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Winter term 2016/17
January 19, 2016

Functional Analysis II
Assignment 12

Problem 45. *(Unbounded multiplication operators)*

Let X be a metric space and μ a positive measure on the Borel σ-algebra of X such that $\mu(\Lambda) < \infty$ for any bounded Borel set $\Lambda \subset X$. For a (possibly unbounded) measurable function $f : X \to \mathbb{C}$ consider the linear map M_f in $L^2(X, \mu)$ defined by

$$\begin{aligned} \mathcal{D}(M_f) &:= \left\{ \varphi \in L^2(X,\mu) \,\middle|\, f\varphi \in L^2(X,\mu) \right\} \\ M_f \varphi &:= f\varphi \,. \end{aligned}$$

Prove:

(a) $\mathcal{D}(M_f)$ is dense in $L^2(X, \mu)$.

(b) $(M_f)^* = M_{\overline{f}}$.

(c) $\sigma(M_f) = \operatorname{essran} f = \{\lambda \in \mathbb{C} \,|\, \forall \varepsilon > 0 : \mu(\{x \in X \,|\, |\lambda - f(x)| < \varepsilon\}) > 0\}$.

(d) λ is an eigenvalue of M_f iff $\mu(f^{-1}(\{\lambda\})) > 0$.

(e) Let $X = \mathbb{R}$, let μ be Lebesgue measure on $\mathbb{R}$, and let $f(x) := x \ \forall x \in \mathbb{R}$. Then the position operator $q := M_f$ is self-adjoint, has no eigenvalues, and $\sigma(q) = \mathbb{R}$.

Problem 46. *(Properties of the adjoint)*

Let A and B be densely defined operators on a Hilbert space $\mathcal{H}$. Prove:

(a) $(\alpha A)^* = \overline{\alpha} A^* \quad \forall \alpha \in \mathbb{C}$.

(b) If $\mathcal{D}(A+B) = \mathcal{D}(A) \cap \mathcal{D}(B)$ and $\mathcal{D}(A^*+B^*) = \mathcal{D}(A^*) \cap \mathcal{D}(B^*)$ are dense in $\mathcal{H}$, then $(A+B)^* \supset A^*+B^*$.

(c) If $\mathcal{D}(AB)$ is dense, then $(AB)^* \supset B^* A^*$.

(d) If $A \subset B$, then $A^* \supset B^*$.

(e) If A is self-adjoint, then A has no symmetric extensions.

(f) $N(A^*) = R(A)^\perp$.

Problem 47. *(Cyclic Vectors II)*

Consider the self-adjoint operators A, B in $L^2([-1,1])$ discussed in Problem 44, i.e. $Af(x) = xf(x)$ and $Bf(x) = x^2 f(x)$. Prove:

(a) $L^2([-1,1]) \cong \mathcal{H}_1 \oplus \mathcal{H}_2$, where $\mathcal{H}_1$ and $\mathcal{H}_2$ are Hilbert spaces on which B has a cyclic vector.

(b) $f \in L^2([-1,1])$ is a cyclic vector for A iff $f(x) \neq 0$ almost everywhere.
[Hint: For $\Leftarrow$*, it can be helpful to prove* $\{\phi f : \phi \in M_b([-1;1])\} \subset \overline{span\{A^n f, n \geq 0\}}$*.]*

Problem 48. *(Von Neumann's Theorem)*

(a) Let A be a symmetric operator on a Hilbert space $\mathcal{H}$ and assume there exists a map $C : \mathcal{H} \to \mathcal{H}$ with the following properties:

  (i) C is anti-linear (i.e. $C(\alpha x + y) = \overline{\alpha} C(x) + C(y)$).

  (ii) C is norm-preserving.

  (iii) $C^2 = \mathbb{I}$.

  (iv) $\mathcal{D}(A)$ is invariant under C.

  (v) $AC = CA$ on $\mathcal{D}(A)$.

  [Remark: A map satisfying $(i) - (iii)$ *is called a conjugation.]*

  Prove that A has self-adjoint extensions.

(b) Consider the operator H in $L^2(\mathbb{R}^d)$ given by

$$\begin{aligned} \mathcal{D}(H) &= C_0^\infty(\mathbb{R}^d) \\ (H\psi)(x) &= -\Delta\psi(x) + V(x)\psi(x) \quad \text{for a.e. } x \in \mathbb{R}^d, \end{aligned}$$

where $\Delta = \sum_{j=1}^d \partial_j^2$ and $V \in L^2_{loc}(\mathbb{R}^d)$ is real-valued. Show that H is symmetric and has at least one self-adjoint extension.

This sheet is to be discussed in the exercise class on Thursday, January 26.
For more details please visit http://www.math.lmu.de/~tkoenig/16FA2exercises.php