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Functional Analysis II
Assignment 13

Problem 49. *(Operators with non-empty resolvent set are closed)*

Prove that a densely defined operator T on a Hilbert space $\mathcal{H}$ satisfying $\sigma(T) \subsetneq \mathbb{C}$ is necessarily closed.

Problem 50. *(A commutation relation only satisfied by unbounded operators)*

Let P, Q be densely defined linear operators on a Hilbert space $\mathcal{H}$ such that $\mathcal{D}(PQ) \cap \mathcal{D}(QP)$ is dense in $\mathcal{H}$, and

$$[P,Q] := PQ - QP = i\mathbb{I}\,.$$

(a) Prove that if $P, Q \in \mathcal{B}(\mathcal{H})$, then $P^nQ - QP^n = inP^{n-1}$ for all $n \in \mathbb{N}$.

(b) Prove that at least one of the operators P and Q has to be unbounded.

Problem 51. *(Absolutely continuous functions and weak differentiability)*

Let $[a,b] \subset \mathbb{R}$ be a compact interval. We call a function $f \in C([a,b])$ *absolutely continuous* if there exists $g \in L^1([a,b])$ such that f can be written as

$$f(x) = f(a) + \int_a^x g(y)\,dy \quad \text{for all } x \in [a,b].$$

The vector space of all absolutely continuous functions on $[a,b]$ is denoted by $AC([a,b])$.

The Lebesgue differentiation theorem asserts that if $f \in AC([a,b])$, then f is differentiable almost everywhere with derivative $f'(x) = g(x)$ for a.e. $x \in [a,b]$.

(a) Prove that for every $f, g \in AC([a,b])$ the usual integration by parts formula holds:

$$\int_a^b f(x)g'(x)\,dx = f(b)g(b) - f(a)g(a) - \int_a^b f'(x)g(x)\,dx.$$

From the IBP formula the following extended notion of differentiability can be derived: A function $f \in L^1((a,b))$ is said to be *weakly differentiable* on (a,b) if there exists $g \in L^1((a,b))$ such that for all $\phi \in C_0^\infty((a,b))$ it holds $\int_a^b f(x)\phi'(x)\,dx = -\int_a^b g(x)\phi(x)\,dx$. In that case, g is called the *weak* (or *distributional*) *derivative* of f.

(b) Prove that $f \in AC([a,b])$ if and only if f is weakly differentiable on (a,b).
[Hint: You may use without proof the so-called Fundamental Lemma of the Calculus of Variations: If some function $h \in L^1((a,b))$ fulfills $\int_a^b h\phi = 0$ for all $\phi \in C_0^\infty((a,b))$, then $h \equiv 0$.]

Problem 52. *(Momentum operator on* $[0,2\pi]$*)*

Consider the operators A_0 and A in $L^2([0,2\pi])$ given by

$$\begin{aligned} A_0 f &= -if', \qquad \mathcal{D}(A_0) = \{f \in C^1([0,2\pi]) \,|\, f(0) = f(2\pi) = 0\}, \\ Af &= -if', \qquad \mathcal{D}(A) = \{f \in C^1([0,2\pi]) \,|\, f(0) = f(2\pi)\}. \end{aligned}$$

(a) Prove that A_0 and A are symmetric, and that $A_0 \subset A$.

(b) Prove:

  (i) $D(A_0^*) = \{g \in AC([0,2\pi]) \,|\, g' \in L^2([0,2\pi])\}$ and $A_0^* g = -ig'$ for all $g \in D(A_0^*)$.

  (ii) $D(\overline{A_0}) = \{g \in D(A_0^*) \,|\, g(0) = g(2\pi) = 0\}$ and $\overline{A_0} g = -ig'$ for all $g \in D(\overline{A_0})$.

  (iii) $D(A^*) = \{g \in D(A_0^*) \,|\, g(0) = g(2\pi)\}$, $A^* g = -ig'$ for all $g \in D(A^*)$.

(c) Prove that A is essentially self-adjoint.

(d) Prove that A_0 has no eigenvalues.

(e) Prove that A admits an orthonormal basis of eigenvectors.

(f) Let $\lambda \in \mathbb{S}^1 := \{z \in \mathbb{C} \,|\, |z| = 1\}$. Prove that the operator A_λ given by

$$D(A_\lambda) = \{g \in D(A_0^*) \,|\, g(0) = \lambda g(2\pi)\} \quad \text{and} \quad A_\lambda g = -ig'$$

is a self-adjoint extension of A_0.

(g) Prove that the family of operators $(A_\lambda)_{\lambda \in \mathbb{S}^1}$ consists of *all* the self-adjoint extensions of A_0, i.e. if $B \supset A_0$ is a self-adjoint extension of A_0, then already $B = A_\lambda$ for some $\lambda \in \mathbb{S}^1$.
[Hint: Take a close look at the proof of Thm 3.14 from the lecture.]