Orthogonal Intertwiners for Infinite Particle Systems On The Continuum

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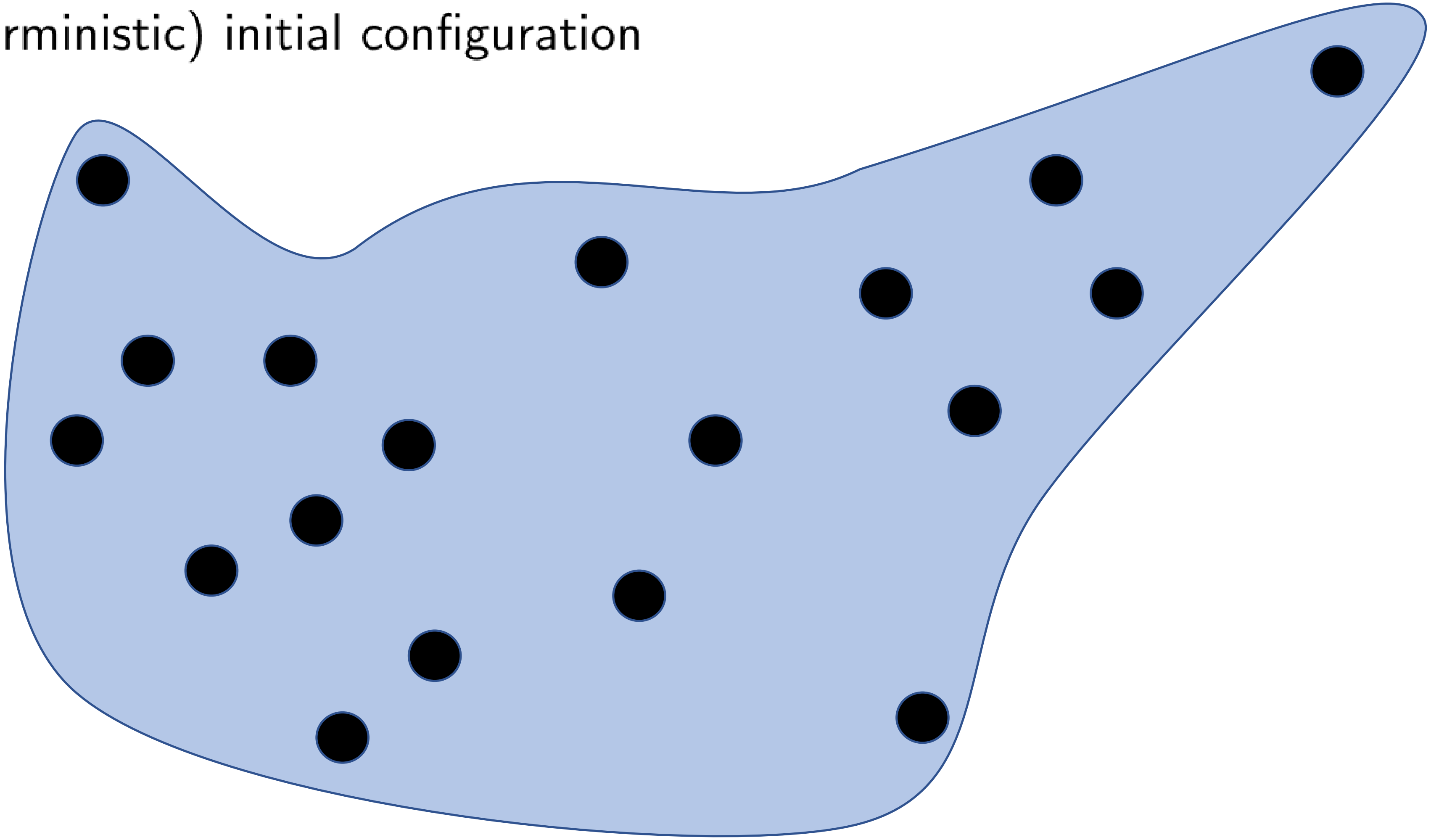
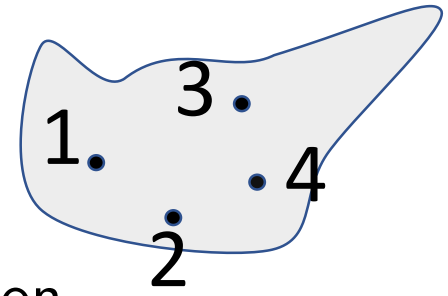
Particle system

$=$ a Markov process describing the (stochastic) evolution of particles in a set E for each (deterministic) initial configuration



Two different notations



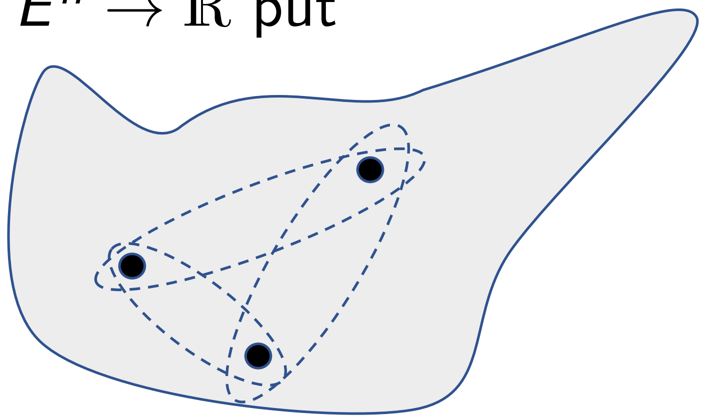
Assumption: total number of particles is preserved

Unlabeled notation	Labeled notation
Markov process	family of Markov processes indexed via number of particles $N \in \mathbb{N}$
values in $\mathbf{N}$, the set of counting measures consisting of $\mu = \sum_{k=1}^{N} \delta_{x_k}$, $x_k \in E$. $\mu(A) =$ number of particles in a subset A	values in E^N
$(P_t)_{t\geq 0}$ Markov semigroup acting on functions $F : \mathbf{N} \to \mathbb{R}$	family of Markov semigroups $(P_t^{[N]})_{t\geq 0}, N \in \mathbb{N}$ acting on functions $f_N : E^N \to \mathbb{R}$

Factorial measure intertwiner

Integral with respect to the n-th factorial measure. For $f_n : E^n \to \mathbb{R}$ put

$$J_n\left(f_n, \sum_{k=1}^N \delta_{x_k}\right) := \sum_{\substack{i_1,\ldots,i_n=1 \\ \text{pairwise different}}}^N f_n(x_{i_1},\ldots,x_{i_n})$$



Examples: $J_1(f_1, \delta_x + \delta_y + \delta_z) = f_1(x) + f_1(y) + f_1(z)$,
$J_2(f_2, \delta_x + \delta_y + \delta_z) = f_2(x,y) + f_2(y,x) + f_2(x,z) + f_2(z,x) + f_2(y,z) + f_2(z,y)$

Theorem 1 (Redig, Jansen, Floreani, W., '21)

finite particle system + number of particles is preserved. Then,

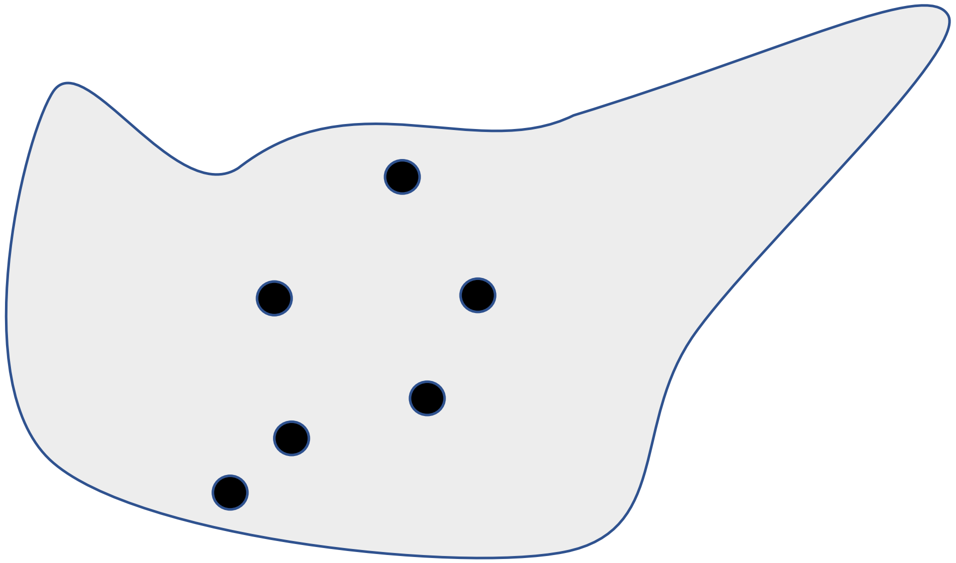
consistency $\iff$ *J_n intertwines $P_t^{[n]}$ and P_t for all t and n, i.e.,*

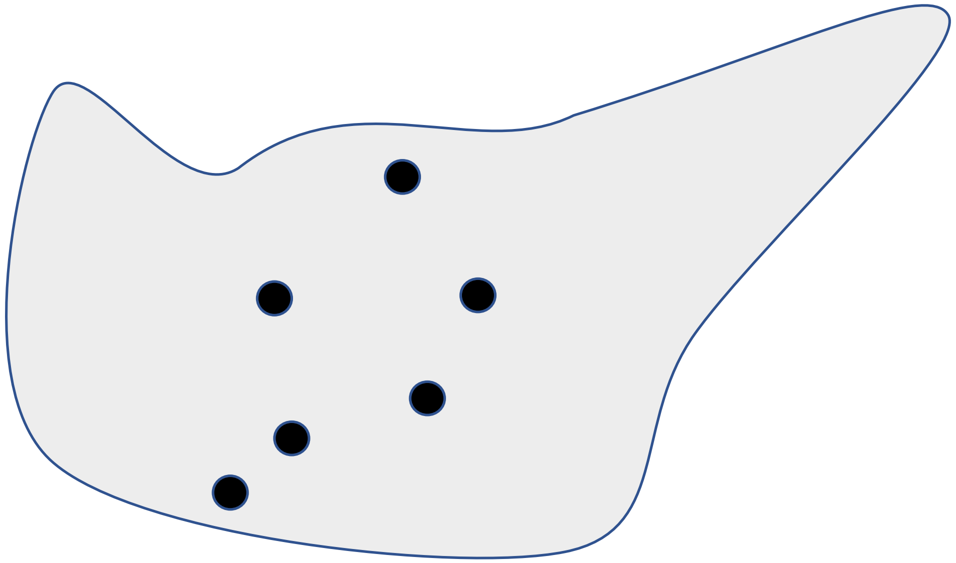
$$P_t J_n(f_n, \cdot)(\mu) = J_n\left(P_t^{[n]} f_n, \mu\right) \text{ for all } f_n : E^n \to \mathbb{R}, \mu \in \mathbf{N}$$

Theorem 1 recovers well-known self-dualities in terms of falling factorial polynomials for interacting particle systems on discrete sets (SIP, SEP, IRW)

Consistency

= *"the action of removing a particle uniformly at random commutes with the dynamic"*





Consistency

= *"the action of removing a particle uniformly at random commutes with the dynamic"*

Examples:

- independent particles
- symmetric inclusion process (SIP), symmetric exclusion process (SEP), independent random walkers (IRW)
- generalized SIP in the continuum / Moran process
- compatible systems, stochastic flows (Le Jan, Reimond)
  - correlated Brownian motions
  - coalescing Brownian motions
  - Howitt-Warren flow (sticky Brownian motions)

Intertwining relation

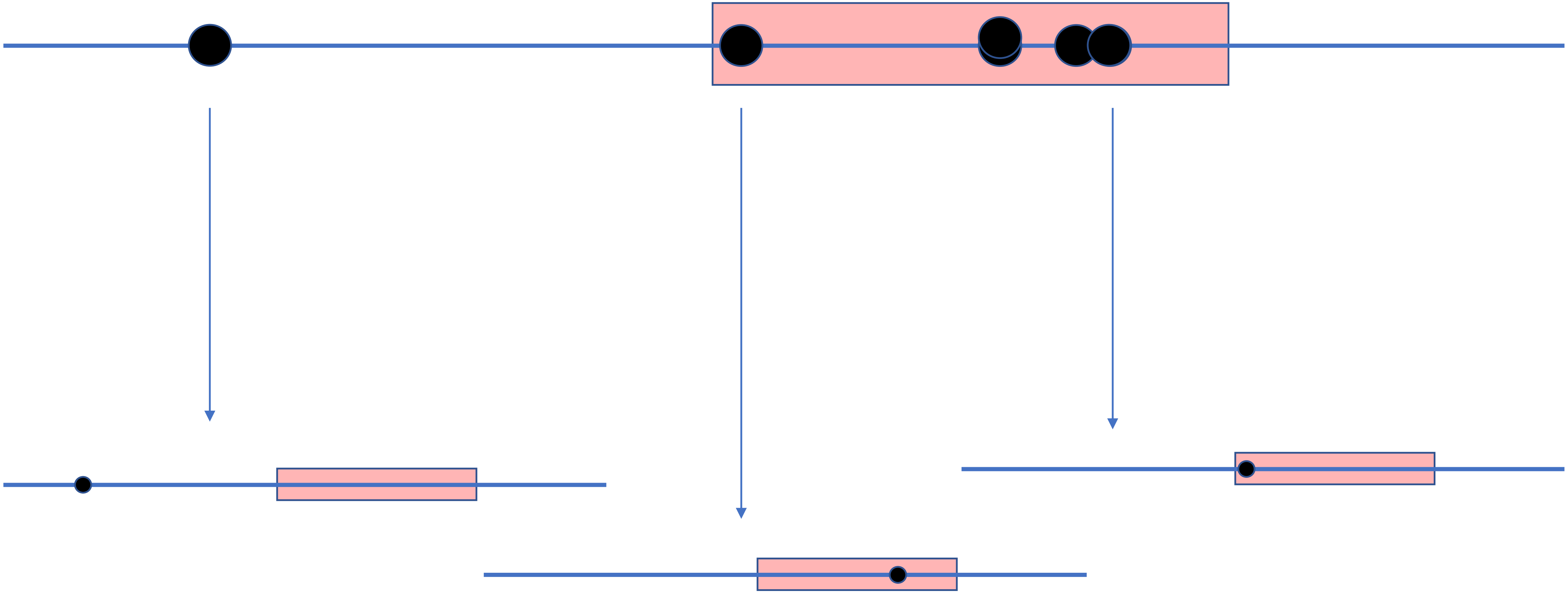
Example: Given an initial configuration μ: What is the expected number of particles in a set A at time t?

$$\text{expected number of particles in a set } A \text{ at time } t \qquad = \qquad P_t J_1 \mathbf{1}_A(\mu)$$

Let each particle evolve the one-particle dynamics separately. Sum of the probabilities

$$\text{that the respective particle is in the set } A \text{ at time } t \qquad = \qquad J_1 P_t^{[1]} \mathbf{1}_A(\mu)$$

Intertwining relation



Infinite-dimensional orthogonal polynomials

Let ρ be a probability measure on $\mathbf{N}$, i.e., the distribution of a point process

$$I_n(f_n, \cdot\,) \quad := \quad \text{orthogonal projection of } J_n(f_n, \cdot\,) \text{ onto } \{J_k(u_k, \cdot\,), u_k : E^k \to \mathbb{R}, 0 \leq k \leq n-1\}^\perp$$

in $L^2(\rho)$.

Keywords: infinite-dimensional orthogonal polynomials, (extended) Fock spaces, chaos decompositions, multiple stochastic integrals, non-Gaussian white noise analysis, Malliavin calculus

Theorem 2 (Redig, Jansen, Floreani, W., '21)

$$J_n \text{ intertwines } P_t^{[n]} \text{ and } P_t \text{ for all } t \text{ and } n$$
$$+$$
$$\rho \text{ reversible}$$
$$\Downarrow$$
$$I_n \text{ intertwines } P_t^{[n]} \text{ and } P_t \text{ for all } t \text{ and } n$$

Together with Theorem 1:

finite number of particles $+\ \rho$ reversible $+$ consistency $\Rightarrow I_n$ intertwines $P_t^{[n]}$ and P_t

Applications:

- Theorem 2 recovers well-known self-dualities in terms of orthogonal polynomials for discrete particle systems (SIP, SEP, IRW)
- independent particles (ρ distribution of the Poisson process)
- generalized SIP (ρ distribution of the Pascal process)

Infinite particle systems?

1. Theorem 1 does not hold: The property of consistency does not establish any link between the dynamics of finite and infinite particles.
2. Reversible measures for the infinite dynamics are difficult to obtain.

Theorem 3 (W., '23)

Let ρ be the distribution of a Poisson or Pascal process

$$J_n \text{ intertwines } P_t^{[n]} \text{ and } P_t \text{ for all } t \text{ and } n$$

$$+$$

$$\text{for each } n\text{: } n\text{-th factorial moment measure of } \rho \text{ is reversible for } n \text{ unlabeled particles}$$

$$\Downarrow$$

$$I_n \text{ intertwines } P_t^{[n]} \text{ and } P_t \text{ for all } t \text{ and } n \qquad + \qquad \rho \text{ reversible}$$

Application to uniform sticky Brownian motions: infinite-dimensional Meixner polynomial of degree n intertwines the dynamics of infinite particles and the dynamics of n particles. As a consequence, the distribution of a Pascal process is reversible.

Thank you!

S. Floreani, S. Jansen, F. Redig, S.W.: *Duality and intertwining for consistent Markov processes* arXiv:2112.11885 [math.PR], 32 pp.