The algebraic approach to duality in non-discrete spaces

Workshop: “Recent Developments in Stochastic Duality”

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MCQST

The algebraic approach to duality

1. **Known facts - the algebraic approach to duality in discrete spaces.**
2. **Non-discrete spaces?**

Joint work with

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- Simone Floreani (University of Oxford)

Symmetric inclusion process (SIP) on a finite set

E finite set, $\alpha_x \in \mathbb{N}_0$, $x \in E$, $c_{x,y} \geq 0$, $x, y \in E$ symmetric, $p \in (0,1)$.

$$Lf(\eta) = \sum_{x,y \in E} c_{x,y} \left(f(\eta - \delta_x + \delta_y) - f(\eta)\right)(\alpha_y + \eta_y)\eta_x = \sum_{\{x,y\} \subset E} c_{x,y} L_{x,y} f(\eta), \quad \eta \in \mathbb{N}_0^E$$

$$L_{x,y} = k_x^+ k_y^- + k_x^- k_y^+ - 2k_x^0 k_y^0 + \frac{\alpha_x \alpha_y}{2}$$

$$k_x^+ f(\eta) = \frac{1}{\sqrt{p}} f(\eta - \delta_x)\eta_x, \qquad k_x^- f(\eta) = \sqrt{p} f(\eta + \delta_x)(\alpha_x + \eta_x)$$

$$k_x^0 f(\eta) = f(\eta)\left(\frac{\alpha_x}{2} + \eta_x\right)$$

- Representation of $su(1,1)$ Lie algebra:

$$[k_x^+, k_y^+] = 0 \qquad [k_x^0, k_y^0] = 0 \qquad [k_x^-, k_y^-] = 0,$$

$$[k_x^-, k_y^+] = \mathbb{1}_{x=y} 2k_x^0, \quad [k_x^0, k_y^+] = \mathbb{1}_{x=y} k_x^+, \quad [k_x^0, k_y^-] = -\mathbb{1}_{x=y} k_x^-, \qquad x, y \in E$$

- k_x^+ and k_x^- adjoint in $L^2(\rho_{p,\alpha})$; reversible $\rho_{p,\alpha} := \otimes_{x \in E} \mathrm{NegativeBinomial}(p, \alpha_x)$.

Duality

$$\sum_{x\in E} k_x^+, \qquad \sum_{x\in E} k_x^0, \qquad \sum_{x\in E} k_x^- \quad \text{commute with } L$$

$$\Rightarrow \quad U_{\xi,\phi} := \exp\left(\xi \sum_{x\in E} k_x^+ - \overline{\xi} \sum_{x\in E} k_x^-\right) \exp\left(2i\phi \sum_{x\in E} k_x^0\right)$$

commutes with L for all $\xi \in \mathbb{C}, \phi \in [0, 2\pi)$

Theorem (Carinci, Franceschini, Giardinà, Groenevelt, Redig, 19')

$H_{\text{cheap}} : (\eta, \nu) \mapsto \mathbb{1}_{\eta=\nu} \frac{1}{\rho(\{\eta\})}$

$\tanh \xi = \sqrt{p}, \quad \phi = 0$

$\Rightarrow \quad (U_{\xi,\phi} H_{\text{cheap}}(\,\cdot\,, \nu)(\eta)) =$ *orthogonal duality function (Meixner polynomials)*

Change of representation

$$\tanh\xi = \sqrt{p}, \quad \phi = 0 \qquad \Rightarrow \qquad U_{\xi,\phi} \text{ intertwines} \qquad k_x^+, k_x^-, k_x^0 \qquad \text{and}$$

$$\begin{aligned}
K_x^- &= \frac{1}{1-p}\left(p k_x^+ - 2\sqrt{p} k_x^0 + k_x^- \right) \\
K_x^0 &= \frac{1}{1-p}\left(-\sqrt{p} k_x^+ + (p+1) k_x^0 - \sqrt{p} k_x^- \right) \\
K_x^+ &= \frac{1}{1-p}\left(k_x^+ - 2\sqrt{p} k_x^0 + p k_x^- \right), \qquad x \in E.
\end{aligned}$$

$(K_x^\sharp U = U k_x^\sharp)$

What happens if E is non-discrete?

$E =$ Polish space (e.g. $\mathbb{R}^d$).

Question. Does the algebraic framework still work?

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Answer. Yes! ☺

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$E =$ Polish space (e.g. $\mathbb{R}^d$).
Question. Does the algebraic framework still work?
Answer. Yes! ☺

Our ideas are motivated by the following communities.

- representation theory of renormalized square of white noise, infinite-dimensional $*$-Lie algebras, current algebras *(Accardi, Boukas, Franz, Skeide)*
- Many body quantum mechanics: *canonical commutation relation*s (CCRs) and the *canonical anti-commutation relations* (CARs) (infinite dimensional Heisenberg algebra group) *(Bratteli, Robinson)*
- Polynomials in infinite dimensions, orthogonal decompositions for Lévy processes, extended Fock spaces, chaos decompositions, Jacobi-Fields *(Lytvynov, Berezansky, Mierzejewski, Bozejko, Rodionova)*
- Point processes, Papangelou kernels *(Last, Penrose)*
- Intertwiners for particle systems in the continuum *(previous talk)*

Generalizing k^+

Finite E : $$k^+(\varphi) := \sum_{x\in E} \varphi_x k_x^+ \qquad \varphi = (\varphi_x)_{x\in E}, \quad \varphi_x \in \mathbb{C}$$

$$\Rightarrow k^+(\mathbb{1}_{\{x\}}) = k_x^+$$

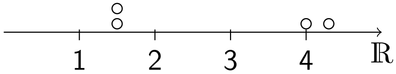
$$\Rightarrow k^+(\varphi) f(\eta) = \frac{1}{\sqrt{p}} \sum_{x\in E} \varphi_x f(\eta - \delta_x)\eta_x, \qquad \eta \in \mathbb{N}_0^E, \quad f : \mathbb{N}_0^E \to \mathbb{C}$$

General E : $$k^+(\varphi) f(\eta) := \frac{1}{\sqrt{p}} \int_E \varphi(x) f(\eta - \delta_x)\, \mathrm{d}\eta(x), \quad \varphi : E \to \mathbb{C}, \quad f : \mathbf{N} \to \mathbb{C}$$

$\eta \in \mathbf{N} = \left\{ \sum_{k=1}^{N} \delta_{x_k} : x_k \in E, N \in \mathbb{N}_0 \cup \{\infty\} \right\}$

Modern notation for point processes (Last, Penrose)

Example: $E = \mathbb{R}, \quad \eta = 2\delta_{1.5} + \delta_4 + \delta_{4.3} =$



$$\Rightarrow \sqrt{p} k^+(\varphi) f(\eta) = 2\varphi(1.5) f(\delta_{1.5} + \delta_4 + \delta_{4.3}) + \varphi(4) f(2\delta_{1.5} + \delta_{4.3}) + \varphi(4.3) f(2\delta_{1.5} + \delta_4)$$

Definition of k^+, k^0, k^-

Fix σ-finite measure α on E, $\qquad p \in (0,1)$.

$$k^+(\varphi)f(\eta) := \frac{1}{\sqrt{p}} \int \varphi(x) f(\eta - \delta_x)\eta(\mathrm{d}x),$$

$$k^0(\varphi)f(\eta) := f(\eta) \int \varphi(x) \left(\eta + \frac{1}{2}\alpha\right)(\mathrm{d}x),$$

$$k^-(\varphi)f(\eta) := \sqrt{p} \int \overline{\varphi(x)} f(\eta + \delta_x)\,(\alpha + \eta)\,(\mathrm{d}x), \quad \eta \in \mathbf{N}_{<\infty}, \quad f : \mathbf{N}_{<\infty} \to \mathbb{C},$$

$\varphi \in \mathcal{C}$ = set of measurable, bounded $\varphi : E \to \mathbb{C}$ satisfying $\alpha[\varphi \neq 0] < \infty$.

The measures $w_{p,\alpha}$ and $\rho_{p,\alpha}$

- $w_{p,\alpha}$ = measure on finite configurations $\mathbf{N}_{<\infty}$:

$$\int f \,\mathrm{d}w_{p,\alpha} = \sum_{n=0}^{\infty} \frac{p^n}{n!} \int f(\delta_{x_1} + \ldots + \delta_{x_n})\lambda_n(\mathrm{d}(x_1, \ldots, x_n)), \qquad f : \mathbf{N}_{<\infty} \to [0, \infty)$$

  λ_n = measure on E^n:

$$\int f_n \,\mathrm{d}\lambda_n = \int \cdots \int f_n(x_1, \ldots, x_n)(\alpha + \delta_{x_1} + \ldots + \delta_{x_{n-1}})(\mathrm{d}x_n) \cdots (\alpha + \delta_{x_1})\alpha(\mathrm{d}x_1).$$

  Notice that $(1-p)^{-a}\mathrm{NegativeBinomial}(p, a)(\{n\}) = \frac{p^n}{n!}(a+n-1)\cdots(a+1)a$.
- $\rho_{p,\alpha}$ = measure on $\mathbf{N}$: distribution of the Pascal process, i.e., $\xi \sim \rho_{p,\alpha}$ satisfies:
  - $\xi(A_1), \ldots, \xi(A_N)$ are independent for pairwise disjoint $A_1, \ldots, A_N$,
  - $\xi(A)$ has a negative binomial distribution with parameters $\alpha(A)$ and p.

$$\alpha(E) < \infty \qquad \Rightarrow \qquad \rho_{p,\alpha} = (1-p)^{\alpha(E)} w_{p,\alpha}.$$

Properties

Hilbert space $H = L^2(w_{p,\alpha})$.

Theorem (Floreani, Jansen, W., '22+)

1. $k^+(\varphi)$, $k^0(\varphi)$, $k^-(\varphi)$ *are (unbounded) linear operators* $D \to D$. D *is dense in* H.
2. $k^+(\varphi)$ *and* $k^0(\varphi)$ *are linear in* φ *whereas* $k^-(\varphi)$ *is antilinear in* φ.
3. *Commutation relations:*

$$[k^+(\varphi), k^+(\theta)] = 0, \qquad [k^0(\varphi), k^0(\theta)] = 0, \qquad [k^-(\varphi), k^-(\theta)] = 0,$$
$$[k^-(\varphi), k^+(\theta)] = 2k^0(\overline{\varphi}\theta), \quad [k^0(\varphi), k^+(\theta)] = k^+(\varphi\theta), \quad [k^0(\varphi), k^-(\theta)] = -k^-(\overline{\varphi}\theta).$$

Representation of a current algebra (Accardi, Franz, Skeide)

4. $\psi(\eta) := \mathbb{1}_{\eta=0}$ has the property $k^-(\varphi)\psi = 0$ for all $\varphi \in \mathcal{C}$. We call it *vacuum*.
5. $\left\langle f, k^0(\varphi)g \right\rangle = \left\langle k^0(\varphi)f, g \right\rangle, \quad \langle f, k^+(\varphi)g\rangle = \langle k^-(\varphi)f, g\rangle, \quad \varphi \in \mathcal{C} \quad f, g \in D$.
6. Orthogonality:

$$\left\langle k^+(\varphi_1)\cdots k^+(\varphi_n)\psi, k^+(\theta_1)\cdots k^+(\theta_m)\psi \right\rangle$$
$$= \delta_{n,m} n! \int \mathrm{Symmetrization}\,(\varphi_1 \otimes \cdots \otimes \varphi_n)\,\overline{\mathrm{Symmetrization}\,(\theta_1 \otimes \cdots \otimes \theta_m)}\lambda_n(\mathrm{d}x)$$
$$\varphi_1, \ldots, \varphi_n, \quad \theta_1, \ldots, \theta_m \in \mathcal{C}, \qquad n, m \in \mathbb{N}_0 \ .$$

We say: *Fock representation on the Hilbert space H of the current algebra of $su(1,1)$ with vacuum $\psi \in H$*

Application to the generalized symmetric inclusion process (gSIP)

$$Lf(\eta) = \iint \left(f(\eta - \delta_y + \delta_x) - f(\eta)\right) c(x,y)(\eta+\alpha)(\mathrm{d}x)\eta(\mathrm{d}y)$$

$$\Rightarrow \qquad L = k^+(\varphi_1)k^-(\varphi_2) + k^-(\varphi_1)k^+(\varphi_2) - 2k^0(\varphi_1)k^0(\varphi_2) - C\mathrm{id}$$

with

$$c(x,y) := \varphi_1(x)\varphi_2(y) + \varphi_2(x)\varphi_1(y),$$
$$C := \int \varphi_1\varphi_2 \,\mathrm{d}\alpha - \frac{1}{2}\int \varphi_1 \,\mathrm{d}\alpha \int \varphi_2 \,\mathrm{d}\alpha,$$

non-negative $\varphi_1, \varphi_2 \in \mathcal{C}$.

Commutation relations $\quad + \quad (k^+(\varphi))^* = k^-(\varphi)$

$\Rightarrow$ consistency and reversibility

(gSIP) introduced by (Floreani, Jansen, Redig, W., '21)

Extended Fock-space

$$f \in L^2(\rho_{w,\alpha}) \xleftrightarrow[f(\delta_{x_1}+\ldots+\delta_{x_n})=f_n(x_1,\ldots,x_n)]{\text{identify}} f = (f_n)_{n\in\mathbb{N}_0} \in \bigoplus_{n=0}^{\infty} \frac{p^n}{n!} L^2_{\text{PermutationInvariant}}(\lambda_n)$$

Example:

$$k^+(\varphi)f(\eta) = \frac{1}{\sqrt{p}} \int \varphi(x) f(\eta - \delta_x)\eta(\mathrm{d}x)$$

$$\longleftrightarrow (k^+(\varphi)f)_n = \frac{n}{\sqrt{p}} \text{Symmetrization}(f_{n-1} \otimes \varphi)$$

$\Rightarrow$ *creation operator* in the context of Fock spaces.

extended Fock space and creation operators: Bozejko, Lytvynov, Rodionova

Chaos decomposition of the Pascal process

The operator

$$\mathfrak{U} : L^2(w_{p,\alpha}) = \bigoplus_{n=0}^{\infty} \frac{p^n}{n!} L^2_{\text{PermutationInvariant}}(\lambda_n) \to L^2(\rho_{p,\alpha}),$$

$$f = (f_n)_{n\in\mathbb{N}_0} \mapsto \sum_{n=0}^{\infty} \frac{(1-p)^n}{n!} \mathcal{P}_n(\,\cdot\,; f_n)$$

is unitary.

$\Rightarrow$ Each square integrable functional G has a unique chaos decomposition f_n with

$$G = \sum_{n=0}^{\infty} \frac{(1-p)^n}{n!} \mathcal{P}_n(\,\cdot\,; f_n).$$

Orthogonal decompositions Pascal processes: Lytvynov

A second representation

We put $K^\sharp(\varphi) := \mathfrak{U} k^\sharp(\varphi)\mathfrak{U}^{-1}$.

Theorem (Floreani, Jansen, W., '22+)

The operators $K^+(\varphi)$, $K^0(\varphi)$, $K^-(\varphi)$, $\varphi \in \mathcal{C}$ are a Fock representation on $L^2(\rho_{p,\alpha})$ of the current algebra of $su(1,1)$ with vacuum $\mathbb{1}$. Moreover,

$$
\begin{aligned}
K^-(\varphi) &= \frac{1}{1-p}\Big(p k^+(\overline{\varphi}) - 2\sqrt{p} k^0(\overline{\varphi}) + k^-(\varphi) \Big)\\
K^0(\varphi) &= \frac{1}{1-p}\Big(-\sqrt{p} k^+(\varphi) + (p+1) k^0(\varphi) - \sqrt{p} k^-(\overline{\varphi}) \Big)\\
K^+(\varphi) &= \frac{1}{1-p}\Big(k^+(\varphi) - 2\sqrt{p} k^0(\varphi) + p k^-(\overline{\varphi}) \Big).
\end{aligned}
$$

In particular: $K^+(\varphi_1)\cdots K^+(\varphi_n)\mathbb{1} = c^n \mathcal{P}_n\left(\,\cdot\,, \varphi_1 \otimes \cdots \otimes \varphi_n\right)$ where $c = \frac{1}{\sqrt{p}} - \sqrt{p}$.

Interpretation:

$K^\sharp(\varphi)$ acts on $G \in L^2(\rho_{p,\alpha})$ $\quad\longleftrightarrow\quad$ $k^\sharp(\varphi)$ acts on the chaos decomposition of G.

Probabilistic interpretations

If $\varphi \geq 0$, then both

$$\frac{1-p}{\sqrt{p}} K^{-}(\varphi) G(\eta)$$
$$= \int \varphi(x) \left(G(\eta - \delta_x) - G(\eta)\right) \eta(\mathrm{d}x) + \int \varphi(x) \left(G(\eta + \delta_x) - G(\eta)\right) (\eta + \alpha)(\mathrm{d}x)$$

and

$$-(1-p) \left(K^0(\varphi) G(\eta) - \frac{1}{2} G(\eta) \int \varphi \, \mathrm{d}\alpha \right)$$
$$= \int \varphi(x) \left(G(\eta - \delta_x) - G(\eta)\right) \eta(\mathrm{d}x) + p \int \varphi(x) \left(G(\eta + \delta_x) - G(\eta)\right) (\eta + \alpha)(\mathrm{d}x)$$

are generators of birth-death processes.

Poisson case: Last

The unitary operator $\mathfrak{U}$: a closer look

Let α be finite ($\Rightarrow \rho_{p,\alpha} = (1-p)^{\alpha(E)} w_{p,\alpha}$).

1. The operators

$$U_{\xi,\phi} := \exp\left(\xi k^+(\mathbb{1}) - \overline{\xi} k^-(\mathbb{1})\right) \exp\left(2\mathrm{i}\phi k^0(\mathbb{1})\right), \qquad \xi \in \mathbb{C}, \quad \phi \in [0, 2\pi)$$

   are unitary (unitary representation of the Lie group $SU(1,1)$).

2. $W : f \mapsto (1-p)^{-\frac{\alpha(E)}{2}} f$ is a unitary operator from $L^2(w_{p,\alpha})$ to $L^2(\rho_{p,\alpha})$

Theorem (Floreani, Jansen, W., '22+)

$$\phi = 0, \quad \tanh \xi = \sqrt{p} \qquad \Rightarrow \qquad \mathfrak{U} = W U_{\xi,\phi}$$

Generalization of Carinci, Franceschini, Giardinà, Groenevelt, Redig's result

Application for consistent processes

Theorem (Floreani, Jansen, W., '22+)

η_t, $t \geq 0$ *Markov process with values in* $\mathbf{N}$, *finite measure* α, $p \in (0,1)$.
Assume

1. *consistency;*
2. *the number of particles is preserved;*
3. *the distribution of the Pascal process is reversible.*

Then, $U_{\xi,\phi}$ *is self-intertwiner for Markov semigroup* P_t, $t \geq 0$ *for all* $\xi \in \mathbb{C}$, $\phi \in [0, 2\pi)$*:*

$$U_{\xi,\phi} P_t = P_t U_{\xi,\phi}.$$

$\phi = 0$, $\tanh \xi = \sqrt{p}$ $\Rightarrow$ $U_{\xi,\phi}$ recovers the self-intertwiner in terms of generalized Meixner polynomials.

Proof (sketch)

1. Consistency $\Leftrightarrow P_t k^+(\mathbb{1}) = k^+(\mathbb{1}) P_t$
2. $k^0(\mathbb{1}) =$ multiplication operator by $\eta \mapsto \left(\eta(E) + \frac{\alpha(E)}{2}\right)$. Thus,

   $$\text{number of particles is preserved} \quad \Rightarrow \quad P_t k^0(\mathbb{1}) = k^0(\mathbb{1}) P_t$$

3. Adjoining $P_t k^+(\mathbb{1}) = k^+(\mathbb{1}) P_t$ yields

   $$\left(k^+(\mathbb{1})\right)^* (P_t)^* = (P_t)^* \left(k^+(\mathbb{1})\right)^*$$

   By reversibility and $(k^-(\mathbb{1}))^* = k^+(\mathbb{1})$:

   $$k^-(\mathbb{1}) P_t = P_t k^-(\mathbb{1})$$

Hence, also $U_{\xi,\phi} = \exp\left(\xi k^+(\mathbb{1}) - \overline{\xi} k^-(\mathbb{1})\right) \exp\left(2\mathrm{i}\phi k^0(\mathbb{1})\right)$ commutes with P_t.

Summary

	discrete	non-discrete
Repr.	k_x^+, k_x^0, k_x^- acts on lattice site x	$k^+(\varphi), k^0(\varphi), k^-(\varphi), \varphi$ test function
	$k_x^+ f(\eta) = \frac{1}{\sqrt{p}} f(\eta - \delta_x)\eta_x$	$k^+(\varphi) f(\eta) = \frac{1}{\sqrt{p}} \int \varphi(x) f(\eta - \delta_x)\eta(\mathrm{d}x)$
	$\eta \in \mathbb{N}_0^E$	$\eta = \sum_{k=1}^{N} \delta_{x_k}, \quad x_k \in E, \quad N \in \mathbb{N}_0$
SIP	$\sum_{x,y \in E} c_{x,y} \left(f(\eta - \delta_x + \delta_y) - f(\eta)\right)(\alpha_y + \eta_y)\eta_x$	$\iint c(x,y)\left(f(\eta - \delta_y + \delta_x) - f(\eta)\right)(\eta + \alpha)(\mathrm{d}x)\eta(\mathrm{d}y)$
	$\sum_{\{x,y\} \subset E} c_{x,y} L_{x,y} k_x^+ k_y^- + k_x^- k_y^+ - 2k_x^0 k_y^0 + \frac{\alpha_x \alpha_y}{2}$	$k^+(\varphi_1)k^-(\varphi_2) + k^-(\varphi_1)k^+(\varphi_2)$ $-2k^0(\varphi_1)k^0(\varphi_2) - C\mathrm{id}$
Intertw.	$U = \exp\left(\xi \sum_{x \in E} k_x^+ - \bar{\xi} \sum_{x \in E} k_x^-\right), \arctan\xi = \sqrt{p}$	$U = \exp\left(\xi k^+(\mathbb{1}) - \bar{\xi} k^-(\mathbb{1})\right), \arctan\xi = \sqrt{p}$
	$(UH_{\text{cheap}}(\,\cdot\,,\nu)(\eta))$ = Meixner polynomials	U = infinite dimensional Meixner polynomials

Thank you!

Notations

- E is a Polish space
- $\mathbf{N} = \left\{ \sum_{k=1}^{N} \delta_{x_k} : x_k \in E, N \in \mathbb{N}_0 \cup \{\infty\} \right\}$
- α is a measure on $(E, \mathcal{E})$
- $p \in (0,1)$
- $k^+(\varphi) f(\eta) = \frac{1}{\sqrt{p}} \int \varphi(x) f(\eta - \delta_x) \eta(\mathrm{d}x)$,
  $k^0(\varphi) f(\eta) = f(\eta) \int \varphi(x) \left(\eta + \frac{1}{2}\alpha\right)(\mathrm{d}x)$,
  $k^-(\varphi) f(\eta) = \sqrt{p} \int \overline{\varphi(x)} f(\eta + \delta_x)\,(\alpha + \eta)\,(\mathrm{d}x)$
- $\mathcal{C}$ is the set of measurable, bounded functions $\varphi : E \to \mathbb{C}$ satisfying $\alpha[\varphi \neq 0] < \infty$.
- $\int f \,\mathrm{d}w_{p,\alpha} = \sum_{n=0}^{\infty} \frac{p^n}{n!} \int f(\delta_{x_1} + \ldots + \delta_{x_n}) \lambda_n(\mathrm{d}(x_1, \ldots, x_n)), f : \mathbf{N}_{<\infty} \to [0, \infty)$
- $\int f_n \,\mathrm{d}\lambda_n = \int \cdots \int f_n(x_1, \ldots, x_n)(\alpha + \delta_{x_1} + \ldots + \delta_{x_{n-1}})(\mathrm{d}x_n) \cdots (\alpha + \delta_{x_1}) \alpha(\mathrm{d}x_1)$
- $\rho_{p,\alpha}$ = distribution of the pascal process
- $\mathfrak{U} : (f_n)_{n \in \mathbb{N}_0} \mapsto \sum_{n=0}^{\infty} \frac{(1-p)^n}{n!} \mathcal{P}_n(\,\cdot\,; f_n)$