LUDWIG-
MAXIMILIANS-
UNIVERSITÄT
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MATHEMATISCHES INSTITUT

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Winter term 2016/17
December 1, 2016

Functional Analysis II
Assignment 7

Problem 25. *(Decomposition into sums of self-adjoint resp. unitary operators)*

Let $\mathcal{H}$ be a complex Hilbert space and let $A \in \mathcal{B}(\mathcal{H})$. Prove:

(a) There exist unique self-adjoint operators $R_A, I_A \in \mathcal{B}(\mathcal{H})$ such that $A = R_A + iI_A$.

(b) A is normal iff $[R_A, I_A] := R_A I_A - I_A R_A = \mathbb{O}$.

(c) A is unitary iff A is normal and $R_A^2 + I_A^2 = \mathbb{I}$.

(d) If $T = T^*$ and $\|T\| \leqslant 1$, then $U := T + i\sqrt{\mathbb{I} - T^2}$ is unitary and $T = \frac{1}{2}(U + U^*)$.

(e) There exist unitary operators $U_1, \ldots, U_4$ and $a_1, \ldots, a_4 \in \mathbb{C}$ such that

$$A = a_1 U_1 + a_2 U_2 + a_3 U_3 + a_4 U_4\,,$$

and $|a_j| = \|A\|/2$ for all $j = 1, 2, 3, 4$.

Problem 26. *(Weyl sequences II)*

Let $\mathcal{H}$ be a Hilbert space and let $T \in \mathcal{B}(\mathcal{H})$. Prove:

(a) If $\lambda \in \sigma(T)$ then there exists a Weyl sequence for T at λ *or* for T^* at $\overline{\lambda}$.

(b) If T is normal , then $\lambda \in \sigma(T)$ iff T has a Weyl sequence at λ.

(c) If T is self-adjoint and λ is an isolated point in $\sigma(T)$ then λ is an eigenvalue of T.

Problem 27. *(Min-max principle for compact operators)*

Let A be a positive compact self-adjoint operator on a Hilbert space $\mathcal{H}$. For $n \in \mathbb{N}$, let λ_n denote its eigenvalues labeled such that they may be repeated due to multiplicity and

$$0 \leqslant \cdots \leqslant \lambda_2 \leqslant \lambda_1.$$

Prove that for each $n \in \mathbb{N}$,

$$\lambda_n = \inf_{V \in \mathcal{H}_{n-1}} \sup_{\substack{x \perp V \\ \|x\|=1}} \langle x, Ax \rangle$$

where $\mathcal{H}_{n-1}$ is the collection of all $(n-1)$-dimensional subspaces of $\mathcal{H}$.

If A has also negative eigenvalues, how can one obtain those via a similar variational formula?

Problem 28. *(Volterra integral operator II)*

Let $V : L^2([0,1]) \to L^2([0,1])$ be the Volterra integral operator introduced in Problem 20, i.e. $Vf(x) = \int_0^x f(y)\,dy$.

(a) Show that if $f \in L^2([0,1])$ is an eigenfunction of the operator V^*V with eigenvalue λ, then $\lambda > 0$, f is twice differentiable a.e., and $\lambda f'' + f = 0$ a.e. in $[0,1]$.

(b) Find explicitly the collections $\{\lambda_n\}_{n=1}^\infty$ and $\{f_n\}_{n=1}^\infty$ of all eigenvalues and corresponding eigenfunctions of V^*V. Prove that $\{f_n\}_{n=1}^\infty$ is (up to normalization) an ONB in $L^2([0,1])$ without using the spectral theorem for compact operators.

[Hint: You may use without proof that $\{e_k\}_{k\in\mathbb{Z}}$ with $e_k(x) = e^{2\pi ikx}$ forms an ONB of $L^2([0,1])$. Try to find a unitary map $L^2 \to L^2$ which maps $\{e_k\}$ to $\{f_n\}$.]

(c) Deduce from (b) that $\|V\| = \frac{2}{\pi}$.

This sheet is to be discussed in the exercise class on Thursday, December 8.
For more details please visit http://www.math.lmu.de/∼tkoenig/16FA2exercises.php