The large scale geometry and degenerations of Higgs bundle moduli spaces

Jan Swoboda (LMU München)

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Asymptotic geometry of the Higgs bundle moduli space

Based on:

Asymptotic geometry of the Higgs bundle moduli space

**Goals:**
Investigation of moduli spaces of solutions of Higgs bundles and related objects *in the large*:

- degenerations of solutions to the self-duality equations defining the moduli spaces
- asymptotics of underlying geometric structures such as their hyperkähler metrics
- identification of the resulting geometry *at infinity*
The Kobayashi-Hitchin correspondence

Geometric setup

**Geometric setup:**

- $(\Sigma, J)$ closed Riemann surface, genus $\gamma \geq 2$
- $\pi : E \to \Sigma$ complex vector bundle, rank $r_E \geq 1$, degree $d_E = \int_\Sigma c_1(E) \in \mathbb{Z}$, slope $\mu_E = d_E / r_E$
- hermitian bundle metric $h$ on $E$
- $\mathcal{A}(E, h)$ space of $h$-unitary connections on $E$
The Kobayashi-Hitchin correspondence

Holomorphic vector bundles

**Basic facts:**

- Every unitary connection $A \in \mathcal{A}(E, h)$ induces a $\bar{\partial}$-operator on $E$ via $d_A = \partial_A + \bar{\partial}_A \mapsto \bar{\partial}_A : \Omega^0(E) \to \Omega^{0,1}(E)$

- Conversely, every $\bar{\partial}$-operator $\bar{\partial}_E$ extends to a unique unitary connection

- Every choice of $\bar{\partial}$-operator $\bar{\partial}_E$ endows $E$ with the additional structure of a holomorphic vector bundle

- We consider two holomorphic structures on $E$ equivalent, if they differ by a bundle automorphism (gauge transformation) of $E$: $\bar{\partial}_2 = g^{-1} \circ \bar{\partial}_1 \circ g$ for some $g \in \mathcal{G}^c(E) = \Gamma(\text{GL}(E))$
**Question:** Given a holomorphic vector bundle \((E, \bar{\partial}_E)\), is there a preferred unitary connection \(A \in A(E, h)\) in its complex gauge orbit?

We may look for a gauge transformation \(g \in \mathcal{G}^c(E)\) such that the unitary connection

\[
A = (g^{-1} \circ \bar{\partial}_E \circ g) - (g^{-1} \circ \bar{\partial}_E \circ g)^*\]

has curvature \(F_A\) as “constant” as possible.
The Kobayashi-Hitchin correspondence

Hermitian-Yang-Mills connections

Definition

A unitary connection $A \in \mathcal{A}(E, h)$ is called **Hermitian-Yang-Mills** if it satisfies the PDE

$$F_A = -2\pi i \mu_E \text{id}_E \omega. \quad (1)$$

The prefactor $-2\pi i \mu_E$ is due to topological reasons, since

$$\int_{\Sigma} \text{Tr} F_A = -2\pi i d_E$$

by Chern-Weil theory.
Existence and uniqueness (up to unitary gauge transformations) of Hermitian-Yang-Mills connections in a given $G^c(E)$-orbit? Suppose that $F \subset E$ is a proper holomorphic subbundle such that

$$\bar{\partial}_E = \begin{pmatrix} \bar{\partial}_F & \beta \\ 0 & \bar{\partial}_Q \end{pmatrix} \quad \text{and} \quad d_A = \begin{pmatrix} d_{AF} & \partial \beta \\ -\bar{\partial} \beta^* & d_{AQ} \end{pmatrix}.$$

Then one has the lower bound

$$\| F_A + 2\pi i \mu_E \text{id}_E \omega \|_{L^2(\Sigma)} \geq 2\pi \left| \mu_F - \mu_E + \| \beta \|^2 \right| + 2\pi \left| \mu_E - \mu_Q + \| \beta \|^2 \right|.$$

It follows that Eq. (1) does not have a solution if $\mu_F > \mu_E$ ($\Leftrightarrow \mu_Q < \mu_E$).
The Kobayashi-Hitchin correspondence

The theorem of Narasimhan-Seshadri

**Definition**

A holomorphic vector bundle \((E, \bar{\partial}_E)\) is called **stable** if \(\mu_F < \mu_E\) holds for every proper holomorphic subbundle \(F \subset E\).

**Theorem (Narasimhan-Seshadri)**

*Let \((E, \bar{\partial}_E)\) be a holomorphic vector bundle. Then its complex gauge orbit contains an irreducible solution \(A\) of Eq. (1) if and only if it is stable. In this case, \(A\) is unique up to unitary gauge transformations.*

The theorem establishes one instance of the so-called **Kobayashi-Hitchin correspondence**. It admits generalizations in various directions, including:

- stable vector bundles over general Kähler manifolds
  (Uhlenbeck-Yau theorem)
- Higgs bundles
**Definition**

A **Higgs bundle** on \( E \) is a pair \((\bar{\nabla}_E, \Phi)\), where \( \Phi \in \Omega^{1,0}(\Sigma, \text{End}(E)) \) with \( \bar{\nabla}_E \Phi = 0 \). It is called stable if \( \mu_F < \mu_E \) holds for all proper \( \Phi \)-invariant holomorphic subbundles \( F \subset E \).

**Example:** Let \( K = \mathcal{A}^{1,0}(\Sigma) \cong T^*\Sigma \) be the canonical line bundle and consider the holomorphic vector bundle \( E = K^{-\frac{1}{2}} \oplus K^{\frac{1}{2}} \). Together with the Higgs field

\[
\Phi = \begin{pmatrix} 0 & 1 \\ q & 0 \end{pmatrix} \quad (q \in H^0(K^2), \text{holomorphic quadratic differential})
\]

it is a stable Higgs bundle.
Higgs bundles
The Hitchin-Narasimhan-Seshadri theorem

The replacement of the Hermitian-Yang-Mills equations are Hitchin’s self-duality equations

\[
\begin{align*}
0 &= \bar{\partial}_A \Phi \\
-2\pi i \mu_E \text{id}_E \omega &= F_A + [\Phi \wedge \Phi^*]
\end{align*}
\] (2)

for a unitary connection \(A\) and a Higgs field \(\Phi\).

**Theorem (Hitchin-Narasimhan-Seshadri)**

The complex gauge orbit of every stable Higgs bundle \((\bar{\partial}_E, \Phi)\) contains an irreducible solution of Eq. (2). It is unique up to unitary gauge transformations.
From now on, we focus on rank-2 Higgs bundles and consider their moduli space

\[ \mathcal{M} = \mathcal{M}_{r=2,d} = \frac{\{(A, \Phi) \mid \text{irreducible solution of (2)}\}}{G(E, h)}. \]

**Basic properties:**
- \( \mathcal{M} \) is a 12(\( \gamma - 1 \))-dimensional manifold
- it is noncompact
- it arises as a hyperkähler quotient
- the resulting \( L^2 \) hyperkähler metric is complete if \( d \) is odd
Simplifying assumption: in the following all holomorphic quadratic differentials are simple, i.e. only have simple zeroes.

**Definition**

A **limiting configuration** for a simple $q \in H^0(K^2)$ is a (singular) pair $(A_\infty, \Phi_\infty)$ satisfying on $\Sigma^\times = \Sigma \setminus q^{-1}(0)$ the decoupled self-duality equations

\[
\begin{cases}
0 = \bar{\partial}_{A_\infty} \Phi_\infty \\
0 = F_{A_\infty}^\perp \\
0 = [\Phi_\infty \wedge \Phi_\infty^*]
\end{cases}
\]
Lemma (Mazzeo, S., Weiß, Witt)

For every simple $q$, the space $T_q$ of limiting configurations is diffeomorphic to the torus

$$T_q \cong \frac{\mathcal{H}_{A_\infty}}{\Lambda_q}$$

of dimension $\frac{1}{2} \dim \mathcal{M}$. Here $(A_\infty, \Phi_\infty) \in T_q$ is some base point and $\mathcal{H}_{A_\infty}$ denotes the vector space of $d_{A_\infty}$-harmonic 1-forms commuting with $\Phi_\infty$. 
Higgs bundles
Compactification by limiting configurations

The disjoint union $\bigsqcup_{q \in B} T_q$ of limiting tori yields a partial compactification of the moduli space $\mathcal{M}$:

**Theorem (Mazzeo, S., Weiß, Witt)**

For every $(A_1, \Phi_1) \in T_q$ there exists a 1-parameter family $(A_t, t\Phi_t)$ of smooth solutions to the self-duality equations, such that $\det \Phi_t = q$ and

$$(A_t, \Phi_t) \rightarrow (A_\infty, \Phi_\infty)$$

in a suitable sense as $t \rightarrow \infty$. 
The moduli space $\mathcal{M}$ carries the natural Weil-Petersson type metric

$$G_{L^2}([A, \Phi])(\nu, \nu) = \text{Re} \int_\Sigma \text{Tr}(\nu \wedge \nu^*),$$

where $\nu \in \ker \mathcal{H}$ is contained in the infinitesimal deformation space of Eq. (2) at $(A, \Phi)$ and is $L^2$-orthogonal to the unitary gauge orbit. The metric $G_{L^2}$ is **hyperkähler**, i.e. has holonomy contained in $\text{Sp}(m)$. One wants to understand the geometric structure of the “ends” of $\mathcal{M}$ w.r.t. this metric.
The union $\bigsqcup_{q \in B^\times} T_q$ of limiting tori can be endowed with an $L^2$-metric, where each torus fibre $T_q$ is intrinsically flat. A formal variation of the parameter $q$ yields that the induced metric on the base $B^\times$ is

$$G_{SK}(q)(\dot{q}, \ddot{q}) = \int_{\Sigma} \frac{|\dot{q}|^2}{|q|} \, dA.$$ 

Note that $G_{SK}$ is singular at $B \setminus B^\times$. 

**Metric asymptotics**

The $L^2$-hyperkähler metric on $\mathcal{M}$
Viewing $\mathcal{M}$ as a completely integrable system, $G_{SK}$ is the **special Kähler metric** associated with it. Here the integrable system structure is induced by

- a holomorphic symplectic form $\eta$ on $\mathcal{M}^\times$
- the holomorphic map $\det: \mathcal{M}^\times \to \mathcal{B}^\times$ with Lagrangian fibres $\det^{-1}(q)$
Metric asymptotics
Integrable systems and special Kähler geometry

An equivalent description of Higgs bundles: **Spectral data.** Any Higgs bundle \((\overline{\partial}_E, \Phi)\) with simple determinant \(q = \det \Phi\) is uniquely determined by:

- **the spectral curve** \(S_q := \{\psi \in K | \psi^2 = q\}\) (a branched 2:1 cover of \(\Sigma\))
- **the decomposition of** \(\pi^*E\) into **eigenline bundles** \(L_{\pm}\) w.r.t. \(\pi^*\Phi\)

The eigenlines satisfy \(\sigma^* L_{\pm} = L_{\mp}\), where \(\sigma\) is the involution on \(S_q\) interchanging the sheets of the cover. So

\[
\sigma^* (L_{\pm} \otimes K^{\frac{1}{2}}) = (L_{\pm} \otimes K^{\frac{1}{2}})^*,
\]

which implies that \(L_{\pm} \otimes K^{\frac{1}{2}} \in \text{PPrym}(S_q)\), the Picard-Prym subtorus of the Jacobi variety of holomorphic line bundles of degree \(d = 0\) over \(S_q\).
Fix a symplectic basis \( \{ \alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_m \} \) of \( H_1(S_q; \mathbb{Z}) \). Integration of the Seiberg-Witten differential \( \lambda_{SW}(q) \) (the pullback to \( S_q \) of the Liouville 1-form on \( K \)) yields two sets of complex coordinates on \( \mathcal{B}^\times \):

\[
z_i = \int_{\alpha_i} \lambda_{SW}(q) \quad \text{and} \quad w_i = \int_{\beta_i} \lambda_{SW}(q)
\]

They combine to yield the special Kähler form

\[
\omega_{SK} := \sum_i dz_i \wedge dw_i
\]

and Kähler metric \( G_{SK} = \omega_{SK}(\cdot, i\cdot) \). It has the structure of a cone metric via the isometry

\[
\phi: (\mathbb{R}_+ \times S^\times, dt^2 + t^2 g_{SK}|_{S^\times}) \to (\mathcal{B}^\times, g_{SK}), \quad (t, q) \mapsto t^2 q.
\]
Using the Riemann bilinear relations, the metric $G_{SK}$ can be rewritten in the form

$$G_{SK}(q)(\dot{q}, \dot{q}) = \int_{\Sigma} \frac{|\dot{q}|^2}{|q|} dA.$$ 

This shows:

**Theorem (Mazzeo, S., Weiß, Witt)**

The $L^2$ metric on the space of limiting configuration coincides with the special Kähler metric associated with the Hitchin integrable system.
Metric asymptotics
Asymptotics of the $L^2$ metric

There is a standard way to construct a **semiflat hyperkähler metric** (i.e. a metric with intrinsically flat fibres) on $\mathcal{M}^\times$ from $G_{SK}$ (Freed 1999).

Verifying a conjecture due to Gaiotto–Moore–Neitzke, one can show that on the regular part $\mathcal{M}^\times$ of the moduli space the initially defined metric $G_{L^2}$ is asymptotic to $G_{SF}$:

**Theorem (Mazzeo, S., Weiß, Witt)**

*There is a complete asymptotic expansion*

$$G_{L^2}(x, t) = G_{SF}(x, t) + \sum_{j=0}^{\infty} t^{(4-j)/3} G_j(x) + O(e^{-\beta t}).$$
Degenerating surfaces

Setup

Until now: the Riemann surface \((\Sigma, J)\) has been kept fixed. We now consider the moduli space \(\mathcal{M}(\Sigma, J)\) as being parametrized by the complex structure \(J\).

Of particular interest: \textbf{degenerating} families of smooth surfaces \((\Sigma, J_t) \to (\Sigma, J_0)\), where \((\Sigma, J_0)\) is a \textbf{Riemann surface with nodes} \(\mathcal{p} = \{p_1, \ldots, p_m\}\).
The unique hyperbolic metric $g_t$ in its conformal class develops longer and longer hyperbolic “necks”:

$$g_t = ds^2 + \rho^2 \cosh^2(s) \, d\theta^2$$

in Fermi coordinates around $p_i \in \mathfrak{p}$, where $\rho = |t|$. By the conformal invariance of the self-duality equations, we can work with any conformally related metric. In the following, we endow the infinite cylinders with the flat metric $ds^2 + d\theta^2$. 
Degenerating surfaces

Singular solutions

It is natural to allow for singular solutions of the self-duality equations. The theory of parabolic Higgs bundles provides for singularity models of varying growth order:

- Simpson: logarithmic singularities
  \[(A, \Phi) = \left( \frac{A_0}{z}, \frac{\Phi_0}{z} \right) + \mathcal{O}(r^{-1+\delta})\]

- Biquard–Boalch: higher order singularities

We here only allow for logarithmic singularities. Note that under the conformal change of coordinates \( z = \exp(i\varphi) \), solutions in this class are of the form

\[(A(s, \theta), \Phi(s, \theta)) = (A_0, \Phi_0) + \exp(-\delta s).\]
Degenerating surfaces

Main result

Every singular solution arises as the limit of a sequence of smooth solutions under the degeneration of $\Sigma$ to a noded surface:

**Theorem (Gluing theorem)**

Let $(A_0, \Phi_0)$ be a solution of the self-duality equations with logarithmic singularities in $\mathcal{P}$. Let $(\Sigma, J_i)$ be a sequence of smooth Riemann surfaces converging uniformly to $(\Sigma, J_0)$. Then, for every sufficiently large $i$, there exists a smooth solution $(A_i, \Phi_i)$ of the self-duality equations on $(\Sigma, J_i)$ such that $(A_i, \Phi_i) \to (A_0, \Phi_0)$ as $i \to \infty$, uniformly on compact subsets of $\Sigma \setminus \mathcal{P}$. 
Degenerating surfaces

Outline of proof

Let \((A_0, \Phi_0)\) be a solution on \((\Sigma, J_0)\) with logarithmic singularities in \(p\).

▶ Using that along the infinite hyperbolic cylinders \((A_0, \Phi_0)\) is exponentially close to some translation invariant model solution, a cutoff function argument yields an approximate solution \((A_{i, \text{app}}, \Phi_{i, \text{app}})\) on each smooth surface \((\Sigma, J_i)\).

▶ We want to obtain a nearby exact solution by writing it as a fixed point of the map

\[
\gamma \mapsto -L_i^{-1}(\mathcal{H}(A_{i, \text{app}}, \Phi_{i, \text{app}}) + Q_i(\gamma)) : B_\rho(H^2(\Sigma)) \to B_\rho(H^2(\Sigma)),
\]

where

▶ \(\mathcal{H}\) is the nonlinear Hitchin operator
▶ \(L_i\) is the linearization of \(\mathcal{H}\) at \((A_{i, \text{app}}, \Phi_{i, \text{app}})\)
▶ \(Q_i\) denote quadratic and higher order terms in \(\gamma\)
Degenerating surfaces

Outline of proof

- For $\mathcal{H}(A_i^{\text{app}}, \Phi_i^{\text{app}})$ an exponential error bound holds by construction.

- To conclude, a polynomial upper bound for the operator norm $L_i^{-1}: L^2(\Sigma) \to H^2(\Sigma)$ w.r.t. the length $T$ ($T \to \infty$) of the long cylindrical pieces of $(\Sigma, J_i)$ suffices.

- Need to show that the smallest eigenvalue of $L_i$ decays at the expected rate $\lambda_T \leq CT^{-2}$ (but not faster).

- This is the content of the Lee-Cappell-Miller gluing theorem for Dirac type operators on manifolds with cylindrical ends: “large” eigenvalues decay at exactly this rate; “small” eigenvalues may exist and come from the kernel of the limiting operator $L_0$.

- Can show that $\ker L_0 = 0$ in this setup.
Degenerating surfaces
Application: Maximal components of $\text{Sp}(4, \mathbb{R})$ surface group representations

Donaldson (rank-2 Higgs bundles) and Corlette (higher rank) have established a bijective correspondence between the moduli spaces $\mathcal{M}_{r,d}$ of irreducible solutions to the self-duality equations and the character varieties

$$\chi(\Sigma)_r = \{\rho : \pi_1(\Sigma) \rightarrow \text{SL}(r, \mathbb{C})\}$$

of conjugation classes of irreducible representations of the fundamental group.
*Real* representations $\rho : \pi_1(\Sigma) \rightarrow \text{SL}(r, \mathbb{R})$ fall into several (3 or 6, depending on the genus $\gamma$) connected components, which can be distinguished by Higgs bundle methods, due to a result of Hitchin.
Degenerating surfaces
Application: Maximal components of $\text{Sp}(4, \mathbb{R})$ surface group representations

Connected components of representations into the Lie group $\text{Sp}(4, \mathbb{R})$ are labeled by a topological invariant $|d| \leq 2\gamma - 2$. Maximal representations ($|d| = 2\gamma - 2$) fall into one of the following:

- reductions to $\text{SL}(2, \mathbb{R})$-representations ($2^{2\gamma}$ Hitchin-Teichmüller components);
- reductions to $\text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R})$-representations ($2^{2\gamma+1} - 1$ components);
- $2\gamma - 3$ further components.
Degenerating surfaces
Application: Maximal components of $\text{Sp}(4, \mathbb{R})$ surface group representations

Results:

- Burger-Iozzi-Labourie-Wienhard (2010): new topological invariants which allow to distinguish all connected components of maximal representations
- Guichard-Wienhard (2010): construct one model representation for each of the above $2\gamma - 3$ components
- Kydonakis (ongoing Ph.D. project): construct a Higgs bundle with monodromy representation in a prescribed component using the above gluing theorem