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Functional Analysis II
Assignment 11

Problem 41. *(Operator convex functions)*

Let $\mathcal{H}$ be a Hilbert space. A continuous real-valued function f defined on an interval $I \subset \mathbb{R}$ is called *operator convex* (on I) if for any pair of self-adjoint operators $A, B \in \mathcal{B}(\mathcal{H})$ with $\sigma(A), \sigma(B) \subset I$ and any $\lambda \in [0,1]$ the inequality $f(\lambda A + (1-\lambda)B) \leqslant \lambda f(A) + (1-\lambda) f(B)$ holds in the operator sense. Prove:

(a) A continuous real-valued function f is operator convex iff $f(\frac{A+B}{2}) \leqslant \frac{1}{2}(f(A)+f(B))$ for any pair $A, B \in \mathcal{B}(\mathcal{H})$ of self-adjoint operators with spectrum in I.

(b) $f : \mathbb{R} \to \mathbb{R}, t \mapsto t^2$ is operator convex on every interval.

(c) $f : [0,\infty) \to \mathbb{R}, t \mapsto t^3$ is *not* operator convex on $[0,\infty)$.

(d) $f : \mathbb{R} \to \mathbb{R}, t \mapsto |t|$ is *not* operator convex on any interval that contains a neighbourhood of zero.

(e) $f : (0,\infty) \to \mathbb{R}, t \mapsto t^{-1}$ is operator convex on $(0,\infty)$.

Problem 42. *(A unitary group of operators)*

Let $\mathcal{H}$ be a Hilbert space and $A \in \mathcal{B}(\mathcal{H})$ be self-adjoint. Prove:

(a) The operator $U(t) := e^{itA}$ constructed via the functional calculus is unitary for all $t \in \mathbb{R}$, and

$$U(t)^* = U(-t)\,, \qquad U(t)U(s) = U(t+s) \quad \forall t, s \in \mathbb{R}\,.$$

(b) The operator-valued function $t \mapsto U(t)$ defined in (a) is a differentiable map between the normed spaces $\mathbb{R}$ and $\mathcal{B}(\mathcal{H})$ with derivative $U'(t) = iAU(t)$ for all $t \in \mathbb{R}$.

(c) For $\lambda \notin \sigma(A)$ we have $\|(A-\lambda\mathbb{I})^{-1}\| = \operatorname{dist}(\lambda, \sigma(A))^{-1}$.

Problem 43. *('Diagonalizing' an integral operator)*

Let A be the integral operator on $L^2([0,1])$ given by

$$Af(x) = \int_0^1 \min(x,y)\, f(y)\, dy\,.$$

(a) Prove that A is bounded and self-adjoint.

(b) Find a measure space (M,μ), an isomorphism $U : L^2([0,1]) \to L^2(M,\mu)$, and a bounded measurable function $F : M \to \mathbb{R}$ such that $UAU^* : L^2(M,\mu) \to L^2(M,\mu)$ is the operator of multiplication by F.

Problem 44. *(Cyclic vectors I)*

(a) An $N \times N$ Hermitian matrix has a cyclic vector iff its eigenvalues are all distinct.

(b) Consider the self-adjoint operators A, B on $L^2([-1,1])$, where A is multiplication by $x \mapsto x$ and B is multiplication by $x \mapsto x^2$. Prove:

   (*i*) $f : [-1,1] \to \mathbb{R}, x \mapsto 1$ is a cyclic vector of A.

   (*ii*) The characteristic function $\chi_{[0,1]}$ is not a cyclic vector of A.

   (*iii*) B does not have any cyclic vectors.

This sheet is to be discussed in the exercise class on Thursday, January 19.
For more details please visit http://www.math.lmu.de/~tkoenig/16FA2exercises.php