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Functional Analysis II
Assignment 5

Problem 17. *(Spectral mapping theorem for polynomials)*

Let X be a Banach space and $T \in \mathcal{B}(X)$. Prove for any polynomial p on $\mathbb{C}$ of degree $n \geqslant 1$ that

$$\sigma(p(T)) = p(\sigma(T)) .$$

Problem 18. *(Square root of positive semidefinite operators - 'by hand')*

Let $\mathcal{H}$ be a Hilbert space and let $T \in \mathcal{B}(\mathcal{H})$ be positive semidefinite. If T were compact then the Spectral Theorem for compact operators would allow to construct the square root of T in a straightforward way (see lecture). Even though we do not have the Spectral Theorem for self-adjoint operators at our disposal yet, we can still construct $\sqrt{T}$ in this case from scratch, as will be done in this exercise. Prove:

(a) The power series $\sqrt{1-x} = \sum_{n=0}^{\infty} c_n x^n$ converges absolutely for $|x| \leqslant 1$, where

$$c_n = \frac{1}{n!} \frac{d^n}{dx^n}\bigg|_{x=0} \sqrt{1-x} .$$

(b) The series $S := \sqrt{\|T\|} \sum_{n=0}^{\infty} c_n \left(I - \frac{1}{\|T\|} T\right)^n$ converges in $\mathcal{B}(\mathcal{H})$, $S \geqslant 0$, and $S^2 = T$.

(c) The operator $S \in \mathcal{B}(\mathcal{H})$ with the properties S self-adjoint, $S \geqslant 0$ and $S^2 = T$ is unique.

Problem 19. *(Norm-preserving linear maps on a Hilbert space)*

Let $\mathcal{H}$ be a Hilbert space and $U \in \mathcal{B}(\mathcal{H})$. Recall that U is called an *isometry* if $\|Ux\| = \|x\|$ for all $x \in \mathcal{H}$, and U is called *unitary* if U is a surjective isometry. Moreover, U is called a *partial isometry* if $\|Ux\| = \|x\|$ for all $x \in N(U)^{\perp}$. Prove:

(a) U is unitary iff $U^*U = UU^* = I$.

(b) U is an isometry iff $U^*U = I$.

(c) If $U \neq 0$ is a partial isometry then $R(U)$ is closed and $\|U\| = 1$.

(d) The adjoint of a partial isometry is again a partial isometry.

(e) U is a partial isometry iff U^*U is an orthogonal projection.

(f) U is a partial isometry iff $U = UU^*U$.

Problem 20. *(Volterra integral operator)*

Let $V : L^2([0,1]) \to L^2([0,1])$ be given by

$$(Vf)(x) := \int_0^x f(y)\, dy \,.$$

Prove the following:

(a) V is a well-defined, bounded operator in $L^2([0,1])$.

(b) V is compact.

(c) $\sigma(V) = \sigma_c(V) = \{0\}$.

(d) $V+V^*$ is an orthogonal projection with $\dim R(V+V^*) = 1$.

This sheet is to be discussed in the exercise class on Thursday, November 24.
For more details please visit http://www.math.lmu.de/~tkoenig/16FA2exercises.php