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Functional Analysis II

Assignment 9

Problem 33. *(Operator monotonicity of the inverse)*

Let $\mathcal{H}$ be a Hilbert space and let $A, B \in \mathcal{B}(\mathcal{H})$ be self-adjoint. Assume that $\mathbb{O} \leqslant A \leqslant B$ and $\lambda > 0$. Prove:

(a) $A+\lambda\mathbb{I}$ and $B+\lambda\mathbb{I}$ are invertible, and $(B+\lambda\mathbb{I})^{-1} \leqslant (A+\lambda\mathbb{I})^{-1}$ for all $\lambda > 0$.

(b) If A is invertible then B is invertible too, and $B^{-1} \leqslant A^{-1}$.

[Remark: The result from (b) can be rephrased by saying that the function $t \to -t^{-1}$ is operator monotone on the interval $(0, \infty)$. More on this topic in Assignment 10.]

Problem 34. *(Stone's formula)*

Let $\mathcal{H}$ be a Hilbert space and let $A \in \mathcal{B}(\mathcal{H})$ be self-adjoint. Prove for $a, b \in \mathbb{R}$, $a < b$, that

$$\frac{1}{\pi i} \int_a^b \Big((A-\lambda-i\varepsilon)^{-1} - (A-\lambda+i\varepsilon)^{-1} \Big)\, d\lambda \;\longrightarrow\; \chi_{[a,b]}(A) + \chi_{(a,b)}(A)$$

strongly, as $\varepsilon \to 0^+$. Both sides are defined via the measurable functional calculus.

Problem 35. *(Discrete Laplacian)*

Let $-\Delta : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ be given by

$$(-\Delta x)_n := \sum_{\substack{m\in\mathbb{Z},\\ |m-n|=1}} (x_n - x_m) \quad \forall n \in \mathbb{Z}.$$

(a) Show that $-\Delta$ is bounded and self-adjoint.

(b) Show that $\mathbb{O} \leqslant -\Delta \leqslant 4\mathbb{I}$

(c) Compute $\|-\Delta\|$.

(d) Determine $\sigma(-\Delta)$. *[Hint: Construct appropriate Weyl sequences with the help of the sequences $(x_n = e^{in\alpha})_{n\in\mathbb{Z}}$, where $\alpha \in \mathbb{R}$.]*

(e) Find a measure space $(\mathcal{M}, \mu)$, an isomorphism $U : \ell^2(\mathbb{Z}) \to L^2(\mathcal{M}, \mu)$, and a function $F : \mathcal{M} \to \mathbb{R}$ such that $U(-\Delta)U^{-1}$ is the operator of multiplication by F.

Problem 36. *(A 'spectral puzzle')*

Let $A \in \mathcal{B}(L^2([0,1]))$ be such that $A \geqslant 0$ and

$$A^{2016} e^A f(x) = e f(x) + e \int_0^x f(y)\, dy\,,$$

where $e = \exp(1)$. Find $\sigma(A)$.