LMU LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN

MATHEMATISCHES INSTITUT

Prof. T. Ø. Sørensen PhD
T. König

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Functional Analysis II

Assignment 0 - Warm-up sheet

Problem 1. *(Constructing new Banach spaces as quotients of Banach spaces)*

Let $(X, \|\cdot\|)$ be a Banach space (over $\mathbb{C}$) and let $Y \subset X$ be a *closed* subspace. The goal of this exercise is to prove that the quotient vector space X/Y, endowed with a 'natural' norm inherited from $\|\cdot\|$, is again a Banach space.

We first construct X/Y as a vector space.

(a) Define the relation $\sim$ on X by $x \sim x' :\Leftrightarrow x - x' \in Y$. Prove that $\sim$ is an equivalence relation.

(b) Denote by X/Y the set of equivalence classes with respect to $\sim$, i.e. of elements $[x] = \{x + y : y \in Y\}$. Prove that on X/Y, addition and scalar multiplication are well-defined by $[x] + [x'] := [x + x']$ respectively $\lambda[x] := [\lambda x]$ for $x, x' \in X$, $\lambda \in \mathbb{C}$.

Next, we define a norm on X/Y with respect to which X/Y is complete.

(c) Define $\|[x]\|_\sim := \inf_{y \in Y} \|x - y\|$. Prove that $\|\cdot\|_\sim$ is well-defined and that $\|\cdot\|_\sim$ is indeed a norm on X/Y.

(d) Define the projection $\pi : X \to X/Y$ by $\pi(x) := [x]$. Prove that the normed space structures on X and X/Y are compatible in the sense that π is bounded linear of norm $|||\pi||| = 1$.

(e) Prove that $(X/Y, \|\cdot\|_\sim)$ is a Banach space.

Let $T : V \to W$ be a bounded surjective linear map between Banach spaces V and W and denote by $\pi : V \to V/\ker T$ the canonical projection. As an application of the above, we will prove that $V/\ker T$ and W are isomorphic as Banach spaces:

(f) Prove that there is a unique bounded linear map $\tilde{T} : V/\ker T \to W$ such that $T = \tilde{T} \circ \pi$. Prove that $\tilde{T}$ is bijective.

(g) Using the previous results and the open mapping theorem, prove that $\tilde{T}$ is a Banach space isomorphism between the spaces $V/\ker T$ and W, i.e. $\tilde{T}$ has a bounded linear inverse $\tilde{T}^{-1} : W \to V/\ker T$.

Problem 2. *(A continuity property for a sequence of orthogonal projections)*

Let $\mathcal{H}$ be a Hilbert space, $V_1 \supset V_2 \supset \cdots \supset V_n \supset V_{n+1} \supset \cdots$ be a sequence of closed linear subspaces. Define $V := \bigcap_{n\in\mathbb{N}} V_n$ and let $P_n : \mathcal{H} \to \mathcal{H}$ resp. $P : \mathcal{H} \to \mathcal{H}$ be the orthogonal projection operators onto V_n resp. V. Prove that for every $x \in \mathcal{H}$, $\lim_{n\to\infty} P_n x = Px$.

Problem 3. *(A compact integral operator)*

For $k \in C([0,1] \times [0,1])$, consider the integral operator

$$T : L^2([0,1]) \to C([0,1]), \quad (Tf)(x) := \int_0^1 k(x,y) f(y) dy \text{ for } x \in [0,1],$$

where $L^2([0,1])$ is equipped with the norm $\|\cdot\|_2$ given by $\|u\|_2^2 = \int_0^1 |u(x)|^2 dx$. Prove:

(a) The operator T is well-defined.

(b) If we equip $C([0,1])$ with the norm $\|\cdot\|_2$, then T is bounded with

$$|||T||| \leq \Big(\int_0^1 \int_0^1 |k(x,y)| dx\, dy \Big)^{1/2} = \|k\|_{L^2([0,1]\times[0,1])}.$$

(c) If we equip $C([0,1])$ with the norm $\|\cdot\|_\infty$ given by $\|u\|_\infty = \sup_{x\in[0,1]} |u(x)|$, then T is compact.

This sheet is to be discussed in the exercise class on Thursday, October 20.

For more details please visit http://www.math.lmu.de/~tkoenig/16FA2exercises.php