}$ is said to be *strongly continuous* if the map $t \mapsto U(t)\varphi$ is continuous for all $\varphi \in \mathcal{H}$.

Let A be a densely defined self-adjoint operator on $\mathcal{H}$. Prove:

(a) $U(t) := e^{-itA}$ defines a strongly continuous unitary group $\{U(t)\}_{t\in\mathbb{R}}$.

(b) For all $t \in \mathbb{R}$, $U(t)$ leaves $\mathcal{D}(A)$ invariant and fulfills $U(t)A = AU(t)$ on $\mathcal{D}(A)$.

(c) For all $\varphi \in \mathcal{D}(A)$, the map $t \mapsto U(t)\varphi$ is differentiable with $\frac{d}{dt}(U(t)\varphi) = -iAU(t)\varphi$.

Problem 56. *(Unitary groups are generated by s.a. operators aka Stone's theorem)*

Let $\{U(t)\}_{t\in\mathbb{R}}$ be a strongly continuous unitary group on a Hilbert space $\mathcal{H}$. The operator A given by

$$\mathcal{D}(A) = \{\psi \in \mathcal{H} \,|\, \lim_{t\to 0} \frac{i}{t}(U(t)\psi - \psi) \text{ exists in } \mathcal{H} \text{ (in norm sense)}\}, \quad A\psi = \lim_{t\to 0} \frac{i}{t}(U(t)\psi - \psi)$$

is called the *generator* of the unitary group $\{U(t)\}_{t\in\mathbb{R}}$. This is justified by what shall be proved in this exercise: A is a densely defined self-adjoint operator and $U(t) = e^{-itA}$.

(a) Given $\psi \in \mathcal{H}$, and $\tau > 0$, let $\psi_\tau := \int_0^\tau U(t)\psi\, dt$, where the integral is a Hilbert-space-valued Riemann integral. Prove that $\psi_\tau \in \mathcal{D}(A)$. Conclude from this that $\mathcal{D}(A)$ is dense in $\mathcal{H}$.
[Hint: It is helpful to prove that for any $B \in \mathcal{B}(\mathcal{H})$, $B\int_0^\tau U(t)\psi\, dt = \int_0^\tau BU(t)\psi\, dt$.]

(b) Prove that A is symmetric.

(c) Prove that $U(t)$ leaves $\mathcal{D}(A)$ invariant and that $AU(t)\psi = U(t)A\psi$ for all $\psi \in \mathcal{D}(A)$.

(d) Prove that for all $\psi \in \mathcal{D}(A)$, we have

$$\frac{d}{dt}U(t)\psi = -iU(t)A\psi \quad \text{and} \quad U(t)\psi - \psi = \frac{1}{i}\int_0^t U(s)A\psi\, ds$$

where the integral is again a Riemann integral. Use this to prove that A is closed.

(e) Suppose that $\varphi \in N(A^* \pm i)$. Prove that for any $\psi \in D(A)$,

$$\frac{d}{dt}(\varphi, U(t)\psi) = \pm(\varphi, U(t)\psi).$$

Use this to prove that necessarily $\varphi = 0$. Conclude that A is self-adjoint.

(f) Prove that $U(t) = e^{-itA}$.
[Hint: Set $w(t) := U(t)\varphi - e^{-itA}\varphi$ for $\varphi \in \mathcal{D}(A)$ and check that $\frac{d}{dt}\|w(t)\|^2 = 0$.]

This sheet is to be discussed in the exercise class on Thursday, February 9.
For more details please visit http://www.math.lmu.de/~tkoenig/16FA2exercises.php