Seminar Winter 2020/21

Characteristic Classes

Characteristic classes are cohomological invariants of vector/principal bundles over topological spaces. More precisely, to such a bundle one associates a cohomology class of the base in a natural way. Characteristic classes are important in many branches of differential geometry and also appear in physics.

In the seminar we will first discuss several constructions of vector bundles. In particular, we will see that under mild assumptions each vector bundle can be obtained from a universal bundle over a so-called classifying space. Then we will introduce Stiefel-Whitney classes, the Euler class and Chern classes and study some of their properties and applications. We may then discuss Chern-Weil theory, which describes how characteristic classes can be computed in terms of the curvature of a principal bundle.

Students should have some background in algebraic topology. Knowledge about differential geometry is also helpful.

The seminar will take place in a week in October before the lectures of the winter term start.

Organizational meeting: ...

Talks

(Talks with a star* may be skipped.)

(1) **Review and Outlook** (C.L. & J.S.)
   - Smooth manifolds, vector bundles, ...
   - Literature: [MS], p.3–24, ...
(2) **Construction of vector bundles**
   - Whitney sum, induced bundles (pull back), tensor product, dual bundle
   - Literature: [MS], p.25–36
(3) **Universal bundles**
   - Grassmann manifolds, tautological bundles, paracompact spaces, infinite Grassmannians and universal bundles
   - Literature: [MS], p.55–68
(4) **Cell structures for Grassmanians**
   - Construction of CW-structure on Grassmannians
   - Literature: [MS], p.68–70, 73–81
(5) **Stiefel-Whitney classes I**
   - Axiomatic description of Stiefel-Whitney classes, show that $H^*(G_n, \mathbb{Z}_2)$ is a polynomial algebra over $\mathbb{Z}_2$ freely generated by the Stiefel Whitney classes
   - Literature: [MS], p.37–47, 83–87

(6) **Stiefel-Whitney classes II***
  - Parallelizability, immersions, boundaries
  - Literature: [MS], p.48-53 (perhaps plus parts of the previous talk), ...

(7) **Oriented bundles, Thom isomorphism, Euler class**
  - Oriented vector bundles, Euler class defined using the Thom isomorphism from the next talk, properties of the Euler class.
  - Literature: [MS], p.95–103

(8) **Proof of Thom isomorphism****
  - Thom class and Thom isomorphism
  - Literature: [MS], p.105–114

(9) **Existence of Stiefel Whitney classes***
  - Existence of Stiefel Whitney classes via Steenrod squares and/or projectivation construction
  - Literature: [MS], p.89–95, [K], §8

(10) **Smooth manifolds**
  - Euler class of smoooth manifolds
  - Literature: [MS], p.115–130

(11) **Complex vector bundles, Chern classes I**
  - Definitions, conjugation, complex Grassmannians
  - Literature: [MS], p. 149–159, [K], §8

(12) **Chern classes II**
  - Chern classes generate cohomology complex Grassmannians, Product theorem for Chern classes
  - Literature: [MS], p. 159–173

(13) **Chern-Weyl construction** (possibly two talks)
  - Connection, Curvature, Gauß-Bonnet
  - Literature: [MS], p. 289–314

[MS] J. W. Milnor and J. D. Stasheff, *Characteristic classes*, Princeton University Press, 1974.
[K] M. Kreck, *Differential algebraic topology*, Graduate Studies in Mathematics, 110, AMS, 2010.