LMU LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN
MATHEMATISCHES INSTITUT

Prof. T. Ø. Sørensen PhD
T. König

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Functional Analysis II

Assignment 4

Problem 13. *(Spectrum of the product)*

Let X be a Banach space and $S, T \in \mathcal{B}(X)$.

(a) Prove that $\sigma(TS) \cup \{0\} = \sigma(ST) \cup \{0\}$.
[*Hint: Given* $0 \neq \lambda \in \rho(ST)$, *formally (!) expand* $(TS - \lambda)^{-1}$ *into a geometric series to express* $(TS - \lambda)^{-1}$ *in terms of* λ, S, T *and* $(ST - \lambda)^{-1}$.]

(b) Show that $\sigma(TS) = \sigma(ST)$ is not true in general.

Problem 14. *(Spectrum of self-adjoint operators)*

Let A be a bounded self-adjoint operator on a Hilbert space $\mathcal{H}$, i.e. $A^* = A$. Prove the following:

(a) $\sigma(A) \subset [m, M] \subset \mathbb{R}$, where $m = \inf_{x \in \mathcal{H}, \|x\|=1} \langle x, Ax \rangle$, $\quad M = \sup_{x \in \mathcal{H}, \|x\|=1} \langle x, Ax \rangle$.

(b) $m, M \in \sigma(A)$.

(c) $\sigma_r(A) = \emptyset$.

(d) If $x, y \in \mathcal{H}$ and $\lambda \neq \mu$ are such that $Ax = \lambda x$ and $Ay = \mu y$ then $\langle x, y \rangle = 0$.

(e) If $\sigma(A) = \{0\}$, then $A = \mathbb{O}$.

Problem 15. *(Weyl sequences)*

Let X be a Banach space and $T \in \mathcal{B}(X)$. A sequence $(x_n)_{n \in \mathbb{N}}$ in X is called a *Weyl sequence* of T at $\lambda \in \mathbb{C}$, if $\|x_n\| = 1$ for all $n \in \mathbb{N}$ and $\|Tx_n - \lambda x_n\| \to 0$ as $n \to \infty$. Prove:

(a) If T has a Weyl sequence at $\lambda \in \mathbb{C}$ then $\lambda \in \sigma(T)$.

(b) If $\lambda \in \partial\sigma(T)$ then T has a Weyl sequence at $\lambda \in \mathbb{C}$.

Now let $\mathcal{H}$ be a Hilbert space and let $T \in \mathcal{B}(\mathcal{H})$ be self-adjoint.

(c) Prove that T has a Weyl sequence at λ iff $\lambda \in \sigma(T)$.

Problem 16. *(Multiplication operators on general measure spaces)*

Let (X, μ) be a σ-finite measure space, let $1 \leqslant p < \infty$, and for a measurable function $h : X \to \mathbb{C}$ let

$$D_h := \{f \in L^p(X, \mu) : hf \in L^p(X, \mu)\}.$$

Let $M_h : D_h \to L^p(X, \mu)$, $f \mapsto hf$.

(a) Prove that $M_h \in \mathcal{B}(L^p(X, \mu))$ iff $h \in L^\infty(X, \mu)$.

Assuming $h \in L^\infty(X, \mu)$ prove the following:

(b) $\sigma_p(M_h) = \big\{\lambda \in \mathbb{C} : \mu(\{x \in X : h(x) = \lambda\}) > 0\big\}$.

(c) $\rho(M_h) = \big\{\lambda \in \mathbb{C} : \exists c > 0 \text{ such that } |\lambda - h(x)| \geqslant c \;\; \mu\text{-a.e.}\big\}$.

This sheet is to be discussed in the exercise class on Thursday, November 17.
For more details please visit http://www.math.lmu.de/~tkoenig/16FA2exercises.php