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Functional Analysis II

Assignment 2

Problem 5. *(Orthogonal projections and their spectrum)*

Let V be a vector space and let P be a *projection* on V, that is, a linear map $P : V \to V$ such that $P^2 = P$. Prove:

(a) $R(P) = N(I - P)$.

(b) $V = R(P) \oplus N(P)$, where $\oplus$ denotes the direct sum.

Let $\mathcal{H}$ be a Hilbert space. A projection $P : \mathcal{H} \to \mathcal{H}$ is called *orthogonal* if $\mathrm{R}(P) \perp \mathrm{N}(P)$.

(c) Let $P : \mathcal{H} \to \mathcal{H}$ be a projection. Prove that P is orthogonal iff $P \in \mathcal{B}(\mathcal{H})$ and $P^* = P$.

(d) Let A be a linear subspace of $\mathcal{H}$. Show that there exists a unique orthogonal projection $P_A : \mathcal{H} \to \mathcal{H}$ with $R(P_A) = \overline{A}$. [*Hint: Projection Theorem.*]

Let $P : \mathcal{H} \to \mathcal{H}$ be a non-trivial orthogonal projection (i.e. $R(P) \neq \mathcal{H}, N(P) \neq \mathcal{H}$).

(e) Prove that $\sigma_p(P) = \sigma(P) = \{0, 1\}$.
[*Hint: Find an explicit expression for* $(P - \lambda I)^{-1}$ *whenever* $\lambda \in \mathbb{C} \setminus \{0, 1\}$.]

Problem 6. *(Multiplication operators acting on a sequence space)*

For $w \in \ell^\infty(\mathbb{N})$ let $T_w : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ be the componentwise multiplication by $w = (w_1, w_2, \dots)$, i.e.

$$T_w x := (w_1 x_1, w_2 x_2, \dots)$$

(a) Prove that T_w is bounded and calculate its norm.

(b) Find the explicit action of the adjoint T_w^*.

(c) Characterize the sequences $w \in \ell^\infty(\mathbb{N})$ for which

  (*i*) $T_w^* T_w = T_w T_w^*$ (such operators are called *normal*).

  (*ii*) $T_w = T_w^*$.

  (*iii*) T_w is compact.

(d) Determine $\sigma_p(T_w)$ and prove that $\overline{\sigma_p(T_w)} = \sigma(T_w)$.

Problem 7. *(Spectrum of the inverse operator)*

Let X be a Banach space and let $T \in \mathcal{B}(X)$ be bijective. Prove:

(a) $\sigma(T^{-1}) = \frac{1}{\sigma(T)} := \{\lambda^{-1} \in \mathbb{C} \,|\, \lambda \in \sigma(T)\}$.

(b) If $Tx = \lambda x$ for some $\lambda \neq 0$ and $x \in X$, then $T^{-1}x = \lambda^{-1}x$.

Problem 8. *(Resolvent formulas, power series expansions of the resolvent map)*

Let X be a Banach space, let $T \in \mathcal{B}(X)$, let $\rho(T) \subset \mathbb{C}$ be the resolvent set of T and for $\lambda \in \rho(T)$ let $R_\lambda(T) = (T - \lambda I)^{-1}$ be the resolvent of T at λ.

(a) Prove the following two useful identities, also known under the names of *first* resp. *second resolvent formula*:

(i) $R_\lambda(T) - R_\mu(T) = (\lambda - \mu)\, R_\lambda(T)\, R_\mu(T)$ for all $\lambda, \mu \in \rho(T)$.

(ii) $R_\lambda(T) - R_\lambda(S) = R_\lambda(T)\,(S - T)\, R_\lambda(S)$ for all $S \in \mathcal{B}(X)$ and $\lambda \in \rho(T) \cap \rho(S)$.

(b) Using the Neumann series known from the lecture, prove the following power series expansions for the resolvent map $\rho(T) \to \mathcal{B}(X), \lambda \mapsto R_\lambda(T)$:

(i) If $\lambda \in \mathbb{C}$ is such that $|\lambda - \lambda_0| < \|R_{\lambda_0}(T)\|^{-1}$ for some $\lambda_0 \in \rho(T)$, then $\lambda \in \rho(T)$ and

$$R_\lambda(T) = \sum_{n=0}^{\infty} (\lambda - \lambda_0)^n R_{\lambda_0}(T)^{n+1}\,.$$

(ii) $R_\lambda(T) = -\sum_{n=0}^{\infty} \lambda^{-1-n}\, T^n$ for $|\lambda| > \|T\|$.

(c) Use the previous results to prove the following facts about $R_\lambda(T)$:

(i) $\|R_\lambda(T)\| \geqslant (\mathrm{dist}(\lambda, \sigma(T)))^{-1}$ for all $\lambda \in \rho(T)$.

(ii) The resolvent map $\lambda \mapsto R_\lambda(T)$ is continuous.

(iii) The resolvent map $\lambda \mapsto R_\lambda(T)$ has a complex derivative, in the sense that

$$\frac{d}{d\lambda} R_\lambda(T) := \lim_{h \to 0, h \in \mathbb{C}} \frac{1}{h}\big(R_{\lambda+h}(T) - R_\lambda(T)\big)$$

exists in $\mathcal{B}(X)$. In fact, $\frac{d}{d\lambda} R_\lambda(T) = R_\lambda(T)^2$.

This sheet is to be discussed in the exercise class on Thursday, November 3.
For more details please visit http://www.math.lmu.de/~tkoenig/16FA2exercises.php