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Functional Analysis II
Assignment 1

Problem 1. *(Examples of compact and non-compact operators)*

Let X and Y be normed spaces, and let $C([0,1])$ be equipped with $\|\cdot\|_\infty$. Decide which of the following operators are compact:

(a) $T : C([0,1]) \to C([0,1]), \quad Tf(x) = f(0) + xf(1)$.

(b) $\mathrm{id} : X \to X$, $x \mapsto x$.

(c) $F \in \mathcal{B}(X,Y)$ with $\dim \operatorname{Ran}(F) < \infty$ (such F are called *finite-rank* operators).

Problem 2. *(Some properties of compact operators)*

Let X, Y and Z be Banach spaces. Prove the following statements:

(a) $\mathcal{K}(X,Y)$ is a closed subspace of $\mathcal{B}(X,Y)$.

(b) For $A \in \mathcal{B}(X,Y)$ and $B \in \mathcal{B}(Y,Z)$, we have $BA \in \mathcal{K}(X,Z)$ if A or B is compact.

(c) If $\dim X = \infty$ and $T \in \mathcal{K}(X)$, then $0 \in \sigma(T)$.

(d) If $T \in \mathcal{K}(X,Y)$, then $\operatorname{Ran}(T)$ is closed if and only if T is a finite-rank operator. [*Hint: For '$\Rightarrow$', use the open mapping theorem and the fact that the unit ball is compact in finite dimensions only.*]

Problem 3. *(Approximating compact operators by finite-rank operators)*

Let $\mathcal{H}$ be a separable Hilbert space, let $\{\varphi_n\}_{n\in\mathbb{N}}$ be an orthonormal basis of $\mathcal{H}$, and let P_N be the orthogonal projection onto the span of $\{\varphi_1, \ldots, \varphi_N\}$. Prove for $T \in \mathcal{K}(\mathcal{H})$ that

$$T \circ P_N \xrightarrow{|||\cdot|||} T \quad \text{in operator norm as } N \to \infty.$$

Problem 4. *(Multiplication operators)*

For $1 \leqslant p < \infty$ and $h \in L^\infty([0,1])$ let the operator of multiplication by h be denoted by

$$M_{p,h} : L^p([0,1]) \to L^p([0,1]), \quad M_{p,h}f(x) := h(x)f(x).$$

(a) Find the adjoint $M'_{p,h}$ of $M_{p,h}$ as an operator on $L^q([0,1])$, where $1/p + 1/q = 1$.

(b) Prove that $M_{p,h}$ is compact if and only if $h = 0$. [*Hint: Find a closed subspace $V \subset L^p([0,1])$ such that $M_{p,h}\big|_V : V \to V$ is surjective.*]

This sheet is to be discussed in the exercise class on Thursday, October 27.
For more details please visit http://www.math.lmu.de/~tkoenig/16FA2exercises.php