Convexity and constructive infima

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May 13, 2016

Abstract

We show constructively that every quasi-convex uniformly continuous function $f : C \to \mathbb{R}^+$ has positive infimum, where C is a convex compact subset of $\mathbb{R}^n$. This implies a constructive separation theorem for convex sets.

1 Introduction and main results

A well-known reformulation of Brouwer's fan theorem for detachable bars states that every uniformly continuous function $f : [0,1] \to \mathbb{R}^+$ has positive infimum, i.e. $\inf f > 0$, see [1, 7]. Here $\mathbb{R}^+ = \{x \in \mathbb{R} \mid x > 0\}$. It is also well-known that Brouwer's fan theorem does not hold in constructive mathematics in the tradition of Errett Bishop [6]. In this paper we prove within Bishop's constructive mathematics [3, 4] that under the additional assumption of quasi-convexity of f we have $\inf f > 0$. More generally, this holds for arbitrary convex domains. Note that a function $f : \mathrm{C} \to \mathbb{R}$, where C is a convex subset of $\mathbb{R}^n$, is called *quasi-convex* if

$$f(\lambda x + (1-\lambda)y) \leq \max\{f(x), f(y)\}$$

for all $\lambda \in [0,1]$ and $x, y \in \mathrm{C}$. Hence, in particular, any *convex* function $f : \mathrm{C} \to \mathbb{R}$, i.e.

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$$

for all $\lambda \in [0,1]$ and $x, y \in \mathrm{C}$, is quasi-convex. Our main result is:

Theorem 1. *Fix a convex and compact subset* C *of* $\mathbb{R}^n$ *and suppose that* $f : \mathrm{C} \to \mathbb{R}^+$ *is uniformly continuous and quasi-convex. Then* $\inf f > 0$.

We will prove Theorem 1 in Section 2. In [2, Proposition 1] we showed that any uniformly continuous convex function

$$f : \mathcal{X}_n = \left\{ (x_1, \ldots, x_n) \in \mathbb{R} \mid \sum_{i=1}^{n} x_i = 1 \text{ and } x_i \geq 0 \text{ for all } i \right\} \to \mathbb{R}^+$$

has positive infimum. The generalization to the class of quasi-convex functions, which is by far richer than the class of convex functions, and in particular the generalization of the domain $\mathcal{X}_n$ to arbitrary convex compact subsets C of $\mathbb{R}^n$ provides interesting new insights. For instance, as a consequence of Theorem 1 we obtain Theorem 2, a separation result for convex sets in $\mathbb{R}^n$ which is almost as powerful as its classical counterpart. Here *classical mathematics* refers to mathematics with the law of excluded middle as an admitted proof tool and the usual weaker existential quantifier. Before we formulate Theorem 2, we like to clarify our notation and to recall a few standard definitions from constructive mathematics.

Set $\mathbb{N} := \{1, 2, 3, \ldots\}$, and for $n \in \mathbb{N}$ set $I_n := \{1, \ldots, n\}$. Moreover, for $x, y \in \mathbb{R}^n$ we set

- $\langle x, y \rangle := \sum_{i \in I_n} x_i \cdot y_i$
- $\|x\| := \sqrt{\langle x, x \rangle}$
- $d(x, y) := \|y - x\|$.

Fix $\varepsilon > 0$ and sets $D \subseteq \mathrm{C} \subseteq \mathbb{R}^n$. The set D is an *ε-approximation* of C if for every $x \in \mathrm{C}$ there exists $y \in D$ with $d(x, y) < \varepsilon$. The set C is

- *inhabited*, if it has an element
- *totally bounded* if for every n there exist elements $x_1, \ldots, x_m$ of C such that $\{x_1, \ldots, x_m\}$ is a $1/n$-approximation of C
- *closed* or *complete* if every Cauchy sequence in C has a limit in C
- *compact* if it is totally bounded and complete
- *convex* if $\lambda x + (1 - \lambda) y \in \mathrm{C}$ for all $x, y \in \mathrm{C}$ and $\lambda \in [0, 1]$
- *located* if it is inhabited and if for every $x \in \mathbb{R}^n$ the distance

  $$d(x, \mathrm{C}) := \inf \{ d(x, c) \mid c \in \mathrm{C} \}$$

  exists.

The following Lemma [5, Corollary 2.2.7] provides an important criterion for the existence of infima and suprema.

Lemma 1. *If* C *is totally bounded, and* $f : \mathrm{C} \to \mathbb{R}$ *is uniformly continuous, then the infimum of* f*,*

$$\inf f = \inf \{f(y) \mid y \in \mathrm{C}\}$$

does exist. The same holds for the supremum

$$\sup f = \sup \{f(y) \mid y \in \mathrm{C}\} .$$

Theorem 2. *Let* $\mathrm{C}, Y \subseteq \mathbb{R}^n$ *such that*

1. C *is convex and compact;*
2. Y *is convex, closed, and located;*
3. $d(c, y) > 0$ *for all* $c \in \mathrm{C}$ *and* $y \in Y$ *.*

Then there exist $p \in \mathbb{R}^n$ *and reals* α, β *such that*

$$\langle p, c\rangle \ < \ \alpha \ < \ \beta \ < \ \langle p, y\rangle$$

for all $c \in \mathrm{C}$ *and* $y \in Y$*.*

We prove this theorem throughout Section 3.

2 Proof of Theorem 1

We start with some technical lemmas. For a subset C of $\mathbb{R}^n$, $i \in I_n$, and $t \in \mathbb{R}$ set

$$\mathrm{C}_i^t = \{x \in \mathrm{C} \mid x_i = t\} .$$

Lemma 2. *Fix a convex subset* C *of* $\mathbb{R}^n$*,* $t \in \mathbb{R}$ *and* $i \in I_n$*. Suppose further that there are* $y, z \in \mathrm{C}$ *with* $y_i < t < z_i$*. Then there exists* $\lambda \in (0, 1)$ *such that*

$$\lambda y + (1 - \lambda)z \in \mathrm{C}_i^t .$$

Proof. Set $\lambda = \frac{z_i - t}{z_i - y_i}$. □

We call C_i^t *admissible* if there exist $y, z \in \mathrm{C}$ with $y_i < t < z_i$.

Lemma 3. *Assume that $n > 1$. Fix a subset C of $\mathbb{R}^n$ and suppose that C_i^t is convex and compact. Then there exists a convex compact subset $\hat{\mathrm{C}}$ of $\mathbb{R}^{n-1}$ and a uniformly continuous bijection*

$$g : \hat{\mathrm{C}} \to \mathrm{C}_i^t$$

which is affine *in the sense that*

$$g(\lambda x + (1-\lambda)y) = \lambda g(x) + (1-\lambda)g(y)$$

for all $\lambda \in [0,1]$ and $x, y \in \hat{\mathrm{C}}$.

Proof. We can assume that $i = 1$. Set

$$\hat{\mathrm{C}} = \left\{(x_2, \ldots, x_n) \in \mathbb{R}^{n-1} \mid (t, x_2, \ldots, x_n) \in \mathrm{C}_1^t\right\}$$

and

$$g(x_2, \ldots, x_n) = (t, x_2, \ldots, x_n)\,.$$

□

The next lemma is crucial for the proof of Theorem 1, and of interest of its own. Its proof is based on the fact that if $\mathrm{C} \subseteq \mathbb{R}^n$ is totally bounded and $f : \mathrm{C} \to \mathbb{R}^m$ is uniformly continuous, then

$$f(C) = \{f(c) \mid c \in \mathrm{C}\}$$

is also totally bounded, see [4, Chapter 4, Proposition 4.2].

Lemma 4. *If $\mathrm{C} \subseteq \mathbb{R}^n$ is convex and compact and C_i^t is admissible, then C_i^t is convex and compact.*

Proof. Let $\mathrm{C} \subseteq \mathbb{R}^n$ be convex and compact and let C_i^t be admissible. Without loss of generality, we may assume that $t = 0$ and $i = 1$. There exist $y, z \in C$ with $y_1 < 0 < z_1$. Let

$$\mathcal{M} = \mathrm{C}_1^0\,, \quad \mathcal{L} = \{x \in \mathrm{C} \mid x_1 \leq 0\}\ , \text{ and } \quad \mathcal{R} = \{x \in \mathrm{C} \mid x_1 \geq 0\}\,.$$

We show that the sets $\mathcal{L}$, $\mathcal{R}$ and $\mathcal{M}$ are convex and compact. It is clear that these sets are convex and complete. It remains to show that they are totally bounded. We start with the case of $\mathcal{R}$. Set

$$\kappa : \mathbb{R} \to \mathbb{R},\ s \mapsto \max(-s, 0)$$

and

$$f : \mathbb{R}^n \to \mathbb{R}^n,\ x \mapsto \frac{z_1}{z_1 + \kappa(x_1)}x + \frac{\kappa(x_1)}{z_1 + \kappa(x_1)}z$$

and note that

- f is uniformly continuous
- f maps C onto $\mathcal{R}$.

In order to prove the latter, we proceed step by step and show that

i) $f(\mathrm{C}) \subseteq \mathrm{C}$

ii) $f(\mathrm{C}) \subseteq \mathcal{R}$

iii) $f(\mathrm{C}) = \mathcal{R}$.

The property i) follows from the convexity of C. In order to show ii), fix $x \in \mathrm{C}$. We show that the assumption that the first component of $f(x)$ is negative is contradictive. So assume that

$$\frac{z_1}{z_1 + \kappa(x_1)} x_1 + \frac{\kappa(x_1)}{z_1 + \kappa(x_1)} z_1 < 0.$$

Then $x_1 < 0$ and therefore $\kappa(x_1) = -x_1$. We obtain

$$z_1 \cdot x_1 - x_1 \cdot z_1 < 0,$$

a contradiction. The property iii) follows from the fact that f leaves the elements of $\mathcal{R}$ unchanged. So we have shown that $\mathcal{R}$ is totally bounded. Analogously, we can show that $\mathcal{L}$ is totally bounded. Next, we show that

$$\mathcal{M} = f(\mathcal{L}),$$

which implies that $\mathcal{M}$ is totally bounded as well. To this end, fix $x \in \mathcal{L}$. Then $\kappa(x_1) = -x_1$ and therefore

$$\frac{z_1}{z_1 + \kappa(x_1)} x_1 + \frac{\kappa(x_1)}{z_1 + \kappa(x_1)} z_1 = 0,$$

which implies that $f(x) \in \mathcal{M}$. □

The following Lemma 5 basically already proves Theorem 1.

Lemma 5. *Fix a convex compact subset* C *of* $\mathbb{R}^n$ *and suppose that*

$$f : \mathrm{C} \to \mathbb{R}^+$$

is convex and uniformly continuous. Assume further that

$$\inf \left\{ f(x) \mid x \in \mathrm{C}_i^t \right\} > 0$$

for every admissible C_i^t*. Then* $\inf f > 0$*.*

Proof. Note that $\inf f$ exists by Lemma 1. We define a sequence (x^m) in C and an weakly increasing binary sequence (λ^m) such that

- $\lambda^m = 0 \;\Rightarrow\; f(x^m) < \min\{2^{-m}, f(x^{m-1})\}$
- $\lambda^m = 1 \;\Rightarrow\; \inf f > 0$ and $x^m = x^{m-1}$

for every $m \geq 2$. Note that under these conditions the sequence $(f(x^m))$ is weakly decreasing.

Let x^1 be an arbitrary element of C and set $\lambda^1 = 0$. Assume that x^m and λ^m have already been defined.

<u>case 1</u> If $0 < \inf f$ or $\lambda^m = 1$, set $x^{m+1} = x^m$ and $\lambda^{m+1} = 1$.

<u>case 2</u> If

$$\inf f < \min\left\{2^{-(m+1)}, f(x^m)\right\},$$

choose x^{m+1} in C with

$$f(x^{m+1}) < \min\left\{2^{-(m+1)}, f(x^m)\right\}$$

and set $\lambda^{m+1} = 0$.

We show that the sequence (x^m) converges.

It is sufficient to show that for each component $i \in I_n$ the sequence $(x_i^m)_{m\in\mathbb{N}}$ is a Cauchy sequence. We consider the case $i = 1$. Fix $\varepsilon > 0$. Let D be the image of C under the projection onto the first component, i.e.

$$D = \{\mathrm{pr}_1(x) \mid x \in \mathrm{C}\}.$$

Note that D is a totally bounded interval. Denote its infimum by a and its supremum by b.

<u>case 1</u> If $b - a < \varepsilon$, then $\left|x_1^k - x_1^l\right| \leq \varepsilon$ for all k, l.

<u>case 2</u> If $b - a > 0$, there exists a finite $\frac{\varepsilon}{2}$-approximation F of (a, b). Note that for every t with $a < t < b$ the set $\mathcal{C}_1^t$ is admissible. Hence, we can choose an l_0 such that

$$f(x) > 2^{-l_0}$$

for all $t \in F$ and all $x \in \mathcal{C}_1^t$. Fix $k, l \geq l_0$. We show that $\left|x_1^k - x_1^l\right| \leq \varepsilon$.

<u>case 2.1</u> If $\lambda^{l_0} = 1$, then $x^k = x^l$.

case 2.2 If $\lambda^{l_0} = 0$, then $f(x^k) < 2^{-l_0}$ and $f(x^l) < 2^{-l_0}$. Suppose that $x_1^k - x_1^l > \varepsilon$. Then there exists $t \in F$ with $x_1^k < t < x_1^l$. According to Lemma 2 there is $\lambda \in [0,1]$ such that $\lambda x^k + (1-\lambda)x^l \in \mathrm{C}_1^t$, and by quasi-convexity of f we obtain

$$f(\lambda x^k + (1-\lambda)x^l) \leq \max\{f(x^k), f(x^l)\} < 2^{-l_0}$$

which contradicts the construction of l_0. Therefore, $x_1^k - x_1^l \leq \varepsilon$, and similarly also $x_1^l - x_1^k \leq \varepsilon$.

Let $x \in \mathrm{C}$ be the limit of the sequence (x^m). There exists an l such that

$$f(x) > 2^{-l}$$

and a k such that

$$d(x,y) < 2^{-k} \;\Rightarrow\; |f(x) - f(y)| < 2^{-(l+1)}$$

for all $y \in \mathrm{C}$. Finally, pick $N > l$ such that

$$d(x, x^N) < 2^{-k}.$$

Then $f(x^N) \geq 2^{-N}$, therefore $\lambda_N = 1$, therefore $\inf f > 0$. □

of Theorem 1. We conduct induction over the dimension n.

If $n = 1$, then every admissible set C_1^t equals $\{t\}$, so $\inf f > 0$ follows from Lemma 5.

Now fix $n > 1$ and assume the assertion of Theorem 1 holds for $n-1$. Furthermore, let C be a convex compact subset of $\mathbb{R}^n$, and suppose that

$$f : \mathrm{C} \to \mathbb{R}^+$$

is convex and uniformly continuous. Fix an admissible subset C_i^t of C. By Lemma 4, C_i^t is convex and compact. Using Lemma 3 construct the convex compact set $\hat{\mathrm{C}} \subseteq \mathbb{R}^{n-1}$ and the uniformly continuous affine bijection

$$g : \hat{\mathrm{C}} \to \mathrm{C}_i^t.$$

Then $F : \hat{\mathrm{C}} \to \mathbb{R}^+$ given by $F = f \circ g$ is quasi-convex and uniformly continuous. The induction hypothesis now implies that

$$\inf \left\{ f(x) \mid x \in \mathrm{C}_t^i \right\} = \inf \left\{ F(x) \mid x \in \hat{\mathrm{C}} \right\} > 0\,.$$

Thus, $\inf f > 0$ follows from Lemma 5. □

3 Proof of Theorem 2

We use the following result from [2].

Lemma 6. *Let $\mathcal{Y}$ be an inhabited convex subset of $\mathbb{R}^n$ and $x \in \mathbb{R}^n$ such that $d = d(x, \mathcal{Y})$ exists. Then there exists a unique $a \in \overline{\mathcal{Y}}$ such that $\|a - x\| = d$. Furthermore, we have*

$$\langle a - x, c - a \rangle \geq 0$$

and therefore

$$\langle a - x, c - x \rangle \geq d^2$$

for all $c \in \mathcal{Y}$.

of Theorem 2. Since

$$|d(x, Y) - d(y, Y)| \leq d(x, y)$$

for all $x, y \in \mathbb{R}^n$, the function

$$f : \mathrm{C} \to \mathbb{R},\ c \mapsto d(c, Y)$$

is uniformly continuous. Since Y is closed, Lemma 6 implies that for every $c \in \mathrm{C}$ there is a unique $y \in Y$ with

$$f(c) = d(c, y).$$

Therefore, f is positive-valued and also convex, as we can see as follows. Fix $c_1, c_2 \in \mathrm{C}$ and $\lambda \in [0, 1]$. There are $y_0, y_1, y_2 \in Y$ such that

$$f(c_1) = d(c_1, y_1)\,, \quad f(c_2) = d(c_2, y_2)\,,$$

and

$$f(\lambda c_1 + (1 - \lambda)c_2) = d(\lambda c_1 + (1 - \lambda)c_2, y_0)\,.$$

We obtain

$$\begin{aligned} f(\lambda c_1 + (1 - \lambda)c_2) &= d(\lambda c_1 + (1 - \lambda)c_2, y_0) \\ &\leq d(\lambda c_1 + (1 - \lambda)c_2, \lambda y_1 + (1 - \lambda)y_2) \\ &\leq \lambda d(c_1, y_1) + (1 - \lambda)d(c_2, y_2) \\ &= \lambda f(c_1) + (1 - \lambda)f(c_2)\,. \end{aligned}$$

By Theorem 1, $\inf f > 0$. The set

$$Z = \{y - c \mid x \in \mathrm{C},\ y \in Y\}$$

is inhabited and convex. Since we have

$$\inf \{ \|y - c\| \mid x \in \mathrm{C},\, y \in Y\} = \inf f\,,$$

we can conclude that $\delta = d(0, Z)$ exists and is positive. By Lemma 6, there exists $p \in \mathbb{R}^n$ such that

$$\langle p, y\rangle \geq \delta^2 + \langle p, c\rangle$$

for all $y \in Y$ and $c \in \mathrm{C}$. Setting $\eta = \sup\{\langle p, c\rangle \mid c \in \mathrm{C}\}$ (which exists by Lemma 1),

$$\alpha = \frac{\delta^2}{3} + \eta \;\text{ and }\; \beta = \frac{\delta^2}{2} + \eta\,,$$

we obtain

$$\langle p, c\rangle < \alpha < \beta < \langle p, y\rangle$$

for all $c \in \mathrm{C}$ and $y \in Y$. □

Acknowledgement We thank Christian Ittner for helpful comments.

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