Prof. T. Ø. Sørensen PhD
T. König

Winter term 2016/17
November 24, 2016

Functional Analysis II
Assignment 6

Problem 21. *(Hilbert-Schmidt operators)*

Let $d \geqslant 1$ and $k \in L^2(\mathbb{R}^d \times \mathbb{R}^d)$. For $f \in L^2(\mathbb{R}^d)$ let

$$Tf(x) := \int_{\mathbb{R}^d} k(x,y) f(y)\, dy \qquad \text{for a.e. } x \in \mathbb{R}^d. \tag{1}$$

(a) Prove that this defines $T \in \mathcal{B}(L^2(\mathbb{R}^d))$ and find an upper bound for $\|T\|$.

(b) Prove that T is compact.

(c) Prove for any orthonormal basis $\{\varphi_n\}_{n=1}^\infty$ of $L^2(\mathbb{R}^d)$ that $\sum_{n=1}^\infty \|T\varphi_n\|_2^2 = \|k\|_2^2$.

(d) Prove that $\dim N(T-I) \leqslant \|k\|_2^2$.

[Remark: Operators with the property that $\sum_{n=1}^\infty \|T\varphi_n\|_2^2 < \infty$ for any orthonormal basis $\{\varphi_n\}_{n=1}^\infty$ of $L^2(\mathbb{R}^d)$ are called Hilbert-Schmidt operators. They form an important class of compact operators. It can be shown that every Hilbert-Schmidt operator on $L^2(\mathbb{R}^d)$ is in fact of the form (1) for some $k \in L^2(\mathbb{R}^d \times \mathbb{R}^d)$.]

Problem 22. *(A non-compact integral operator)*

Consider the so-called *Abel integral operator* $A : C([0,1]) \to C([0,1])$ given by

$$Af(t) = \begin{cases} \int_0^t \frac{f(s)}{\sqrt{t^2-s^2}} ds & \text{for } t \in (0,1], \\ \frac{\pi}{2} f(0) & \text{for } t = 0. \end{cases}$$

(a) Prove that A is well-defined and bounded.

(b) Prove that A is not compact. (Compare with Problem 3 on the warm-up sheet!) *[Hint: For $\alpha > 0$, consider the functions $f_\alpha(t) = t^\alpha$.]*

Problem 23. *(No triangle inequality for operators)*

(a) Show that $|A+B| \leqslant |A| + |B|$ is *not* true for arbitrary compact operators A and B.

(b) Prove for compact operators A, B on a Hilbert space $\mathcal{H}$ that $\frac{1}{2}|A+B|^2 \leqslant |A|^2 + |B|^2$.

Problem 24. *(Perturbation of the spectrum by compact operators)*

(a) Let X be a Banach space and let $S, T \in \mathcal{B}(X)$ be such that $T-S$ is compact. Prove that $\sigma(T) \setminus \sigma_p(T) \subset \sigma(S)$. [*Hint: Fredholm Alternative.*]

(b) Let $X = X_1 \oplus X_2$ be a Banach space which is the direct sum of two closed subspaces X_1 and X_2. For $i = 1, 2$, let operators $A_i \in \mathcal{B}(X_i)$ be given and define the direct sum operator $A = A_1 \oplus A_2$ by $A((x_1, x_2)) := (A_1 x_1, A_2 x_2)$.
Prove that $\sigma(A) = \sigma(A_1) \cup \sigma(A_2)$.

(c) Let $\mathcal{H}$ be a Hilbert space and let $U \in \mathcal{B}(\mathcal{H})$ be a unitary operator. Prove that

$$\sigma(U) \subset \{\lambda \in \mathbb{C} : |\lambda| = 1\}.$$

(d) The fact proved in (a) does not exclude that the two spectra may look considerably different. As an example, find a bounded operator A and a compact operator K on a Hilbert space $\mathcal{H}$ such that

$$\sigma(A) \subset \{\lambda \in \mathbb{C} : |\lambda| = 1\}, \quad \sigma(A+K) = \{\lambda \in \mathbb{C} : |\lambda| \leqslant 1\}.$$

[*Hint: One way is to consider shift operators on $\ell^2(\mathbb{Z})$. The results from (b) and (c) can be helpful.*]

This sheet is to be discussed in the exercise class on Thursday, December 1.
For more details please visit http://www.math.lmu.de/~tkoenig/16FA2exercises.php