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UNIVERSITÄT
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MATHEMATISCHES INSTITUT

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# Functional Analysis II

## Assignment 3

**Problem 9.** *(Shift operator)*

Consider the *left shift operator* $T : \ell^1(\mathbb{N}) \to \ell^1(\mathbb{N})$, $(x_1, x_2, x_3, \dots) \mapsto (x_2, x_3, x_4, \dots)$.

(a) Prove that $T \in \mathcal{B}(\ell^1(\mathbb{N}))$ and determine $\|T\|$.

(b) Find the adjoint $T'$ (domain and action).

(c) Determine the spectra, point spectra and the residual spectra of $T$ and $T'$.

**Problem 10.** *(Existence and continuity of projections)*

(a) Show that for every linear subspace $W \subset V$ of a linear space $V$ there exists a projection $P : V \to V$ with $R(P) = W$. [*Hint: Zorn's Lemma.*]

(b) Find a normed space $X$ and a projection $P : X \to X$ that is not continuous.

(c) Suppose that $X$ is a Banach space and that $P : X \to X$ is a projection such that $R(P)$ and $N(P)$ are closed. Prove that $P$ is continuous.

**Problem 11.** *(Complemented subspaces, codimension of a subspace)*

Let $X$ be a Banach space. A closed subspace $Y \subset X$ is called a *complemented subspace of* $X$ (or *complemented in* $X$) if there exists a closed subspace $Z \subset X$ such that $X = Y \oplus Z$.

(a) Prove that every subspace $Y \subset X$ of finite dimension is complemented in $X$.
[*Hint: Consider suitable functionals defined on $Y$ and use the Hahn-Banach theorem to extend them to elements of $X'$.*]

Define the codimension of a subspace $Y \subset X$ to be the dimension of the quotient space $X/Y$ (possibly $\infty$).

(b) Prove that every closed subspace $Y \subset X$ of finite codimension is complemented in $X$. If $Z \subset X$ is a closed subspace such that $X = Y \oplus Z$, prove that $\dim Z = \dim X/Y$.

*[Remark: This shows that the codimension of a closed subspace with finite-dimensional complement, as it was introduced in the lecture, is indeed well-defined.]*

**Problem 12.** *(Compact vs. closed range)*

Let $X$ and $Y$ be Banach spaces and let $T : X \to Y$ be a compact operator. The goal of this exercise is to draw and illustrate two useful consequences of the fact that $R(T)$ cannot be closed unless $\dim R(T) < \infty$ (Problem 2.d).

(a) Assume that $\dim X = \infty$.

  (*i*) Prove that there cannot exist $c > 0$ such that $\|Tx\|_Y \geq c\|x\|_X$ for all $x \in X$.

  (*ii*) If $T$ is injective, construct a non-injective compact operator arbitrarily close in norm to $T$.

(b) Assume that $\dim Y = \infty$.

  (*i*) Prove that $T$ cannot be surjective.

  (*ii*) Let $1 \leq p \leq \infty$ and $T : \ell^p(\mathbb{N}) \to \ell^p(\mathbb{N})$, $x \mapsto (x_1, \frac{x_2}{2}, \frac{x_3}{3}, \dots)$. Find $y \in \ell^p(\mathbb{N})$ such that $Tx = y$ has no solution $x \in \ell^p(\mathbb{N})$.

*This sheet is to be discussed in the exercise class on Thursday, November 10.*
*For more details please visit* http://www.math.lmu.de/~tkoenig/16FA2exercises.php