LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN

MATHEMATISCHES INSTITUT

Prof. T. Ø. Sørensen PhD
T. König

Winter term 2016/17
December 8, 2016

Functional Analysis II

Assignment 8

Problem 29. *(The $\leqslant$-relation for a self-adjoint operator)*

Let $\mathcal{H}$ be a Hilbert space and $A = A^* \in \mathcal{B}(\mathcal{H})$. Prove:

(a) $A \leqslant \|A\| \, \mathbb{I}$.

(b) If $A \geqslant \mathbb{O}$ then $\sigma(A) \subset [0, \|A\|]$.

(c) If $\sigma(A) \subset [0, R]$ for some $R > 0$, then $\mathbb{O} \leqslant A \leqslant R \, \mathbb{I}$.

Problem 30. *(The $\leqslant$-relation for two self-adjoint operators)*

Let $\mathcal{H}$ be a Hilbert space and let $A, B \in \mathcal{B}(\mathcal{H})$ be self-adjoint. Prove:

(a) If $A \leqslant B$ then $C^*AC \leqslant C^*BC$ for all $C \in \mathcal{B}(\mathcal{H})$.

(b) If $\mathbb{O} \leqslant A \leqslant B$ then $\|A\| \leqslant \|B\|$.

(c) If $A \geqslant \mathbb{O}$, then A is invertible iff $A \geqslant c\mathbb{I}$ for some $c > 0$.

Problem 31. *(Commuting operators under measurable functional calculus)*

Let $\mathcal{H}$ be a Hilbert space, let $S, T \in \mathcal{B}(\mathcal{H})$ be self-adjoint, and assume that $TS = ST$. Prove for any bounded Borel function $f \in \mathcal{M}_b(\sigma(T))$ that $f(T)S = Sf(T)$.

Problem 32. *(Some properties of normal operators)*

Let $\mathcal{H}$ be a Hilbert space and let $T \in \mathcal{B}(\mathcal{H})$ be normal. Prove:

(a) $N(T) = N(T^*)$.

(b) $\overline{R(T)} = \overline{R(T^*)}$, and if $R(T)$ is closed then $R(T^*) = R(T)$.

(c) If T has a bounded one-sided inverse, then T is invertible.

This sheet is to be discussed in the exercise class on Thursday, December 15.
For more details please visit http://www.math.lmu.de/~tkoenig/16FA2exercises.php