The Inclusion Process on General Spaces: Reversible Measures, Consistency and Self-Duality

YoungStatS Webinar: Inclusion Process and Sticky Brownian Motions

Stefan Wagner (LMU Munich)

February 9th, 2022

Overview

1. **Known facts - discrete spaces.**
   Self-dualities of Markov processes describing the evolution of particles on a discrete set.
2. **My Research - general spaces.**
   What happens if we replace the discrete space by a much more general space?

Joint work with

- Sabine Jansen (LMU Munich)
- Frank Redig (TU Delft)
- Simone Floreani (TU Delft)

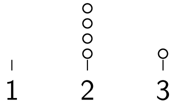
Some notations: Particle configurations

Let E be a countable set (e.g. $E = \{1, \dots, N\}$, $E = \mathbb{Z}^d$, graph). Consider

$$\mathbb{N}_0^E := \{(x_k)_{k \in E} : x_k \in \mathbb{N}_0\}.$$

x_k = "number of particles at position k".

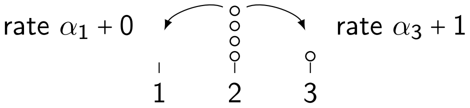
Example: $E = \{1, 2, 3\}$: $x = (x_1, x_2, x_3) = (0, 4, 1) =$



The Symmetric Inclusion Process (SIP)

$$Lf(x) = \sum_{k \in E} \sum_{l \in E} (f(x - \delta_k + \delta_l) - f(x))(\alpha_l + x_l)x_k$$

A single particle at Position k jumps to a position l with rate $\alpha_l + x_l$



Consistency

"the action of removing a particle uniformly at random commutes with the dynamic"
More precisely: The lowering operator $\mathcal{A}f(x) := \sum_{k\in E} x_k f(x - \delta_k)$ commutes with the generator (and equiv. with the semigroup).

Examples:

- SIP (also with optional weight function)
- Symmetric exclusion process
- Independent random walkers

All models are conservative, i.e., conserve the number of particles.

Reversible measures

Fix $p \in (0,1)$. Then

$$\varrho = \bigotimes_{k \in E} \varrho_{\alpha_k}$$

with

$$\varrho_a = \mathrm{NegativeBinomial}(p, a)$$

is a reversible measure for SIP.

Duality with falling factorials

Theorem (Carinci, Giardinà, Redig)
Put $(n)_k := n(n-1)\cdots(n-k+1)$. Then, $H(x,y) := \frac{1}{\varrho(\{x\})}\prod_{k\in E}(y_k)_{x_k}\frac{1}{(x_k)!}$ is a self-duality function for SIP, i.e.

$$\mathbb{E}_{X_0} H(X_t, Y_0) = \mathbb{E}^{Y_0} H(X_0, Y_t)$$

for all $X_0, Y_0 \in \mathbb{N}_0^E$. Thereby, X, Y are Markov-Processes of SIP starting in X_0, Y_0.

Orthogonal Polynomials

Let $(M_n(\,\cdot\,; a))_{n\in\mathbb{N}_0}$ be the Meixner Polynomials. Consider the multivariate polynomials

$$H_y(x) := \prod_{k\in E} M_{y_k}(x_k;\alpha_k), x, y \in \mathbb{N}_0^E.$$

which are orthogonal with respect to ϱ.

Orthogonal Duality

Theorem (Franceschini, Giardinà)
$(x, y) \mapsto H_y(x)$ is a self-duality function for SIP.

Generalization?

Replace the discrete E by a much more general space E (e.g. $\mathbb{R}^d$, topological space)
My Question: Generalization of all the objects and the resulting theorems?

We are now looking at all the slides again and see what happens at each step.

Some notations: Particle configurations

Let E be a countable set (e.g. $E = \{1, \ldots, N\}$, $E = \mathbb{Z}^d$, graph). Consider

$$\mathbb{N}_0^E := \{(x_k)_{k \in E} : x_k \in \mathbb{N}_0\}.$$

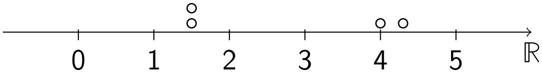
x_k = "number of particles at position k".

Example: $E = \{1, 2, 3\}$: $x = (x_1, x_2, x_3) = (0, 4, 1) =$

Let E be a Borel space. Consider the set of measures

$$\mathbf{N}(E) := \left\{ \sum_{k=1}^{n} \delta_{x_k} : x_k \in E, n \in \mathbb{N}_0 \cup \{\infty\} \right\}$$

Example: $E = \mathbb{R}$: $2\delta_{1.5} + \delta_4 + \delta_{4.3} =$



Therefore, we look at measure-valued Markov processes.

The Symmetric Inclusion Process (SIP)

$$Lf(x) = \sum_{k \in E} \sum_{l \in E} (f(x - \delta_k + \delta_l) - f(x))(\alpha_l + x_l)x_k$$

A single particle at Position k jumps to a position l with rate $\alpha_l + x_l$

Let α be a finite measure on E

$$Lf(\mu) = \iint (f(\mu - \delta_x + \delta_y) - f(\mu))(\mu + \alpha)(\mathrm{d}y)\mu(\mathrm{d}x)$$

A single particle at position x jumps to a position y with rate $\alpha(\mathrm{d}y) + \mu(\mathrm{d}y)$.
Interpretation: The particle jumps either to a "new" position $y \sim \frac{\alpha(\mathrm{d}y)}{\alpha(E)}$ at rate $\alpha(E)$ or to an already occupied position y with rate $\mu(\{y\})$

Consistency

"the action of removing a particle uniformly at random commutes with the dynamic"

More precisely: The lowering operator $\mathcal{A}f(x) := \sum_{k\in E} x_k f(x-\delta_k)$ commutes with the generator (and equiv. with the semigroup).

Examples:

- SIP (also with optional weight function)
- Symmetric exclusion process
- Independent random walkers

All models are conservative, i.e., conserve the number of particles.

Generalization: $\mathcal{A}f(\mu) := \int f(\mu-\delta_x)\mu(\mathrm{d}x)$, $\mu\in\mathbf{N}(E)$, $f:\mathbf{N}(E)\to\mathbb{R}$.

Theorem (Floreani, Jansen, Redig, W.)

The Generalized SIP is consistent.

More Examples:

- Independent (arbitrary) Markov processes
- Sticky Brownian Motion

Reversible measures

Fix $p \in (0,1)$. Then

$$\varrho = \bigotimes_{k \in E} \varrho_{\alpha_k}$$

with

$$\varrho_a = \mathrm{NegativeBinomial}(p, a)$$

is a reversible measure for SIP.

The Pascal process is a random element ζ with values in $\mathbf{N}(E)$ satisfying

1. $\xi(A_1), \ldots, \xi(A_N)$ are independent for pairwise disjoint measurable sets $A_1, \ldots, A_n \subset E$,
2. $\xi(A) \sim \mathrm{NegativeBinomial}(p, \alpha(A))$ for each measurable $A \subset E$.

Theorem (Floreani, Jansen, Redig, W.)
The distribution of ζ is a reversible measure for SIP.

Duality with falling factorials

Theorem (Carinci, Giardinà, Redig)
Put $(n)_k := n(n-1)\cdots(n-k+1)$. Then, $H(x,y) := \frac{1}{\varrho(\{x\})}\prod_{k\in E}(y_k)_{x_k}\frac{1}{(x_k)!}$ is a self-duality function for SIP, i.e.

$$\mathbb{E}_{X_0} H(X_t, Y_0) = \mathbb{E}^{Y_0} H(X_0, Y_t)$$

for all $X_0, Y_0 \in \mathbb{N}_0^E$. Thereby, X, Y are Markov-Processes of SIP starting in X_0, Y_0.

Falling factorial are generalized with factorial measures, i.e., $J_k(f_k,\mu) := \int f_k \,\mathrm{d}\mu^{(k)}$ for $f_k : E^k \to \mathbb{R}$, $\mu \in \mathbf{N}(E)$

Theorem (Floreani, Jansen, Redig, W.)
Let η be a consistent and conservative Markov process. Denote by $P_t, t \geq 0$ the Markov semigroup of η and by $p_t^{[k]}$ its restriction to k (labeled) particles. Then, we obtain the self-intertwining relation

$$P_t J_k(f_k, \cdot)(\mu) = J_k(p_t^{[k]} f_k, \mu), \quad \mu \in \mathbf{N}(E), \quad f_k : E^k \to \mathbb{R}.$$

Orthogonal Polynomials

Let $(M_n(\cdot\,;a))_{n\in\mathbb{N}_0}$ be the Meixner Polynomials. Consider the multivariate polynomials

$$H_y(x) := \prod_{k\in E} M_{y_k}(x_k;\alpha_k), x, y \in \mathbb{N}_0^E.$$

which are orthogonal with respect to ϱ.

We introduce orthogonal polynomials in infinite dimensions. Let

$$\mathcal{P}_n := \left\{ \sum_{k=0}^{n} J_k(f_k,\cdot\,) : f_k : E^k \to \mathbb{R} \right\}$$

be the space of polynomials of degree $\leq n$. Define

$$I_n(f_n,\cdot\,) := L^2(\varrho)\text{-orthogonal projection of } J_n(f_n,\cdot\,) \text{ onto } \mathcal{P}_{n-1}^{\perp}.$$

Orthogonal polynomials in infinite dimensions are known in the theory of chaos decompositions (Fock spaces, multiple stochastic integrals)

Orthogonal Duality

Theorem (Franceschini, Giardinà)
$(x, y) \mapsto H_y(x)$ is a self-duality function for SIP.

Theorem (Floreani, Jansen, Redig, W.)
Let $(\eta_t)_{t\geq 0}$ be a consistent and conservative Markov process. Assume that a reversible measure is given by the distribution of a Lévy process. Then, we obtain the orthogonal self-duality relation

$$P_t I_n(f_n, \cdot)(\mu) = I_n\left(p_t^{[n]} f_n, \mu\right)$$

for all $f_n : E^n \to \mathbb{R}$, $\mu \in \mathbf{N}(E)$, $t \geq 0$. Additionally, if ϱ is the distribution of a Pascal process, then

$$\int I_n(f_n, \cdot) I_m(g_m, \cdot)\, \mathrm{d}\varrho = \delta_{n,m} \frac{p^n n!}{(1-p)^{2n}} \int f_n g_m \, \mathrm{d}\lambda_n$$

for $f_n : E^n \to \mathbb{R}$, $g_m : E^m \to \mathbb{R}$ permutation invariant functions.

Thank you!

S. Floreani, S. Jansen, F. Redig, S.W.: *Duality and intertwining for consistent Markov processes* arXiv:2112.11885 [math.PR], 32 pp.