MORSE QUASIFLATS

JINGYIN HUANG, BRUCE KLEINER, AND STEPHAN STADLER

Abstract. We introduce the notion of Morse quasiflats, generalizing Morse quasigeodesics to arbitrary dimension. Under appropriate assumptions on the ambient space we show that a number of alternative definitions are equivalent and quasi-isometry invariant; we also show that Morse quasiflats are asymptotically conical and have canonically defined Tits boundaries. We provide some first applications.

Contents

1. Introduction 2
2. Discussion of proofs 15
3. Preliminaries 18
4. Approximating currents by currents with uniform density 30
5. Quasiflats in metric spaces 35
6. Morse quasiflats 45
7. Asymptotic conditions on spaces and conditions on asymptotic cones 53
8. Stability of Morse quasiflats 58
9. Cycles close to Morse quasiflats 63
10. Neck decomposition and its immediate consequences 69
11. Visibility for Morse quasiflats 73
12. Criteria and Examples for Morseness 81
13. Questions and further directions 91

Date: October 30, 2019.

The first author thanks Max-Planck Institute for mathematics at Bonn, where part of work was done. The second author was supported by NSF grant DMS-1711556 and a Simons Collaboration grant a NSF grant DMS-1405899 and a Simons Fellowship. The third author was supported by DFG grant SPP 2026.
1. Introduction

1.1. Background and overview. Gromov-hyperbolicity has been a central concept in geometric group theory since it was first introduced in [Gro87]. Over the years, it has inspired a large literature with variations on the original idea. Roughly speaking two approaches have been used: the first is to identify hyperbolic features in settings which may be very far from hyperbolic, while the second is to relax Gromov-hyperbolicity to geometric conditions which still retain some weakly hyperbolic flavor. Examples of the first include:

- Relative hyperbolicity [Gro87, Far98, Bow12, DS+05, Osi06],
- Notions of directional hyperbolicity, including rank 1/Morse/contracting/sublinear geodesics and subsets, tree graded structure, hyperbolically embedded subgroups etc [Bal82, BF02, DMS10, CS14, Cor17, Sis18, QRT19, DT15, CH17, KKL98, DS+05, DGO17];
- Generalizations of classical small cancellation theory [Ol', D+96, Cha94, Gro03, Osi07, DG08, Osi10, OOS09, Con11, Wis, DGO17, CG19].
- Analyzing certain non-hyperbolic solvable groups using elements of hyperbolic geometry [FM98, FM99, EFW12, EFW13],
- Acylindrical hyperbolicity [BF02, Bow08, Osi16, DGO17],
- The projection complex [BBF15],
- Hierarchically hyperbolicity [MM00, BHS17a, BHS15, Sis17],

Examples of the second approach include:

- Spaces with coning inequalities [G+83, Wen05],
- Combable, automatic and semi-hyperbolic groups [ECH+92, Gro93, AB95],
- Spaces with certain geodesic bicombing, injective metric spaces, CAT(0) spaces and generalizations [Bus12, Isb64, Dre84, Lan13, DL15a, Kar11],
- Combinatorial non-positive curvature [JS06, BC08, Osa13, BCC+13, CCHO14, Hod17, HO19b, JJ19],
- Ptolemy spaces [FLS07, FS11b, FS11a],
- Median graphs, median spaces and coarse median spaces [Ver93, Ger97, Ger98, Che00, BC08, CDH10, Bow16, Bow13],
In this paper we introduce and study the notion of Morse quasiflats\footnote{Our notion of Morse quasiflat should not be confused with a quasiflat which is a \textquotedblleft Morse subset\textquotedblright{} in the sense of \cite{Gen17}. We would be inclined to refer to Morse subspaces as \textquotedblleft Morse quasiconvex subsets\textquotedblright{}.}, which are a higher dimensional generalization of Morse quasi-geodesics \cite{Bal82, KKL98, DMS10}. Like Morse quasigeodesics, the definition of Morse quasiflats axiomatizes the stability properties satisfied by quasi-geodesics in Gromov hyperbolic spaces. A brief overview of the paper is as follows; the remainder of the introduction covers this in more detail.

- We introduce several possible definitions of Morse quasiflats. We find conditions which guarantee that the definitions are equivalent and quasi-isometry invariant.
- We establish stability properties for higher dimensional Morse quasiflats, including a generalization of the Morse lemma.
- In spaces with a convex geodesic bicombing, we prove that Morse quasiflats are asymptotically conical, or, to put it in more analytical terms, they have unique tangent cones at infinity. Consequently they have a well-defined Tits boundary.
- We provide criteria for quasiflats to be Morse, and give examples. In CAT(0) spaces, we give an easy-to-verify half flat space criterion which generalizes \cite{Bal82}.
- We mention a few applications to quasi-isometric rigidity.

We emphasize that although a few applications are discussed, our main objective here is to develop the basic theory of Morse quasiflats. There are clearly many natural examples, and we expect the techniques here to be useful in proving further QI rigidity results.

This paper has some overlap with a recent paper on quasiminimizers by Urs Lang and the second author \cite{KL18}. However, the scope of that paper is rather different, since it aims to exhibit hyperbolic properties of maximal rank quasiminimizers; on the one hand it is more general than the setup here because it considers quasiminimizers instead of quasiflats, while on the other the results do not apply to quasiflats of lower rank. The proofs are quite different; see Section \ref{sec:applications} for more on this.

We mention that geometric measure theory plays an important role in this paper, both as the language used state results, and as a technical tool in the proofs. In the process of developing our ideas, we were led to several questions about geometric measure theory in metric spaces which are not addressed by the existing literature. We anticipate that further work in this topic will help to stimulate a fruitful and deeper
interaction between geometric group theory and geometric measure theory. See Subsection 1.8 for more discussion.

1.2. Definitions of Morse quasiflats. We start by proposing several possible definitions of Morse quasiflats. In Subsection 1.6 we will describe a number of examples.

For the remainder of the introduction $X$ will denote a metric space, and $Q$ an $n$-quasiflat in $X$ – the image of an $(L,A)$-quasi-isometric embedding $\Phi : \mathbb{R}^n \to X$; for simplicity we will assume here that $\Phi$ is $L$-Lipschitz. More general cases where $Q$ may not be represented by a Lipschitz map will be discussed in the main body of the paper. We will denote by $F$ a bilipschitz $n$-flat, i.e. the image of a bilipschitz embedding $\mathbb{R}^n \to Z$, where $Z$ is a metric space.

As a motivation for our first definition, recall that Morse quasi-geodesics may be characterized by the property of superlinear divergence, which (roughly speaking) says that the cost of joining two points at distance $2r$ on a quasi-geodesic, while avoiding the $r$-ball centered at their (intrinsic) midpoint, is superlinear in $r$ [BD+14, DMS10, CS14, ACGH17]. For simplicity, here we state the corresponding higher dimensional version for Lipschitz chains. However, more flexible notions are discussed in the text, cf. Definition 6.19, Definition 6.21.

**Definition 1.1.** The $n$-quasiflat $Q = \Phi(\mathbb{R}^n)$ has $(\delta)$-super-Euclidean divergence, if for any $D > 1$, there exists a function $\delta = \delta_D : [0, \infty) \to [0, \infty)$ with $\lim_{r \to \infty} \delta(r) = +\infty$ such that the following property holds.

Pick $r > 0$ and $x \in \mathbb{R}^n$. Suppose that $\hat{\sigma}$ is an $(n-1)$-dimensional Lipschitz cycle carried by $\mathbb{R}^n \setminus B_x(r)$ which represents a nontrivial class in the reduced homology group $\tilde{H}_{n-1}(\mathbb{R}^n \setminus B_x(r))$, and assume that the mass $M(\hat{\sigma})$ satisfies $M(\hat{\sigma}) \leq D \cdot r^{n-1}$. Then for any Lipschitz chain $\tau$ carried by $X \setminus B_{\Phi(x)}(\frac{r}{D})$ such that $\partial \tau = \Phi_*(\hat{\sigma})$, we have $M(\tau) \geq \delta(r) \cdot r^n$.

The mass $M(\sigma)$ of a Lipschitz $k$-chain $\sigma$ takes into account cancellation; in our present setting, the reader may think of the mass as the $k$-dimensional volume of the image, counted with multiplicity (see Section 3).

Next we will introduce some properties of quasiflats which are expressed in terms of asymptotic cones.

Recall that if $Q \subset X$ is a quasigeodesic then $Q$ is Morse if and only if for every asymptotic cone $F := Q_\omega \subset X_\omega =: Z$ the following holds:
(a) (DMS10) Every point $p \in F$ is a cut point of $Z$ which disconnects $F$ in $Z$, i.e. the induced map in reduced homology

$$\tilde{H}_0(F \setminus \{p\}) \to \tilde{H}_0(Z \setminus \{p\})$$

is injective. Equivalently, if $\alpha : I \to F$ is a topological embedding and $\gamma : I \to Z$ is a path with the same endpoints, then $\gamma(I)$ contains $\alpha(I)$.

This condition generalizes in a straightforward way to higher dimensions:

**Definition 1.2.** (Compare [KKL98, Definition 3.8]) A bilipschitz $n$-flat $F \subset Z$ is rigid, if for every topological embedding $\alpha : D^n \to Z$ of the $n$-disk and every singular $n$-chain $\tau$ with $\partial \tau = k \cdot \alpha_*(\partial [D^n])$ for some $k \neq 0$, we have $\text{Im} \, \alpha \subset \text{Im} \, \tau$. Here $\text{Im} \, \tau$ refers to the union of images of corresponding singular simplices. Equivalently, the inclusion map induces an injective mapping

$$(1.3) \quad \tilde{H}_{n-1}(F \setminus \{p\}) \to \tilde{H}_{n-1}(Z \setminus \{p\})$$

in reduced homology for any $p \in F$.

**Remark 1.4.** Note that when $H_n(Z) = \{0\}$ then injectivity of (1.3) is equivalent to the injectivity of the induced homomorphism in relative homology

$$\tilde{H}_n(F,F \setminus \{p\},Z) \to \tilde{H}_n(Z,Z \setminus \{p\},Z).$$

(Compare [KKL98, Definition 3.8]).

To motivate the next condition, we return to the 1-dimensional case.

Suppose (a) holds, and consider a Lipschitz embedding $\alpha : I \to F$ and a Lipschitz path $\gamma : I \to Z$ with the same endpoints. Then the inverse image $\gamma^{-1}(Z \setminus F)$ will be a disjoint union of a collection $\{I_j\}$ of at most countably many open intervals. Restricting $\gamma$ to the closure of $I_j$ we obtain a path $\gamma_j : \bar{I}_j \to Z$ which must actually be a loop in view of condition (a). We may form new path $\hat{\gamma} : \hat{I} \to Z$ by collapsing each of the closed intervals $\{\bar{I}_j\}$ to a point; note that $\hat{\gamma}(\hat{I}) \subset F$. Heuristically, we would like to assert that we have a decomposition

$$\gamma = \hat{\gamma} + \sum_j \gamma_j$$

where $\sum_j \gamma_j$ is an infinite sum of cycles, and the equality holds up to (possibly infinite) decomposition and reparametrization. Also, since $\hat{\gamma}$ and $\alpha$ are Lipschitz paths in $F \simeq \mathbb{R}$ with the same endpoints, then we should have (heuristically speaking) that $\hat{\gamma} = \alpha$ modulo cancellation;
here the reader may think of this as being analogous to the cancellation which occurs when adding simplicial chains. Summing up, we arrive at a second condition which one may impose on asymptotic cones $F \subset Z$:

(b) (Informal) For every Lipschitz embedding $\alpha : I \to F$ and Lipschitz path $\gamma : I \to Z$ with the same endpoints, we have

$$\text{length}(\gamma) = \text{length}(\alpha) + \text{length}(\gamma - \alpha),$$

where length is measured after cancellation.

To make Condition (b) rigorous and generalize it to higher dimensions we use integral currents and mass rather than Lipschitz chains and length, so that infinite sums become valid and cancellation is taken into account. For the purpose of this introduction, the space of integral $n$-currents $\mathcal{I}_n(X)$ can be thought of as the closure of Lipschitz $n$-chains with respect to an appropriate norm, which is derived from mass. See Subsection 3.3 for precise definitions and basic properties. For now it is only important that integral currents form a chain complex whose homology is isomorphic to singular homology, at least for reasonable spaces, and that a chain $\tau \in \mathcal{I}_n(X)$ comes with a natural associated locally finite mass measure $\| \tau \|$. Our second condition to impose on asymptotic cones is as follows:

**Definition 1.5** (Piece property). An bilipschitz $n$-flat $F$ has the piece property if for any integral $n$-current $\nu$ supported in $F$, every filling $\tau$ of $\partial \nu$ contains $\nu$ as a piece, i.e. the decomposition $\tau = \nu + (\tau - \nu)$ is additive with respect to mass:

$$M(\tau) = M(\nu) + M(\tau - \nu);$$

furthermore, the mass measure $\| \tau - \nu \|$ is concentrated on $Z \setminus F$.

Here a filling of an integral cycle $\alpha$ is an integral current $\beta$ with $\partial \beta = \alpha$.

Returning again to the 1-dimensional case, suppose $F \subset Z$ is a bilipschitz 1-flat satisfying Condition (a). Let $\gamma : I \to Z$ be a Lipschitz path in the complement of $F$ whose endpoints lie in the $\rho$-neighborhood $N_\rho(F)$. Then we may concatenate $\gamma$ with two paths $\alpha_{\pm}$ of length $< \rho$ to obtain a new path $\gamma'$ meeting $F$ only at its endpoints. Condition (a) implies that the endpoints of $\gamma'$ are the same, and hence the concatenation of $\alpha_{\pm}$ gives a path of length $< 2\rho$ joining the endpoints of $\gamma$. The higher dimensional generalization of this is as follows:

**Definition 1.6.** A bilipschitz $n$-flat $F$ in a metric space $Z$ has the neck property, if there exists a constant $C > 0$ such that the following holds
for all $\rho > 0$. If $\sigma$ is an integral $(n-1)$-cycle supported in the $\rho$-tubular neighborhood $N_\rho(F)$ of $F$ and $\sigma$ has a filling supported in $Z \setminus F$, then it has a filling $\tau$ with

$$M(\tau) \leq C \cdot \rho \cdot M(\sigma).$$

In addition to rigidity, the piece property, and the neck property, there other conditions – the full support property and the weak neck property – which we will define in the body of the paper.

As mentioned above, each of the above definitions for bilipschitz flats gives rise to a notion for quasiflats: a quasiflat $Q$ in a metric space $X$ is asymptotically rigid/has the asymptotic piece property/has the asymptotic neck property if the corresponding property holds in every asymptotic cone $Q_\omega \subset X_\omega$.

One may produce still more conditions by converting Definitions 1.2, 1.5, 1.6 to quantitive conditions in the space $X$, without reference to asymptotic cones. Instead of stating each counterpart of the notions above, we refer to the precise definitions in the later chapters: Definition 6.13 (coarse rigidity), Definition 10.4 (coarse neck property) and Definition 7.7 (coarse piece property).

1.3. Relations between different definitions of Morse quasiflats. We now show that the definitions given above become equivalent under appropriate assumptions on the ambient space $X$. We refer the reader to Section 3 for definitions.

In the study of Morse quasi-geodesics, it is often assumed that the underlying metric space is geodesic or quasi-geodesic. When working with $n$-dimensional quasiflats, the analogous assumption is that the metric space satisfies coning inequalities for integral currents up to dimension $(n-1)$; in fact we will sometimes need stronger assumptions.

Definition 1.7. A complete metric space $X$ satisfies coning inequalities up to dimension $n$ (or satisfies condition (CI$_n$)), if there exists a constant $c_0 > 0$, such that for all $0 \leq k \leq n$ and every cycle $S \in I_k(X)$ with bounded support there exists a filling $T \in I_{k+1}(X)$ with

$$M(T) \leq c_0 \cdot \text{diam}(\text{spt } S) \cdot M(S).$$

If in addition, there exists a constant $c_1 > 0$ such that $T$ can always be chosen to fulfill

$$\text{diam}(\text{spt } T) \leq c_1 \cdot \text{diam}(\text{spt } S),$$

then $X$ is said to satisfy strong coning inequalities up to dimension $n$, abbreviated condition (SCI$_n$).
Theorem 1.8 (Theorem 10.5). Let $Q \subset X$ be an $(L,A)$-quasiflat in a proper metric space $X$. Consider the following conditions:

1. $Q$ is $(\mu, b)$-rigid (cf. Definition 6.13).
2. $Q$ has super-Euclidean divergence (cf. Definition 6.21).
3. $Q$ has coarse neck property (cf. Definition 7.7).
4. $Q$ has coarse piece property (cf. Definition 10.4).

Then the following hold:

(a) (1) and (2) are equivalent if $X$ satisfies \( (\text{CI}_n - 1) \);
(b) (1), (2) and (3) are equivalent if $X$ satisfies \( (\text{SCI}_n - 1) \);
(c) (1), (2), (3) and (4) are equivalent if $X$ satisfies \( (\text{SCI}_n) \).

Moreover, all of the above equivalences give uniform control on the parameters (e.g. if $X$ satisfies \( (\text{SCI}_n - 1) \) and $Q$ has super-Euclidean divergence, then $Q$ has the coarse neck property (CNP) with parameters of CNP depending only on parameters of super-Euclidean divergence and $X$).

Theorem 1.9 (Theorem 10.10). Suppose $X$ is a proper metric space satisfying \( (\text{SCI}_n) \). Let $Q \subset X$ be an $n$-dimensional quasiflat. Suppose in addition that any asymptotic cone of $X$ has a Lipschitz bicombing (see Section prelims). Then the following conditions are equivalent and each of them is equivalent to the conditions in Theorem 1.8.

1. $Q \subset X$ has the asymptotic piece property (Definition 6.1).
2. $Q \subset X$ has the asymptotic neck property (Definition 6.4).
3. $Q \subset X$ has the asymptotic weak neck property (Definition 6.5).
4. $Q \subset X$ has the asymptotic full support property (Definition 6.8) with respect to the reduced singular homology.
5. $Q \subset X$ has the asymptotic full support property with respect to the reduced homology induced by Ambrosio-Kirchheim currents.

All asymptotic cones here are taken with base points inside $Q$.

Some of the equivalences are proved under weaker assumptions.

Despite the various equivalence characterizations the above theorems, we formally define a quasiflat in a metric space to be Morse if it has super-Euclidean divergence in the sense of Definition 6.21.

An immediate consequence of Theorem 1.9 is:

Corollary 1.10. A quasi-isometry $X \rightarrow X'$ maps Morse $n$-quasiflats to Morse $n$-quasiflats if $X$ and $X'$ satisfy \( (\text{SCI}_n) \) and their asymptotic cones have Lipschitz bicombings.
See Proposition 6.27 for quasi-isometry invariance of Morse quasiflats under weaker assumptions.

Theorem 1.8 and Theorem 1.9 apply to all combable groups $\text{ECH}^+92$, including all automatic groups. More precisely, for a combable group $G$ and any integer $n > 0$, we can always find a complex $X$ where $G$ acts properly and cocompactly such that $X$ satisfies (SCI$_n$), moreover, the quasi-geodesic bicombing in $G$ passes to a Lipschitz bicombing on the asymptotic cone.

Now we give more criteria for Morse quasiflats in CAT(0) spaces.

**Proposition 1.11 (Proposition 12.2).** Let $Q$ be an $n$-quasiflat in a CAT(0) space $X$. Let $\mathbb{F}$ be a non-trivial abelian group. Consider the following conditions.

1. There exists an asymptotic cone $X_\omega$ of $X$ and $p_\omega \in Q_\omega$ such that the map $\hat{H}_n(Q_\omega, Q_\omega \setminus \{p_\omega\}, \mathbb{F}) \rightarrow \hat{H}_n(X_\omega, X_\omega \setminus \{p_\omega\}, \mathbb{F})$ is not injective.
2. There exists an asymptotic cone $X_\omega$ of $X$ such that the limit $Q_\omega$ of $Q$ is an $n$-flat which bounds a flat half-space in $X_\omega$.
3. There exists an ultralimit $X_\omega = \lim_\omega (X, p_i)$ such that the limit $Q_\omega$ of $Q$ is an $n$-flat which bounds a flat half-space in $X_\omega$.

Then (1) and (2) are equivalent. If we assume in addition that $Q$ is a flat, then all three conditions are equivalent.

### 1.4. Stability of Morse quasiflats and quasidisks.

In the following we use $d_H$ to denote Hausdorff distance.

**Proposition 1.12 (Morse lemma for Morse disks).** Suppose $X$ is a complete metric space satisfying condition (CI$_n$). Given $(\mu, b)$ as in Definition 6.13 and positive constants $L, A, A', n$, there exists $C$ depending only on $\mu, b, L, A, A', n$ and $X$ such that the following holds.

Let $D$ and $D'$ be two $n$-dimensional $(L, A)$-quasi-disks in $X$ such that $d_H(\partial D, \partial D') < A'$ and $D$ is $(\mu, b)$-rigid. Then $d_H(D, D') < C$.

In [KL18, Section 5], stronger versions of Morse lemmas for top rank quasi-minimizers are proved.

**Proposition 1.13.** Let $X$ be a complete metric space satisfying condition (CI$_n$) and let $Q, Q' \subset X$ be $(L, A)$-quasiflats with $\dim Q = n$. 
Suppose that $Q$ is $(\mu,b)$-rigid. Then there exist $A'$ and $\epsilon$ depending only on $X, L, A, n, b$ and $\mu$ such that either $d_H(Q, Q') \leq A'$, or

\begin{equation}
\limsup_{r \to \infty} \frac{d_H(B_p(r) \cap Q, Q')}{r} \geq \epsilon
\end{equation}

for some (hence any) $p \in Q$.

1.5. Visibility. In a metric space with bicombing structure, it is a fundamental question to investigate how the Morse quasiflats interact with the bicombing lines. We use the term visibility for results in this direction, which goes back to [EO73].

In Gromov hyperbolic spaces, it is known that quasi-geodesics are at bounded Hausdorff distance from geodesics. Similar results hold for Morse quasi-geodesics [CST14, Cor17] in an appropriate context. For a top rank quasiflat $Q$ in a space with convex geodesic bicombing, it is proved in [KL18] that $Q$ is sub-linearly close to a geodesic cone and $Q$ has a well-defined Tits boundary at infinite. Now we generalize both results to the context of Morse quasiflats.

For simplicity we will state the results for CAT(0) spaces. However, the results below are actually proved in the more general setting of spaces with a convex geodesic bicombing (see Definition 3.1), which includes Busemann convex spaces and injective metric spaces.

For a base point $p$ in a CAT(0) space $X$ and a subset $A \subset X$, we denote the closure of the union of all geodesics from $p$ to a point in $A$ by $\overline{C}_p(A)$. Let $\partial_T X$ be Tits boundary of $X$.

**Definition 1.15.** Let $\sigma$ be the given convex geodesic bicombing on $X$. Let $Q \subset X$ be a quasiflat. We consider the collection of all $\sigma$-geodesics rays in $\overline{C}_p(Q)$ emanating from $p$, and define the *Tits boundary* of the quasiflat $Q$, denoted $\partial_T Q$, to be the subset of $\partial_T X$ determined by these rays. Note that $\partial_T Q$ does not depend on the choice of $p$.

**Theorem 1.16 (Theorem 11.15).** Let $Q$ be a Morse $(L,A)$-quasiflat in a proper CAT(0) space $X$. Let $p$ be a base point in $X$. Then there exists a function $\delta : [0, \infty) \to [0, \infty)$ depending only on $Q, d(p,Q)$ and $X$ such that $\lim_{r \to \infty} \frac{\delta(r)}{r} = 0$ and the following holds whenever $R \geq r$: $d_H(B_p(R) \cap Q, B_p(R) \cap \overline{C}_p(Q)) \leq \delta(r)R$.

**Theorem 1.17.** Let $Q$ be a Morse $(L,A)$-quasiflat in a proper CAT(0) space $X$. Let $p$ be a base point in $X$. Then
(1) there exists a function \( \delta : [0, \infty) \to [0, \infty) \) depending only on 
\( Q, d(p,Q) \) and \( X \) such that \( \lim_{r \to \infty} \frac{\delta(r)}{r} = 0 \) and \( d_H(B_p(R) \cap Q, B_p(R) \cap C_p(\partial_T Q)) \leq \delta(r)R \) whenever \( R \geq r \); 
(2) \( \partial_T Q \) can also characterized as the subset of \( \partial_T X \) represented by 
geodesic rays \( \ell : [0, \infty) \to X \) which travel sublinearly close to 
\( Q \), i.e. \( \lim_{t \to \infty} \frac{d(\ell(t), Q)}{t} = 0 \); 
(3) The Euclidean cone over \( \partial_T Q \) is bilipschitz homeomorphic to 
\( \mathbb{R}^n \) (here \( \partial_T Q \) is given the induced metric from \( \partial_T X \)). Moreover, \( \partial_T Q \) has full support in \( \partial_T X \) with respect to the 
\((n-1)\)-dimensional reduced local homology with \( \mathbb{Z} \)-coefficients.

Theorem 1.17 is Proposition 11.18 and Proposition 11.19. Theorem 1.16 and Theorem 1.17 are actually proved in the weaker setting of pointed Morse quasiflats (Definition 11.16).

We also have the following uniqueness result.

**Theorem 1.18** (Theorem 11.20). Suppose \( X \) is a proper CAT(0) space, 
and \( Q_1, Q_2 \subset X \) are Morse quasiflats. If \( \partial_T Q_1 = \partial_T Q_2 \) then \( d_H(Q_1, Q_2) < C \) where \( C < \infty \) depends only on \( \text{dim} Q_1 \) and the Morse parameters of 
\( Q_1 \) and \( Q_2 \).

1.6. **Examples of Morse quasiflats.** Recall that we have defined a quasiflat in a metric space \( X \) to be *Morse* if it has super-Euclidean divergence in the sense of Definition 6.19. There are many examples of Morse quasiflats which are neither 1-dimensional nor of top rank. In the following list we assume \( X, X_1 \) and \( X_2 \) satisfy the assumptions of 
Theorem 1.9 (this applies to combable groups, spaces with Lipschitz bicombing etc.), where being Morse can be characterized by other conditions in Theorem 1.9.

- Suppose \( X \) has asymptotic rank \( \leq n \) ([KL18]). Then every 
  \( n \)-dimensional quasiflat in \( X \) is Morse.
- Suppose \( Q_i \subset X_i \) is a Morse quasiflat for \( i = 1, 2 \). Then \( Q_1 \times Q_2 \) is a Morse quasiflat in \( X_1 \times X_2 \) (see Corollary 7.5).
- One can obtain Morse quasiflats from more interesting operations, like amalgamation or taking branched covers under certain conditions. We refer to Section 12.3 for an example using branched covers.

Morse quasiflats also arise naturally in some group theoretic contexts. The following is a special case of Proposition 12.2.

**Corollary 1.19.** Suppose \( X \) is a proper CAT(0) space and \( F \subset X \) is a flat such that the stabilizer of \( F \) in \( \text{Isom}(X) \) acts cocompactly on
Then \( F \) is Morse if and only if \( F \) does not bound an isometrically embedded half-flat.

Let \( G \) be a group. Following [WW17] we say that a finitely generated free abelian subgroup \( H \) of \( G \) is highest if its commensurability class is maximal, i.e. \( H \) does not have a finite index subgroup that is contained in a free abelian subgroup of higher rank.

**Corollary 1.20.** Let \( G \) be a group such that \( G \) has a finite index subgroup which is the fundamental group of a compact special cube complex. Then any highest free abelian subgroup of \( G \) is a Morse quasiflat.

One cannot drop the word “special” from the above corollary because of certain irreducible lattices acting on products of trees [RR05]. For groups acting on higher Euclidean buildings or symmetric spaces, it might happen quite often that highest abelian subgroups are not Morse quasiflats, and it would be interesting to identify which number-theoretical invariants lead to this as in [RR05]. Nevertheless, we conjecture the phenomenon in Corollary 1.20 holds for more examples.

**Conjecture 1.21.** Suppose \( G \) is either a Coxeter group, or an Artin group, or a mapping class group of a surface. Then any highest abelian subgroup in \( G \) is a Morse quasiflat.

### Applications

Theorem 1.17 and Theorem 1.18 reduces the study of Morse quasiflats to the study to certain embedded spheres in the Tits boundary. Once the structure of these spheres are understood, this immediately implies structural results of Morse quasiflats in the space. For example, results on quasiflats in symmetric spaces and Euclidean buildings [KL97, EF97] can be recovered in this way (see [KL18]). The same scheme applies to CAT(0) cube complexes, where there are typically plenty of Morse quasiflats which are neither 1-dimensional nor of top rank. A structural result on certain type of cycles in the Tits boundary was proved in [Hua, Theorem 1.4]. This together with Theorem 1.17 and Theorem 1.18 implies the following.

**Theorem 1.22.** Suppose \( X \) is a finite dimensional proper CAT(0) cube complex. If \( Q \subset X \) is a \( k \)-dimensional Morse quasiflat, then there exists a collection of pairwise disjoint \( k \)-dimensional CAT(0) orthants \( \{O_i\}_{i=1}^{k} \) such that \( d_H(Q, \sqcup_{i=1}^{k} O_i) < \infty \).
If $Q$ is pointed Morse, then there exists a collection of pairwise disjoint $k$-dimensional CAT(0) orthants $\{O_i\}_{i=1}^k$ such that $Q$ and $\sqcup_{i=1}^k O_i$ are sub-linearly close in the sense of Theorem 1.17 (1).

Moreover, in each of the above cases, the CAT(0) orthants are at finite Hausdorff distance from some $\ell^1$-orthants.

We recall that an $\ell^1$-orthant in a CAT(0)-cube complex is a subcomplex isometric by a cubical isomorphism to a standard Euclidean orthant equipped the $\ell^1$-distance.

Remark 1.23. In the both cases of Theorem 1.22, each $O_i$ is actually contained in a convex subcomplex $O'_i$ of $X$ such that $O'_i$ splits as a product of $k$ cubical factors ($k = \dim Q$). In the first case of Theorem 1.22 $O'_i$ and $O_i$ has finite Hausdorff distance, in the second case $O'_i$ and $O_i$ are sublinearly close.

The top rank case of Theorem 1.22 follows from results on top rank quasiflats in [Bow19]. We expect that the work in [Bow19, BHS17b] can be adapted to handle Morse quasiflats in coarse median spaces, and hierarchically hyperbolic spaces.

Combining Theorem 1.22 with the argument in [Hua17, Section 5], we obtain the following Morse lemma for Morse flats.

**Theorem 1.24.** Let $X_1$ and $X_2$ be universal covers of compact special cube complexes. If $q : X_1 \to X_2$ is an $(L,A)$-quasi-isometry, then there is a constant $C$ depending only on $X_1, X_2, L$ and $A$ such that for any Morse flat $F_1 \subset X_1$, there exists a Morse flat $F_2 \subset X_2$ such that $d_H(q(F_1), F_2) < C$.

We conjecture that the above discussion holds true in the context of Davis complexes of Coxeter groups, which is another instance where there are plenty of interesting Morse quasiflats.

**Conjecture 1.25.** Morse quasiflats in the Davis complexes of Coxeter groups are at finite Hausdorff distance from a union of CAT(0) orthants in the sense of Theorem 1.22. Moreover, Theorem 1.24 holds when $X$ and $Y$ are the Davis complexes of some Coxeter groups.

Another potentially interesting case is Artin groups of type FC. They act geometrically on injective metric spaces [HO19a], hence Theorem 1.17 and Theorem 1.18 apply. Moreover, they contain plenty of Morse quasiflats. It is necessary to control the behavior of these quasiflats for understanding their quasi-isometric rigidity, and it is less
likely that one can handle this point by only controlling top rank quasi-flats and using an add-hoc argument to control Morse quasi-flats as in [Hua14].

There are many interesting aspects of Morse quasi-flats yet to be explored. We give an indication of few of these in Section 13.

1.8. The role of geometric measure theory in this paper. We now make a few remarks about analytical aspects of this paper.

In the initial stages of this project, we had considered bypassing integral currents in favor of a technically simpler alternative such as simplicial chains or Lipschitz chains. However, our attempts to implement such alternative approaches failed, either because they could only be used to treat a fraction of our results, or because they would necessitate the replacement of standard technicalities from geometric measure theory by new technical arguments of similar complexity, thereby eliminating the expected benefit of avoiding geometric measure theory. Here are some of the properties that a geometric chain complex should have in order to facilitate the objectives of this paper:

- There should be a notion of “cancellation”, when one adds chains which overlap with opposite orientation.
- Slicing of chains by Lipschitz functions, or at least distance functions, is basic to many comparison arguments.
- Minimizers should be compatible with CAT(0) structure which may not be polyhedral.
- Natural limit spaces (Tits boundaries, asymptotic cones) are not PL in any sense, so one is forced to consider chain complexes in metric spaces.
- Compactness is needed in a number of places.

We were not able to find a viable alternative to integral currents which was any simpler, and still had the above properties.

In addition to providing the basic framework for our results, integral currents helped to simplify our arguments in several places. In particular, the estimates we obtained by solving Plateau problems relative to a Lipschitz submanifold (or quasiflat) were very helpful in Sections 4 and 9 and may be of independent interest. Also, following earlier authors [AK+00, Wen11], we use the Ekeland variational principle in Section 4.
MORSE QUASIFLATS

1.9. Structure of the paper and suggestions for the reader.
In Section 2 we discuss ideas of proofs of some of the main results. Section 3 - Section 5 are preparatory in nature. In Section 6 we discuss some background on metric spaces and metric currents. In Section 4 we discuss a procedure of “regularizing” an integral current for later use. In Section 5 we prove some properties on quasiflats in metric spaces with cone inequalities.

In Section 6 we introduce various definitions of Morse quasiflats. Definitions using asymptotic cones are in Section 6.1 and definitions without asymptotic cones are in Section 6.2. We also prove quasi-isometry invariance of Morse quasiflats in Section 6.3. In Section 7 we show equivalence of one definition from Section 6.1 (full support) to a definition in Section 6.2 (super-Euclidean divergence). In Section 8 we prove some basic properties of Morse quasiflats, including a version of Morse lemma for Morse quasi-disks.

Section 9 and Section 10 are devoted to proving the equivalence of the definitions introduced in Section 6. In particular, we introduce a decomposition lemma in the beginning of Section 10 for later use.

In Section 11 we prove various visibility results claimed in Section 1.5. In Section 12 we prove the half flat criterion (Proposition 1.11) for Morse quasiflats in \( \text{CAT}(0) \) spaces, and discuss several examples/non-examples of Morse quasiflats.

For readers who are more interested in the visibility results in the later part of the paper, we suggest the following route. For these readers, a quasiflat \( Q \) in a metric space \( X \) is Morse if \( Q_\omega \) has full support (cf. Section 6.1) in any asymptotic cone \( X_\omega \) of \( X \). One starts with Section 6.1 goes through Lemma 7.6, Proposition 8.6 and Lemma 10.1 and arrives at Section 11.

2. Discussion of proofs

In this section we will give a rough sketch of some of the key arguments in the paper.

Proof of Theorem 1.8. We only discuss \((2) \Rightarrow (3)\), which is the most interesting part. To avoid technicalities, we only discuss the simple case when \( X \) is \( \text{CAT}(0) \) and \( Q \) is an \( n \)-dimensional flat, where the ideas are more transparent.

Take a base point \( p \in Q \). Let \( \sigma \) be an \((n-1)\)-cycle such that \( \sigma = \partial \tau \) with \( M(\tau) \leq CR^n \), \( \text{spt}(\tau) \in B_p(R) \setminus N_{\rho R}(Q) \), \( \text{spt}(\sigma) \subset N_{2\rho R}(Q) \) and
$M(\sigma) \leq CR^{n-1}$ (one should think $R$ is huge, and $\rho$ is tiny). We want to show $\text{Fill}(\sigma) \lesssim \rho RM(\sigma)$. Let $\pi : X \to Q$ be the nearest point projection and let $\sigma' = \pi_\#(\sigma)$. The geodesic homotopy between $\sigma$ and $\sigma'$ gives rise to $h$ such that $\partial h = \sigma - \sigma'$ and $M(h) \lesssim \rho RM(\sigma)$. Suppose $\tau'$ is the canonical filling of $\sigma'$ in $Q$. Since $\partial(h + \tau') = \sigma$, we are done as long as we can show $M(\tau')$ is very small.

Note that $\sigma' = \partial \tau' = \partial(h + \tau)$. The $(\mu, b)$-rigid condition implies that $\text{spt}(\tau') \subset N_{\epsilon R}(\text{spt}(h + \tau))$. We can assume $\epsilon \ll \rho$ by making $R$ large enough. As $\text{spt}(\tau)$ is further away from $Q$, we know $\text{spt}(\tau') \subset N_{\epsilon R}(\text{spt}(h)) \subset N_{(\epsilon + \rho)R}(\text{spt}(\sigma'))$ (we pretend for the moment that the wrong formula $\text{spt}(\sigma') = \pi(\text{spt}(\sigma))$ holds). This implies that $\sigma'$ can be filled in a small neighborhood of its support, which intuitively suggests that $\text{Fill}(\sigma')$ is small. However, we can not argue in this way as there are examples with small filling radius but large filling volume.

The idea is that if $\sigma'$ has some extra geometric control, more precisely controlled $k$-dimensional Minkowski content with $k < n$, then we can deduce small filling volume from small filling radius (see Lemma 9.13 for a precise formulation of what we need).

In order to arrange that $\sigma'$ has controlled Minkowski content, we solve the following free boundary minimizing problem. Consider the collection of all chains with one boundary component being $\sigma$ and another being a cycle in $Q$. Suppose for the moment that we can find a chain $\alpha$ which minimizes the mass in this collection. Clearly $M(\alpha) \lesssim \rho RM(\sigma)$. Suppose $\partial \alpha = \sigma - \sigma'$. A calculation using the minimizing property of $\alpha$ yields the desired Minkowski content control on $\sigma'$ (see Lemma 9.4 and Lemma 9.11), which finishes the proof.

**Proof of Theorem 1.9.** We only discuss a particular step where having full support in the asymptotic cone with respect to compactly supported integral currents implies super-Euclidean divergence (cf. Lemma 7.3). The proof is another instance that solving a free boundary minimizing problem turns out to be helpful.

The usual argument is to suppose super-Euclidean divergence fails. Then one obtains a sequence of integral currents $\sigma_k$, and argues that their rescaled limit in the asymptotic cone contradicts the full support with respect to the reduced homology induced by compact supported integral currents. The Wenger compactness theorem guarantees a limit, however, the limit may not have compact support. One plausible way
out is to play with the “thick-thin decomposition” as in [Wen11]. However, we introduce a different procedure which we believe is cleaner and might be of independent interest.

The idea is to use a free boundary minimizing problem to “regularize” an integral current $\sigma$. More precisely, given a constant $\kappa > 0$, we consider a functional $F : I_0(X) \to \mathbb{R}$ by $F(\tau) = \kappa M(\tau) + M(\sigma - \partial \tau)$. We can think $\tau$ as a “cylinder” with one end of the cylinder being $\sigma$, and another end being $\sigma - \partial \tau$. The functional involves the mass of the cylinder and the mass of the “free end” of the cylinder. One immediately sees that if $\sigma$ has “fingers”, then cutting off the fingers will decrease the functional $F$. We take a minimizer $\sigma'$ of $F$. A simple computation yields that $\sigma'$ has a lower density bound. Moreover, by making $\kappa$ large, we can assume the flat distance between $\sigma'$ and $\sigma$ is small, and $\text{spt}(\sigma')$ is contained in a small neighborhood of $\text{spt}(\sigma)$, see Proposition 4.2. Back to the proof, we can apply this regularization procedure to each $\tau_k$ to obtain a new sequence with uniformly bounded support, moreover, the supports of elements in the new sequence is not too far away from the original elements, which is enough to conclude the proof.

**Proof of Theorem 1.17 (1).** The proof involves a “drop on scale” induction which goes back to [KL18, Proposition 4.5]. However, the argument in [KL18] breaks down as the sub-linear Euclidean isoperimetric inequality no longer holds in our setting. We do not have control on objects being further away from the Morse quasiflat. The new ingredient is a Decomposition Lemma which cuts certain chains into a piece close to the Morse quasiflat and a piece further away such that the cut locus is a small neck (Lemma 10.1).

We represent the quasiflat $Q$ by a local integral current $T$. Take a base point $p \in Q$ and a number $R_0 \gg 0$. We slice $T$ at distance $R_0$ from $p$, and let $T'$ be the cone over such slice based at $p$. For $r < R_0$, let $S_r$ and $S'_r$ be the slice of $T$ and $T'$ at distance $r$ from $p$. Ideally, we want to prove for any small $\epsilon$, $d_H(\text{spt}(S_r), \text{spt}(S'_r)) \leq \epsilon r$ as long as $r$ is larger than a threshold $R$ depending only on $\epsilon$. As $R$ does not depend on $R_0$, we might need to handle the case when $r \ll R_0$.

Instead of obtaining the Hausdorff estimate directly, we first establish a weaker scale invariant estimate $\text{Fill}(S_r - S'_r) \leq \epsilon r^{n-1}$. This estimate is trivial when $r = R_0$. We want to construct inductively a sequence $\{R_i\}$ with $R_{i+1} \in [0.5R_i, 0.6R_i]$ such that the filling volume estimate holds for each $r = R_i$. The first case is that there is an $r \in [0.5R_0, 0.6R_0]$
such that \( \text{Fill}(S_r - S'_r) \leq \epsilon r^{n-1} \), then we set \( R_1 = r \). Now we assume the contrary holds (case 2).

Apply the Decomposition Lemma to \( T' \), which gives a piece decomposition \( T' = U + V \), an \( n \)-chain \( \gamma \) with mass \( < \delta R_0^n \), and a filling \( W \) of \( U + \gamma - T \) with mass \( < \delta R_0^{n+1} \). Note that \( U \) is the piece closer to \( Q \), \( V \) is further away, and \( \gamma \) fills the "neck" \( \partial V \). We assume \( \delta \ll \epsilon \).

Consider the slices \( W_r, U_r, V_r, \gamma_r \) of \( W, U, V \) and \( \gamma \) respectively. As \( \textbf{M}(W) < \delta R_0^{n+1} \), by the coarea inequality, we can find \( r \in [0.5R_0, 0.6R_0] \) such that \( \textbf{M}(W_r) \lesssim \delta r^n \). As \( \text{Fill}(S_r - S'_r) > \epsilon r^n \), we conclude that \( \text{Fill}(V_r - \gamma_r) > \frac{\epsilon}{2} r^n \). As \( \textbf{M}(\gamma) \) is small, \( \textbf{M}(V_r) \) can not be too small, otherwise \( \text{Fill}(V_r - \gamma_r) \) would be small by the isoperimetric inequality. In summary, we find \( r \in [0.5R_0, 0.6R_0] \) such that \( \textbf{M}(U_r) \) is at most a definite multiple \( \kappa \) of \( r^{n-1} \) less than the mass of \( T'_r \) where \( \kappa \) depends on \( \epsilon \), but is independent of \( R_0 \). Define \( R_1 = r \).

Now it looks like our induction scheme fails. However, what we gain in case 2 is a definite amount of discrepancy between \( \textbf{M}(T'_r) \) and \( \textbf{M}(U_r) \) (in a scale invariant way), which implies that if we keep running into the problematic case 2 when passing to a smaller scale, then the "good piece" \( U_r \) of \( T'_r \) will become extinct after a finite number steps independent of \( R_0 \), leading to a contradiction. This suggests that if we weaken the induction assumption appropriately so that it fits with our induction scheme, then a weaker statement (see Proposition 11.2) can be obtained, which is still good enough for the desired distance estimate (see Corollary 11.11).

3. Preliminaries

3.1. Metric notions. Let \( X = (X,d) \) be a metric space. We write

\[
B_p(r) := \{ x \in X : d(p,x) \leq r \}, \quad S_p(r) := \{ x \in X : d(p,x) = r \}
\]

for the closed ball and sphere with radius \( r \geq 0 \) and center \( p \in X \). We write \( A_p(r_1, r_2) := \{ x \in X : r_1 \leq d(p,x) \leq r_2 \} \) for closed annulus with inner radius \( r_1 \) and outer radius \( r_2 \).

A set \( N \subset X \) is called \( \delta \)-separated, for a constant \( \delta \geq 0 \), if \( d(x,y) > \delta \) for every pair of distinct points \( x, y \in N \). For \( E \subset X \), we call a subset \( N \subset E \) a \( \delta \)-net in \( E \) if the family of all balls \( B_x(\delta) \) with \( x \in N \) covers \( E \). Every maximal (with respect to inclusion) \( \delta \)-separated subset of \( E \) is a \( \delta \)-net in \( E \).

A map \( f : X \to Y \) into another metric space \( Y = (Y,d) \) is \( L \)-Lipschitz, for a constant \( L \geq 0 \), if \( d(f(x), f(x')) \leq L d(x, x') \) for all
$x, x' \in X$. The smallest such $L$ is the Lipschitz constant $\text{Lip}(f)$ of $f$. The map $f : X \to Y$ is an $L$-bi-Lipschitz embedding if $L^{-1}d(x, x') \leq d(f(x), f(x')) \leq L d(x, x')$ for all $x, x' \in X$. For an $L$-Lipschitz function $f : E \to \mathbb{R}$ defined on a set $E \subset X$,

$$\bar{f}(x) := \sup\{f(a) - Ld(a, x) : a \in E\} \quad (x \in X)$$

defines an $L$-Lipschitz extension $\bar{f} : X \to \mathbb{R}$ of $f$. Every $L$-Lipschitz map $f : E \to \mathbb{R}^n, E \subset X$, admits a $\sqrt{n}L$-Lipschitz extension $\bar{f} : X \to \mathbb{R}^n$.

A map $f : X \to Y$ between two metric spaces is called an $(L, A)$-quasi-isometric embedding, for constants $L \geq 1$ and $A \geq 0$, if

$$L^{-1}d(x, x') - A \leq d(f(x), f(x')) \leq L d(x, x') + A$$

for all $x, x' \in X$. A quasi-isometry $f : X \to Y$ has the additional property that $Y$ is within finite distance of the image of $f$. An $(L, A)$-quasi-disk $D$ in a metric space $X$ is the image of an $(L, A)$ quasi-isometric embedding $q$ from a closed metric ball $B$ in $\mathbb{R}^n$ to $X$. $B$ is called the domain of $D$. The boundary of $D$, denoted $\partial D$, is defined to be $q(\partial B)$. An $n$-dimensional quasiflat in $X$ is the image of a quasi-isometric embedding of $\mathbb{R}^n$.

A curve $\rho : I \to X$ defined on some interval $I \subset \mathbb{R}$ is a geodesic if there is a constant $s \geq 0$, the speed of $\rho$, such that $d(\rho(t), \rho(t')) = s|t-t'|$ for all $t, t' \in I$. A geodesic defined on $I = \mathbb{R}_+ := [0, \infty)$ is called a ray.

### 3.2. Metric spaces with convex geodesic bicombing.

**Definition 3.1** (convex bicombing). By a convex bicombing $\sigma$ on a metric space $X$ we mean a map $\sigma : X \times X \times [0, 1] \to X$ such that

1. $\sigma_{xy} := \sigma(x, y, \cdot) : [0, 1] \to X$ is a geodesic from $x$ to $y$ for all $x, y \in X$;
2. $t \mapsto d(\sigma_{xy}(t), \sigma_{x'y'}(t))$ is convex on $[0, 1]$ for all $x, y, x', y' \in X$;
3. $\text{Im}(\sigma_{pq}) \subset \text{Im}(\sigma_{xy})$ whenever $x, y \in X$ and $p, q \in \text{Im}(\sigma_{xy})$.

A geodesic $\rho : I \to X$ is then called a $\sigma$-geodesic if $\text{Im}(\sigma_{xy}) \subset \text{Im}(\rho)$ whenever $x, y \in \text{Im}(\rho)$. A convex bicombing $\sigma$ on $X$ is equivariant if $\gamma \circ \sigma_{xy} = \sigma_{\gamma(x)\gamma(y)}$ for every isometry $\gamma$ of $X$ and for all $x, y \in X$.

Note that in (3), we do not specify the order of $p$ and $q$ with respect to the parameter of $\sigma_{xy}$, in particular $\sigma_{yx}(t) = \sigma_{xy}(1-t)$. In the terminology of [DL15b], $\sigma$ is a reversible and consistent convex geodesic bicombing on $X$. In Section 10.1 of [Kle99], metric spaces with such a structure $\sigma$ are called often convex. This class of spaces includes
all CAT(0) spaces, Busemann spaces, geodesic injective metric spaces, as well as (linearly) convex subsets of normed spaces; at the same time, it is closed under various limit and product constructions such as ultralimits, (complete) Gromov–Hausdorff limits, and $l_p$ products for $p \in [1, \infty]$.

Definition 3.2. A metric space $X$ has an $L$-Lipschitz bicombing if for each pair of points $x, y \in X$, there is an $L$-bilipschitz path $\sigma_{xy} := [0, a_{xy}] \to X$ from $x$ to $y$ such that

$$d(\sigma_{xy}(t), \sigma_{x'y'}(t)) \leq L \cdot \max\{d(x, x'), d(y, y')\}$$

for any $x, y, x', y' \in X$ and any $t \geq 0$. Here $\sigma_{xy}$ is assumed to be extended to domain $\mathbb{R}$ by constant map.

Having $L$-Lipschitz bicombing is closed under taking ultralimits. If $X$ is a metric space with a bounded $(L, C)$–quasi-geodesic combing in the sense of [AB95, Section 1], then any asymptotic cone of $X$ has an $L$-Lipschitz bicombing. Thus the asymptotic cone of any combable group, or semi-hyperbolic group, has an $L$-Lipschitz bicombing.

Let $X$ be a complete metric space with a convex bicombing $\sigma$. The boundary at infinity of $(X, \sigma)$ is defined in the usual way, as for CAT(0) spaces, except that only $\sigma$-rays are taken into account. Specifically, we let $R^\sigma X$ and $R^\sigma_1 X$ denote the sets of all $\sigma$-rays and $\sigma$-rays of speed one, respectively, in $X$. For every pair of rays $\rho, \rho' \in R^\sigma X$, the function $t \mapsto d(\rho(t), \rho'(t))$ is convex, and $\rho$ and $\rho'$ are called asymptotic if this function is bounded. This defines an equivalence relation $\sim$ on $R^\sigma X$ as well as on $R^\sigma_1 X$. The boundary at infinity or visual boundary of $(X, \sigma)$ is the set

$$\partial_\infty X := R^\sigma_1 X/\sim$$

Given $\rho \in R^\sigma_1 X$ and $p \in X$, there is a unique ray $\rho_p \in R^\sigma_1 X$ asymptotic to $\rho$ with $\rho_p(0) = p$. The set

$$\bar{X} := X \cup \partial_\infty X$$

carries a natural metrizable topology, analogous to the cone topology for CAT(0) spaces. With this topology, $\bar{X}$ is a compact absolute retract, and $\partial_\infty X$ is a $Z$-set in $\bar{X}$. See Section 5 in [DL15b] for details.

For a subset $A \subset X$, the limit set $\partial_\infty(A)$ is defined as the set of all points in $\partial_\infty X$ that belong to the closure of $A$ in $\bar{X}$. For a point $p \in X$ we define the geodesic homotopy

$$h_p: [0, 1] \times X \to X$$
by $h_p(\lambda, x) := \sigma_{px}(\lambda)$.

Note that the map $h_p, \lambda : X \to X$ is $\lambda$-Lipschitz. For a set $A \subset X$,

$$C_p(A) := h_p([0, 1] \times A)$$

denotes the geodesic cone from $p \in X$ over $A$, and $\overline{C}_p(A)$ denotes its closure in $X$. Similarly, if $\Lambda \subset \partial_\infty X$, then $C_p(\Lambda) \subset X$ denotes the union of the traces of the rays emanating from $p$ and representing points of $\Lambda$.

The *Tits cone* of $(X, \sigma)$ is defined as the set

$$\mathcal{C}_T X := R^\sigma X/\sim,$$

equipped with the metric given by

$$d_T([\rho], [\rho']) := \lim_{t \to \infty} \frac{1}{t} d(\rho(t), \rho'(t)).$$

Note that $t \mapsto d(\rho(t), \rho'(t))$ is convex, thus $t \mapsto d(\rho(t), \rho'(t))/t$ is non-decreasing if $\rho, \rho'$ are chosen such that $\rho(0) = \rho'(0)$. From this it is easily seen that $\mathcal{C}_T X$ is complete.

The *Tits boundary* of $(X, \sigma)$ is the unit sphere

$$\partial_T X := S_\rho(1) = R^*_1 X/\sim$$

in $\mathcal{C}_T X$, endowed with the topology induced by $d_T$. This topology is finer than the cone topology on the visual boundary $\partial_\infty X$, which agrees with $\partial_T X$ as a set.

### 3.3. Local currents in proper metric spaces.

Reader not familiar with the theory of current may think currents as Lipschitz chains on first reading (see Remark 3.5). However, many useful features of currents, like slicing and various compactness results, is less clear for Lipschitz chains.

Currents of finite mass in complete metric spaces were introduced by Ambrosio and Kirchheim in [AK+00]. There is a localized variant of this theory for locally compact metric spaces, as described in [Lan11]. Both versions of currents are needed in this paper. Here we only provide some background on the theory from [Lan11], and leave it to the reader to identify the corresponding statements in [AK+00] (such comparison is already carried in detail in [Lan11]). To avoid certain technicalities, we will assume the underlying metric space $X$ is proper throughout Section 3.3 hence complete and separable.

For every integer $n \geq 0$, let $\mathcal{D}^n(X)$ denote the set of all $(n+1)$-tuples $(\pi_0, \ldots, \pi_n)$ of real valued functions on $X$ such that $\pi_0$ is Lipschitz with compact support spt($\pi_0$) and $\pi_1, \ldots, \pi_n$ are locally Lipschitz.
case that \( X = \mathbb{R}^N \) and the entries of \((\pi_0, \ldots, \pi_n)\) are smooth, this tuple should be thought of as representing the compactly supported differential \( n \)-form \( \pi_0 \, d\pi_1 \wedge \ldots \wedge d\pi_n \). An \( n \)-dimensional current \( S \) in \( X \) is a function \( S : \mathcal{D}^n(X) \to \mathbb{R} \) satisfying the following three conditions:

1. \( S \) is \((n + 1)\)-linear;
2. \( S(\pi_{0,k}, \ldots, \pi_{n,k}) \to S(\pi_0, \ldots, \pi_n) \) whenever \( \pi_{i,k} \to \pi_i \) pointwise on \( X \) with \( \sup_k \text{Lip}(\pi_{i,k}|_K) < \infty \) for every compact set \( K \subset X \) \((i = 0, \ldots, n)\) and with \( \bigcup_k \text{spt}(\pi_{0,k}) \subset K \) for some such set;
3. \( S(\pi_0, \ldots, \pi_n) = 0 \) whenever one of the functions \( \pi_1, \ldots, \pi_n \) is constant on a neighborhood of \( \text{spt}(\pi_0) \).

We write \( \mathcal{D}_n(X) \) for the vector space of all \( n \)-dimensional currents in \( X \). The defining conditions already imply that every \( S \in \mathcal{D}_n(X) \) is alternating in the last \( n \) arguments and satisfies a product derivation rule in each of these. The definition is further motivated by the fact that every function \( w \in L^1_{\text{loc}}(\mathbb{R}^n) \) induces a current \([w] \in \mathcal{D}_n(\mathbb{R}^n)\) defined by

\[
[w](\pi_0, \ldots, \pi_n) := \int w\pi_0 \det[D_j \pi_i]_{i,j=1}^n \, dx
\]

for all \((\pi_0, \ldots, \pi_n) \in \mathcal{D}^n(\mathbb{R}^n)\), where the partial derivatives \( D_j \pi_i \) exist almost everywhere according to Rademacher’s theorem. Note that this just corresponds to the integration of the differential form \( \pi_0 \, d\pi_1 \wedge \ldots \wedge d\pi_n \) over \( \mathbb{R}^n \), weighted by \( w \). For the characteristic function \( \chi_W \) of a Borel set \( W \subset \mathbb{R}^n \), we put \([W] := [\chi_W]\). (See Section 2 in [Lan11] for details.)

Support, push-forward, and boundary. For every \( S \in \mathcal{D}_n(X) \) there exists a smallest closed subset of \( X \), the support \( \text{spt}(S) \) of \( S \), such that the value \( S(\pi_0, \ldots, \pi_n) \) depends only on the restrictions of \( \pi_0, \ldots, \pi_n \) to this set.

Let \( A \subset X \) be a closed subset and let \( S_A \in \mathcal{D}_n(A) \). Then \( S_A \) naturally defines \( S \in \mathcal{D}_n(X) \) by \( S(\pi_0, \ldots, \pi_n) = S_A(\pi_0|_A, \ldots, \pi_n|_A) \). We have \( \text{spt}(S_A) = \text{spt}(S) \). Conversely, let \( S \in \mathcal{D}_n(X) \) and let \( A \subset X \) be locally compact. If \( \text{spt}(S) \subset A \), then \( S \) uniquely determines an element in \( S_A \in \mathcal{D}_n(A) \) ([Lan11], Proposition 3.3]).

For a proper Lipschitz map \( f : X \to Y \) into another proper metric space \( Y \), the push-forward \( f_\# S \in \mathcal{D}_n(Y) \) is defined simply by

\[
(f_\# S)(\pi_0, \ldots, \pi_n) := S(\pi_0 \circ f, \ldots, \pi_n \circ f)
\]

for all \((\pi_0, \ldots, \pi_n) \in \mathcal{D}^n(Y)\). This definition can be extended to proper Lipschitz maps \( f : \text{spt}(S) \to Y \) by the previous paragraph. In either
case, \(\text{spt}(f_\# S) \subset f(\text{spt}(S))\). For \(n \geq 1\), the boundary \(\partial S \in \mathcal{D}_{n-1}(X)\) of \(S \in \mathcal{D}_n(X)\) is defined by

\[
(\partial S)(\pi_0, \ldots, \pi_{n-1}) := S(\tau, \pi_0, \ldots, \pi_{n-1})
\]

for all \((\pi_0, \ldots, \pi_{n-1}) \in \mathcal{D}^{n-1}(X)\) and for any compactly supported Lipschitz function \(\tau\) that is identically 1 on some neighborhood of \(\text{spt}(\pi_0)\). If \(\tilde{\tau}\) is another such function, then \(\pi_0\) vanishes on a neighborhood of \(\text{spt}((\tau - \tilde{\tau})\) and \(\partial S\) is thus well-defined by (1) and (3). Similarly one can check that \(\partial \circ \partial = 0\). The inclusion \(\text{spt}(\partial S) \subset \text{spt}(S)\) holds, and \(f_\#(\partial S) = \partial(f_\# S)\) for \(f: \text{spt}(S) \to Y\) as above. (See Section 3 in [Lan11].) If \(\partial S = 0\), then we will call \(S\) an \(n\)-cycle.

**Mass.** Let \(S \in \mathcal{D}_n(X)\). A tuple \((\pi_0, \ldots, \pi_n) \in \mathcal{D}^n(X)\) will be called normalized if the restrictions of \(\pi_1, \ldots, \pi_n\) to the compact set \(\text{spt}(\pi_0)\) are 1-Lipschitz. For an open set \(U \subset X\), the mass \(\|S\|(U) \in [0, \infty]\) of \(S\) in \(U\) is then defined as the supremum of \(\sum_j S(\pi_{0,j}, \ldots, \pi_{n,j})\) over all finite families of normalized tuples \((\pi_{0,j}, \ldots, \pi_{n,j}) \in \mathcal{D}^n(X)\) such that \(\bigcup_j \text{spt}(\pi_{0,j}) \subset U\) and \(\sum_j |\pi_{0,j}| \leq 1\). Note that \(\|S\|(U) = 0\) if and only if \(U \cap \text{spt}(S) \neq \emptyset\). This induces a regular Borel measure \(\|S\|\) on \(X\), whose total mass \(\|S\|(X)\) is denoted by \(\mathcal{M}(S)\). For Borel sets \(W, A \subset \mathbb{R}^n\), \(\|W\|(A)\) equals the Lebesgue measure of \(W \cap A\). If \(T \in \mathcal{D}_n(X)\) is another \(n\)-current in \(X\), then clearly

\[
\|S + T\| \leq \|S\| + \|T\|.
\]

We will now assume that the measure \(\|S\|\) is locally finite (and hence finite on bounded sets, as \(X\) is proper). Then it can be shown that

\[
|S(\pi_0, \ldots, \pi_n)| \leq \int_X |\pi_0| \|S\|
\]

for every normalized tuple \((\pi_0, \ldots, \pi_n) \in \mathcal{D}^n(X)\). This inequality allows us to extend the functional \(S\) such that the first entry \(\pi_0\) is allowed to be a compact-supported bounded Borel function [Lan11, Theorem 4.4]. We define the restriction \(S \llcorner A \in \mathcal{D}_n(X)\) of \(S\) to a Borel set \(A \subset X\) by

\[
(S \llcorner A)(\pi_0, \ldots, \pi_n) := \lim_{k \to \infty} S(\tau_k, \pi_1, \ldots, \pi_n)
\]

for any sequence of compactly supported Lipschitz functions \(\tau_k\) converging in \(L^1(\|S\|)\) to \(\chi_A \pi_0\). The measure \(\|S \llcorner A\|\) equals the restriction \(\|S\| \llcorner A\) of \(\|S\|\) (Lan11 Lemma 4.7)]).

A **piece decomposition** of \(S\) is a sum \(S = \sum_i S_i\) such that for any Borel set \(A\), \(\|S\|(A) = \sum_i \|S_i\|(A)\). Each \(S_i\) is a **piece** of \(S\). Suppose \(X = A \cup B\) with \(A\) and \(B\) Borel. Then the previous paragraph implies that \(S = S \llcorner A + S \llcorner B\) is a piece decomposition.
Recall a more general restriction operation as follows \cite[Definition 2.3]{Lan11}. For \( S \in \mathcal{D}_m(X) \) and \((u,v) \in \text{Lip}_{\text{loc}}(X) \times [\text{Lip}_{\text{loc}}(X)]^k \) where \( m \geq k \geq 0 \), define the current \( S \subseteq (u,v) \in \mathcal{D}_{m-k}(X) \) by \( (S \subseteq (u,v))(f,g) := S(uf,v,g) = S(uf,v_1,\ldots,v_k,g_1,\ldots,g_{m-k}) \) for \((f,g) \in \mathcal{D}^{m-k}(X) \). If \( k = 0 \), then the definition simplifies to
\[
(S \subseteq u)(f,g) := S(uf,g) = S(uf,g_1,\ldots,g_m).
\]

If \( f: \text{spt}(S) \to Y \) is a proper \( L \)-Lipschitz map into a proper metric space \( Y \) and \( B \subset Y \) is a Borel set, then \((f\#S) \subseteq B = f\#(S \subseteq f^{-1}(B)) \) and
\[
\|f\#S\|(B) \leq L^n \|S\|(f^{-1}(B)).
\]
(See Section 4 in \cite{Lan11}.)

We say \( \|S\| \) is concentrated on a Borel subset \( A \subset X \) if \( \|S\|(X \setminus A) = 0 \). It is possible that \( A \) is much smaller than \( \text{spt}(S) \).

**Remark 3.3.** We summarize some useful properties which follows directly from \cite{Lan11} Lemma 4.7:

1. if \( S_i \) converges to \( S \) weakly and each \( S_i \) is concentrated on a Borel set \( A \), then \( S \) is concentrated on \( A \);
2. if \( S \) is concentrated on \( A \); then \( S \subseteq (1,v) \in \mathcal{D}_{m-k}(X) \) for \( v \in [\text{Lip}(X)]^k \) is also concentrated on \( A \).

**Slicing.** Let \( S \in \mathcal{D}_n(X) \) be such that both \( \|S\| \) and \( \|\partial S\| \) are locally finite (that is, \( S \) is locally normal, see Section 5 in \cite{Lan11}). Let \( \rho: X \to \mathbb{R} \) be a Lipschitz function. The corresponding slice of \( S \) is the \((n-1)\)-dimensional current
\[
\langle S,\rho,s+ \rangle := \partial(S \subseteq \{\rho \leq s\}) - (\partial S) \subseteq \{\rho \leq s\}
\]
\[
= (\partial S) \subseteq \{\rho > s\} - \partial(S \subseteq \{\rho > s\})
\]
with support in \( \{\rho = s\} \cap \text{spt}(S) \). Similarly, we define
\[
\langle S,\rho,s- \rangle := (\partial S) \subseteq \{\rho \geq s\} - \partial(S \subseteq \{\rho \geq s\})
\]
\[
= \partial(S \subseteq \{\rho < s\}) - (\partial S) \subseteq \{\rho < s\}
\]
When \( \text{spt}(S) \) is separable (this is always true as we are assuming \( X \) is proper), \( (\|S\| + \|\partial S\|)(\pi^{-1}(s)) = 0 \) for a.e. (almost every) \( s \). Thus \( \langle S,\rho,s+ \rangle = \langle S,\rho,s- \rangle \) holds for a.e. \( s \), in this case, we define \( \langle S,\rho,s \rangle \) to be one of them.

For almost all \( s \), \( S \subseteq \{\rho \leq s\} \) is the maximal piece of \( S \) supported in \( \{\rho \leq s\} \) and \( S \) has a piece decomposition \( S = S \subseteq \{\rho \leq s\} + S \subseteq \{\rho > s\} \) (we can also write the piece decomposition as \( S = S \subseteq \{\rho < s\} + S \subseteq \{\rho > s\} \) as \( S \subseteq \{\rho \leq s\} = S \subseteq \{\rho < s\} \) for a.e. \( s \).
For $s \in \mathbb{R}$ and $\delta > 0$, let now $\gamma_{s,\delta} : \mathbb{R} \to \mathbb{R}$ be the piecewise affine $\frac{1}{\delta}$-Lipschitz function with $\gamma_{s,\delta}(-\infty,s] = 0$ and $\gamma_{s,\delta}[s,\infty) = 1$. Then, for $y = (y_1, \ldots, y_k) \in \mathbb{R}^k$ and $\delta > 0$, define $\gamma_{y,\delta} : \mathbb{R}^k \to \mathbb{R}^k$ such that $\gamma_{y,\delta}(z) = (\gamma_{y_1,\delta}(z_1), \ldots, \gamma_{y_k,\delta}(z_k))$ for all $z = (z_1, \ldots, z_k) \in \mathbb{R}^k$.

**Lemma 3.4.** For each $s$, we have

1. $\langle S, \rho, s+ \rangle$ is equal to the weak limit $\lim_{\delta \to 0^+} S \ll (1, \gamma_{s,\delta} \circ \rho)$;
2. $S$ is concentrated on a Borel subset $A \subset X$, then $\langle S, \rho, s+ \rangle$ is concentrated on $A \cap \rho^{-1}(s)$.

**Proof.** The first assertion is [Lan11, Equation (6.4)]. (2) follows from (1) and Remark 3.3. □

For a Lipschitz function $\pi : X \to \mathbb{R}^k$, we define the slice of $S$ at $y \in \mathbb{R}^k$ with respect to $\pi$ as the weak limit $\langle S, \pi, y \rangle := \lim_{\delta \to 0^+} S \ll (1, \gamma_{y,\delta} \circ \pi)$ whenever it exists and defines an element of $\mathcal{D}^{(m-k)}(X)$. By cite[Theorem 6.4]{Lan3}, the weak limit $\lim_{\delta \to 0^+} S \ll (1, \gamma_{y,\delta} \circ \pi)$ exists for a.e. $y \in \mathbb{R}^n$.

For a Borel subset $B \subset \mathbb{R}^k$, the coarea inequality

$$\int_B M(\langle S, \pi, s \rangle) \, ds \leq \text{Lip}(\pi) \|S\|((\pi^{-1}(B))$$

holds by [Lan11, Theorem 6.4 (3)].

**Integral currents.** A current $S \in \mathcal{D}_n(X)$ is called locally integer rectifiable if all the following properties hold:

1. $\|S\|$ is locally finite and is concentrated on the union of countably many Lipschitz images of compact subsets of $\mathbb{R}^n$;
2. for every Borel set $A \subset X$ with compact closure and every Lipschitz map $\phi : X \to \mathbb{R}^n$, the current $\phi_#(S \ll A) \in \mathcal{D}_n(\mathbb{R}^n)$ is of the form $[w]$ for some integer valued $w = w_{A,\phi} \in L^1(\mathbb{R}^n)$.

Then $\|S\|$ turns out to be absolutely continuous with respect to $n$-dimensional Hausdorff measure. Furthermore, push-forwards and restrictions to Borel sets of locally integer rectifiable currents are again locally integer rectifiable.

A current $S \in \mathcal{D}_n(X)$ is called a locally integral current if $S$ is locally integer rectifiable and, for $n \geq 1$, $\partial S$ satisfies the same condition. (Remarkably, this is the case already when $\|\partial S\|$ is locally finite, provided $S$ is locally integer rectifiable; see Theorem 8.7 in [Lan11].) This yields a chain complex of abelian groups $\mathcal{I}_{n,\text{loc}}(X)$. We write $\mathcal{I}_{n,c}(X)$ (resp. $\mathcal{I}_n(X)$) for the respective subgroups of integral currents with compact support (resp. with finite mass).
By [Lan11, Theorem 8.5], if \( \pi : X \to \mathbb{R}^k \) is Lipschitz, then for a.e. \( y \in \mathbb{R}^k \), \( \langle S, \pi, y \rangle \in I_{n-k, \text{loc}}(X) \). In particular, if \( k = 1 \), \( S \perp \{ \pi \leq y \} \in I_{n, \text{loc}}(X) \) for a.e. \( y \in \mathbb{R} \).

**Remark 3.5.** If \( \Delta \subset \mathbb{R}^n \) is an \( n \)-simplex and \( f : \Delta \to X \) is a Lipschitz map, then \( f_\# [\Delta] \in I_{n,c}(X) \). Thus every singular Lipschitz chain in \( X \) with integer coefficients defines an element of \( I_{n,c}(X) \).

There is a canonical chain isomorphism from \( I_{*,c}(\mathbb{R}^N) \) to the chain complex of “classical” integral currents in \( \mathbb{R}^N \) originating from [FF60].

If \( T \in I_{N,\text{loc}}(\mathbb{R}^N) \), then \( T = [u] \) for some function \( u \) of locally bounded variation, moreover \( u \) is integer-valued almost everywhere [Lan11, Theorem 7.2]. The element \([\mathbb{R}^N]\) \( \in I_{N,\text{loc}}(\mathbb{R}^N) \) is the fundamental class of \( \mathbb{R}^n \).

For \( n \geq 1 \), we let \( Z_{n,\text{loc}}(X) \subset I_{n,\text{loc}}(X) \) and \( Z_{n,c}(X) \subset I_{n,c}(X) \) denote the subgroups of currents with boundary zero. An element of \( I_{n,\text{c}}(X) \) is an integral linear combination of currents of the form \([x]\), where \( [x](\tau_0) = \tau_0(x) \) for all \( \tau_0 \in \mathcal{D}^0(X) \). We let \( Z_{0,c}(X) \subset I_{0,c}(X) \) denote the subgroup of linear combinations whose coefficients sum up to zero. The boundary of a current in \( I_{1,c}(X) \) belongs to \( Z_{0,c}(X) \). Given \( Z \in Z_{n,c}(X) \), for \( n \geq 0 \), we will call \( V \in I_{n+1,c}(X) \) a filling of \( Z \) if \( \partial V = Z \).

For each \( S \in I_{N-1,c}(\mathbb{R}^N) \), there exists a unique \( T \in I_{N,c}(\mathbb{R}^N) \) such that \( \partial T = S \). In this case, \( T \) is called the canonical filling of \( S \). By above discussion, \( T = [u] \) for a compact supported integer valued function of bounded variation. Similarly, by an approximation argument, one can define a canonical filling \( T \in I_{N}(\mathbb{R}^N) \) for each \( S \in I_{N-1}(\mathbb{R}^N) \). If \( T = [u] \), then \( M(T) \) is the \( L^1 \)-norm of \( u \) ([Lan11, Equation 4.5]) and \( \| \partial T \| = |Du| \) ([Lan11, Theorem 7.2]).

### 3.4. Ambrosio-Kirchheim currents in general metric spaces.

We will also need metric currents in a complete metric space \( X \) which is not necessarily locally compact [AK+00]. These currents are defined using slightly different axioms, however, all the features discussed in the previous section (e.g. support, boundary, mass, slicing etc) are available, see [AK+00] for more details. We refer to [Lan11, Section 4] for comparison between currents in the sense of Ambrosio and Kirchheim and local currents in the previous section.
Let $X$ be a complete metric space. We use $I_n(X)$ to denote the abelian group of $n$-dimensional integral currents in the sense of Ambrosio and Kirchheim. In the special case when $X$ is proper, each element in $I_{n,\text{loc}}(X)$ defined in Section 3.3 with finite mass gives a unique element in $I_n(X)$, and each element of $I_n(X)$ in the sense of Ambrosio and Kirchheim gives a unique element in $I_{n,\text{loc}}$ with finite mass ([Lan11, Section 4]). So when $X$ is proper, we can treat $I_n(X)$ as the abelian group of local integral currents with finite mass. We use $I_{n,c}(X)$ to integral currents with compact support. When $X$ is proper, these currents can be identified with local integral currents with compact support.

3.5. Homotopies. Let $X$ be a complete metric space. Let $[0, 1] \in I_{1,c}([0, 1])$ denote the current defined by

$$[0, 1](\pi_0, \pi_1) := \int_0^1 \pi_0(t) \pi_1'(t) \, dt.$$ 

Note that $D[0, 1] = [1] - [0]$. We endow $[0, 1] \times X$ with the usual $l_2$ product metric. There exists a canonical product construction

$$S \in I_n(X) \rightsquigarrow [0, 1] \times S \in I_{n+1}([0, 1] \times X)$$

for all $n \geq 0$ (see [Wen05, Definition 2.8]). Suppose now that $Y$ is another complete metric space, $h: [0, 1] \times X \to Y$ is a Lipschitz homotopy from $f = h(0, \cdot)$ to $g = h(1, \cdot)$, and $S \in I_n(X)$. Then $h_\#([0, 1] \times S)$ is an element of $I_{n+1}(Y)$ with boundary

$$\partial h_\#([0, 1] \times S) = g_\#S - f_\#S - h_\#([0, 1] \times \partial S)$$

(for $n = 0$ the last term is zero.) See [Wen05, Theorem 2.9]. If $h(t, \cdot)$ is $L$-Lipschitz for every $t$, and $h(\cdot, x)$ is a geodesic of length at most $D$ for every $x \in \text{spt}(S)$, then

$$M(h_\#([0, 1] \times S)) \leq (n + 1)L^nD M(S).$$

(see [Wen05, Proposition 2.10]). A similar formula holds if instead $h(\cdot, x)$ is a bi-Lipschitz path of length $\leq D$.

An important special case of this is when $R \in Z_n(X)$ and $h(\cdot, x) = \sigma_{px}$ is a geodesic from some fixed point $p \in X$ to $x$ for every $x \in \text{spt}(R)$. Then $h_\#([0, 1] \times R) \in I_{n+1}(X)$ is the cone from $p$ over $R$ determined by this family of geodesics, whose boundary is $R$. If the family of geodesics satisfies the convexity condition

$$d(h(t, x), h(t, x')) = d(\sigma_{px}(t), \sigma_{px'}(t)) \leq t d(x, x')$$

for all $x, x' \in \text{spt}(R)$ and $t \in [0, 1]$, and if $\text{spt}(R) \subset B_p(r)$, then

$$M(h_\#([0, 1] \times R)) \leq r M(R).$$
3.6. Coning inequalities and isoperimetric inequalities.

**Definition 3.6.** A complete metric space $X$ satisfies *coning inequalities* up to dimension $n$, abbreviated condition $(\text{CI}_n)$, if there exists a constant $c_0 > 0$, such that for all $0 \leq k \leq n$ and every cycle $S \in I_k(X)$ with bounded support there exists a filling $T \in I_{k+1}(X)$ with

$$M(T) \leq c_0 \cdot \text{diam}(\text{spt}(S)) \cdot M(S).$$

If in addition, there exists a constant $c_1 > 0$ such that $T$ can always be chosen to fulfill

$$\text{diam}(\text{spt}(T)) \leq c_1 \cdot \text{diam}(\text{spt}(S)),$$

then $X$ is said to satisfy *strong coning inequalities* up to dimension $n$, abbreviated condition $(\text{SCI}_n)$.

Note that if $X$ is proper, then we can equivalently take $S$ and $T$ to be compactly supported.

**Remark 3.7.** By Section 3.5, condition $(\text{CI}_n)$ and $(\text{SCI}_n)$ are satisfied for complete metric space with an $L$-Lipschitz bicombing for any $n$.

**Remark 3.8.** One can alternatively formulate Definition 3.6 by requiring both $S$ and $T$ being singular Lipschitz chains, which leads to a different definition. However, if $X$ is bi-Lipschitz homeomorphic to a finite-dimensional simplicial complex with standard metrics on the simplices, then (by a variant of the Federer–Fleming deformation theorem [FF60]) one may equivalent to take $S$ and $T$ to be simplicial chains or singular Lipschitz chains (with integer coefficients). See [ABD+13, Section 2] for more explanation.

**Remark 3.9.** Any $n$-connected simplicial complex with a properly discontinuous and cocompact simplicial action of a combable group satisfies $(\text{CI}_n)$ and $(\text{SCI}_n)$; see [ECH+92, Section 10.2]. Every combable group, in particular every automatic group, admits such an action.

**Theorem 3.10** (isoperimetric inequality). Let $n \geq 2$, and let $X$ be a complete metric space satisfying condition $(\text{CI}_{n-1})$. Then every cycle $R \in Z_{n-1}(X)$ possesses a filling $T \in I_n(X)$ such that

1. $M(T) \leq b_1 M(R)^{n/(n-1)}$;
2. $\text{spt}(T) \subset N_c(\text{spt}(\partial T))$ where $c = b_2 M(R)^{1/(n-1)},$

for some constants $b_1, b_2 > 0$ depending only on the constants of condition $(\text{CI}_{n-1})$. Moreover, if $R$ has compact support, then we can also require $T$ has compact support.
The first item is proved in [Wen05], see the comment after [Wen05, Theorem 1.2] regarding compact supports. The second item follows from [Wen11, Proposition 4.3 and Corollary 4.4].

We also consider the following condition which is weaker than Definition 3.6 (by Theorem 3.10.)

**Definition 3.11.** A complete metric space $X$ satisfies *Euclidean isoperimetric inequality* up to dimension $n$, abbreviated condition (EII$_n$), if there exists a constant $c > 0$, such that for all $0 \leq k \leq n$ and every cycle $S \in Z_{k,c}(X)$ there exists a filling $T \in I_{k+1,c}(X)$ with

$$M(T) \leq c M(R)^{k+1/k}.$$ 

3.7. **Quasi-minimizer and minimizer.**

**Definition 3.12** (quasi-minimizer). Suppose that $X$ is a complete metric space, $n \geq 1$, and $Q \geq 1$, $a \geq 0$ are constants. For a closed set $Y \subset X$, a cycle $S \in Z_n(X,Y) := \{Z \in I_n(X) : \text{spt}(\partial S) \subset Y\}$ relative to $Y$ will be called $(Q,a)$-quasi-minimizing mod $Y$ if, for all $x \in \text{spt}(S)$ and almost all $r > a$ such that $B_x(r) \cap Y = \emptyset$, the inequality

$$M(S \ll B_x(r)) \leq Q M(T)$$

holds whenever $T \in I_n(X)$ and $\partial T = \partial (S \ll B_x(r))$ (recall that $S \ll B_x(r) \in I_n(X)$ for almost all $r > 0$). A current $S \in I_n(X)$ is $(Q,a)$-quasi-minimizing or a $(Q,a)$-quasi-minimizer if $S$ is $(Q,a)$-quasi-minimizing mod $\text{spt}(\partial S)$, and we say that $S$ is quasi-minimizing or a quasi-minimizer if this holds for some $Q \geq 1$ and $a \geq 0$.

When $X$ is proper, then we can define $(Q,a)$-quasi-minimizing similarly for local cycle $S \in Z_{n,\text{loc}}(X,Y)$ (in this case $T \in I_{n,\text{loc}}(X)$).

**Definition 3.13.** Suppose $X$ is a complete metric space. We say an element $S \in I_n(X)$ is minimizing, or $S$ is a minimizer, if $M(S) \leq M(T)$ for any $T \in I_n(X)$ with $\partial T = \partial S$. For a constant $M \geq 1$, we say $S$ is $M$-minimizing, if for each piece $S'$ of $S$, we have $M(S) \leq M M(T)$ for any $T \in I_n(X)$ with $\partial T = \partial S$. Note that $S$ is minimizing if and only if $S$ is 1-minimizing. A local current $S \in I_{n,\text{loc}}(X)$ with $X$ being proper is minimizing, if each compact supported piece of $S$ is minimizing. We define $M$-minimizing for local currents in a similar way.

Obviously every minimizing $S \in I_n(X)$ is $(1,0)$-quasi-minimizing.

For $S \in Z_n(X)$, we define $\text{Fill}(S) := \inf\{M(T) : T \in I_{n+1}(X), \partial T = S\}$. When $S$ is compactly supported and $X$ satisfies EII$_n$, we can
define $\text{Fill}(S)$ by requiring $T \in I_{n+1,c}(X)$ – this gives rise to the same number (see [Wen11, Proposition 4.3]). The next theorem guarantees the existence of a minimal filling under extra assumptions.

**Theorem 3.14.** [KL18, Theorem 2.4] Let $n \geq 1$, and let $X$ be a proper metric space satisfying condition $(\text{CI}_{n-1})$. Then for every $R \in Z_{n-1,c}(X)$ there exists a filling $S \in I_{n,c}(X)$ of $R$ with mass $M(S) = \text{Fill}(S)$. Furthermore, $\text{spt}(S)$ is within distance at most $(M(S)/\delta)^{1/n}$ from $\text{spt}(R)$ for some constant $\delta > 0$ depending only on $n$ and constants in condition $(\text{CI}_{n-1})$.

**Lemma 3.15** (density). Let $n \geq 1$, let $X$ be a complete metric space satisfying condition $(\text{CI}_{n-1})$, and let $Y \subset X$ be a closed set. If $S \in Z_n(X,Y)$ (or $Z_{n,\text{loc}}(X,Y)$ when $X$ is proper) is $(Q,a)$-quasi-minimizing mod $Y$, and if $x \in \text{spt}(S)$ and $r > 2a$ are such that $B_x(r) \cap Y = \emptyset$, then

$$\frac{1}{r^n} \|S\|(B_x(r)) \geq \delta$$

for some constant $\delta > 0$ depending only on $n$, the constants $c_0$ from Definition 3.6, and $Q$.

This is [KL18, Lemma 3.3]. Only the local current case was proved there, however, the same proof works for Ambrosio-Kirchheim currents in complete metric spaces, using Theorem 3.10.

**Lemma 3.16.** [KL18, Lemma 3.4] Let $n \geq 1$, let $X$ be a complete metric space satisfying condition $(\text{CI}_{n-1})$, and let $Y \subset X$ be a closed set. If $S \in Z_n(X,Y)$ (or $S \in Z_{n,\text{loc}}(X,Y)$ when $X$ is proper) is $(Q,a)$-quasi-minimizing mod $Y$, and if $x \in \text{spt}(S)$ and $r > 4a$ are such that $B_x(r) \cap Y = \emptyset$, then

$$\frac{1}{r^{n+1}} \inf \{M(V) : V \in I_{n+1,c}(X), \text{spt}(S - \partial V) \cap B_x(r) = \emptyset\} \geq c$$

for some constant $c > 0$ depending only on $n$, the constant $\delta$ from Lemma 3.15, and $Q$.

4. **Approximating currents by currents with uniform density**

In this section we describe an approximating procedure which improve the compactness of the support of currents. This will be used in Section 6 and Section 7.
4.1. **The approximation.** Let $X$ be a complete metric space. For $\kappa > 0$ we define a distance function $d_\kappa$ on the set of integral currents $I_n(X)$ by

$$d_\kappa(\tau_1, \tau_2) = \kappa M(\tau_1 - \tau_2) + M(\partial \tau_1 - \partial \tau_2).$$

It follows from [AK+00] that:

**Lemma 4.1.** $(I_n(X), d_\kappa)$ is a Banach space

Suppose $\sigma \in I_{n-1}(X)$ satisfying $M(\sigma) \leq C$ (we allow $\partial \sigma \neq 0$). We define $F : I_n(X) \to \mathbb{R}$ by $F(\tau) = \kappa M(\tau) + M(\sigma - \partial \tau)$. Note that $F$ is lower semicontinuous with respect to $d_\kappa$, since convergence in $I_n(X)$ implies weak convergence of currents, and the mass is lower semicontinuous with respect to weak convergence.

Note that $F(0) = M(\sigma)$. It follows from Ekeland’s variational principle that there exists $\tau$ such that

(a) $F(\tau) \leq F(0) = M(\sigma)$;
(b) The functional $\tau' \to F(\tau') + \frac{1}{2} d_\kappa(\tau, \tau')$ defined on $I_n(X)$ attains its minimal at $\tau' = \tau$.

**Proposition 4.2.** Suppose $X$ is a complete metric space satisfying condition (EII$_{n-1}$) (cf. Definition 3.11). Let $\sigma \in I_{n-1}(X)$. Suppose $\kappa > 1$. Let $\tau$ be a minimal of $F$ as above. Define $\sigma' = \sigma - \partial \tau$. There exist constants $\lambda$ and $\lambda'$ depending only on $X$ and $C$ such that the following holds.

1. Fill($\sigma - \sigma'$) $\leq \frac{M(\sigma)}{\kappa}$ and $M(\sigma') \leq M(\sigma)$.
2. For each point $x \in \text{spt}(\sigma')$, we have $M(\sigma' \sqcup B_x(r)) \geq \lambda r^{n-1}$ for $0 \leq r \leq \min\{\frac{\lambda}{\kappa}, d(x, \text{spt}(\partial \sigma'))\}$.
3. $\text{spt}(\sigma') \subset N_a(\text{spt}(\sigma))$ where $a \leq \frac{\lambda \ln(\kappa)}{\kappa}$.
4. $\text{spt}(\tau) \subset N_b(\text{spt}(\sigma))$ where $b \leq (\frac{C}{\lambda n})^\frac{1}{2} + \frac{\lambda \ln(\kappa)}{\kappa}$.

**Proof.** (1) follows from (a). Now we prove (2). Let $D$ be the constant in condition EII$_{n-1}$.

Let $x$ be as in (2). Let $f(r) = M(\sigma' \sqcup B_x(r))$. Let $\alpha$ be a filling of $\langle \sigma', d_x, r \rangle$ with $M(\alpha) \leq D(M(\langle \sigma', d_x, r \rangle))^\frac{n-1}{n}$. Let $W$ be a filling of $\alpha - \sigma' \sqcup B_x(r)$ such that

$$M(W) \leq D(M(\alpha) + (M(\sigma' \sqcup B_x(r)))^\frac{n}{n-1}).$$
Define $\tau' = \tau + W$. Since $F(\tau') + \frac{1}{2}d_\kappa(\tau, \tau') \geq F(\tau)$, we have

$$0 \leq (F(\tau') - F(\tau)) + \frac{1}{2}d_\kappa(\tau, \tau')$$

$$= (-M(\sigma' \land B_x(r)) + M(\alpha) + \kappa M(W)) + \frac{1}{2}(\kappa M(W) + M(-\sigma' \land B_x(r) + \alpha))$$

$$\leq -\frac{1}{2}M(\sigma' \land B_x(r)) + \frac{3}{2}M(\alpha) + \frac{3\kappa}{2}M(W)$$

By coarea inequality, we have $M(\sigma', d_x, r) \leq f'(r)$ for a.e. $r$. It follows that $M(\alpha) \leq D(f'(r))^{\frac{n-1}{n-2}}$ and $M(W) \leq D(f(r) + M(\alpha))^{\frac{n}{n-1}}$. Thus

$$-f(r) + 3D(f'(r))^{\frac{n-1}{n-2}} + 3\kappa D[f(r) + (f'(r))^{\frac{n-1}{n-2}}]^{\frac{n}{n-1}} \geq 0.$$  

(4.3)

Let $y = f(r)$ and $\mu = (f'(r))^{\frac{n-1}{n-2}}$. Then (4.3) implies that $\mu$ is the unique nonnegative solution to the following equation:

$$-y + 3D\mu + 3\kappa D(y + \mu)^{\frac{n}{n-1}} = 0.$$  

In particular, $-y + 3D\mu \leq 0$. By plugging $\mu \leq \frac{y}{3D}$ into the last term of (4.1), we have $-y + 3D\mu + 3\kappa D(y + \frac{y}{3D})^{\frac{n}{n-1}} \geq 0$. Thus

$$3D\mu \geq y \left[1 - 3\kappa D y^{\frac{n}{n-1}} \left(1 + \frac{1}{3D}\right)^{\frac{n}{n-1}}\right].$$  

It follows that there exists $\lambda_0$ depending only on $D$ such that $3D\mu \geq \frac{y}{2}$ for any $y \in [0, (\frac{1}{\lambda_0})^{n-1}]$. Thus for a.e. $r$, either $f(r) \geq (\frac{1}{\lambda_0})^{n-1}$, or

$$-f(r) + 6D(f'(r))^{\frac{n-1}{n-2}} \geq 0.$$  

(4.4)

As $f(r)$ is non-decreasing and $f(0) = 0$, by solving (4.4), we know there exists $\lambda$ depending only on $D$ such that either $f(r) \geq (\frac{1}{\lambda_0})^{n-1}$, or $f(r) \geq \lambda r^{n-1}$. Now (2) follows as $f$ is non-decreasing.

Now we prove (3). Let $x \in \text{spt}(\sigma')$. Let $f(r) = M(\tau \land B_x(r))$. Then

$$F(\tau - \tau \land B_x(r)) + \frac{1}{2}d_\kappa(\tau, \tau - \tau \land B_x(r)) \geq F(\tau).$$  

(4.5)

for $r < d(x, \text{spt}(\sigma))$. Note that for $r < d(x, \text{spt}(\sigma))$,

$$F(\tau - \tau \land B_x(r)) - F(\tau) = -\kappa M(\tau \land B_x(r)) - M(\sigma' \land B_x(r)) + M(\langle \tau, d_x, r \rangle).$$
This together with (4.5) imply that
\[
0 \leq (F(\tau - \tau \cdot B_x(r)) - F(\tau)) + \frac{1}{2}d_\kappa(\tau, \tau - \tau \cdot B_x(r)) \\
\leq (-\kappa M(\tau \cdot B_x(r))) - M(\sigma' \cdot B_x(r)) + M(\langle \tau, d_x, r \rangle)) \\
+ \frac{1}{2}(\kappa M(\tau \cdot B_x(r))) + M(\sigma' \cdot B_x(r) - \langle \tau, d_x, r \rangle) \\
\leq -\frac{\kappa}{2}M(\tau \cdot B_x(r)) - \frac{1}{2}M(\sigma' \cdot B_x(r)) + \frac{3}{2}M(\langle \tau, d_x, r \rangle)
\]

In particular,
\[
-\frac{\kappa}{2}M(\tau \cdot B_x(r)) + \frac{3}{2}M(\langle \tau, d_x, r \rangle) \geq 0 \text{ for } 0 < r < d(x, spt(\sigma))
\]
\[
-\frac{\kappa}{2}M(\tau \cdot B_x(r)) - \frac{1}{2}\kappa r^{n-1} + \frac{3}{2}M(\langle \tau, d_x, r \rangle) \geq 0 \text{ for } 0 < r \leq \frac{\lambda'}{\kappa}
\]
The coarea inequality implies that $M(\langle \tau, d_x, r \rangle) \leq f'(r)$ for a.e. $r$.

Thus
\[
(4.6) \quad -\kappa f(r) + 3f'(r) \geq 0 \text{ for } 0 < r < d(x, spt(\sigma))
\]
\[
(4.7) \quad -\kappa f(r) - \lambda r^{n-1} + 3f'(r) \geq 0 \text{ for } 0 < r \leq \frac{\lambda'}{\kappa}
\]

Since $\ln(f(r))$ is non-decreasing, by (4.6) we know for $r_2 > r_1$,
\[
\ln(f(r_2)) - \ln(f(r_1)) \geq \int_{r_1}^{r_2} f'(t) dt \geq \int_{r_1}^{r_2} \frac{\kappa}{3} dt = \frac{\kappa}{3}(r_2 - r_1).
\]

Thus $e^{-\frac{\lambda'}{\kappa}r}f(r)$ is non-decreasing. This together with (4.7) imply
\[
e^{-\frac{\lambda'}{\kappa}r}f(r) - e^0 f(0) \geq \int_0^r (e^{-\frac{\lambda'}{\kappa}t}f(t))' dt \geq \frac{1}{3} \int_0^r \lambda e^{-\frac{\lambda'}{\kappa}t}t^{n-1} dt.
\]

Thus
\[
f(\frac{\lambda'}{\kappa}) \geq \frac{\lambda}{3} \int_0^{\lambda'} e^{-\frac{\lambda'}{\kappa}t}t^{n-1} dt = \frac{3^{n-1}\lambda}{\kappa^n} \int_0^{\lambda'} e^{-u}u^{n-1} du.
\]

Define $\lambda'' = \lambda \int_0^{\lambda'} e^{-u}u^{n-1} du$. Then for $\frac{\lambda'}{\kappa} < r < d(x, spt(\sigma))$,
\[
e^{-\frac{\lambda'}{\kappa}r}f(r) \geq e^{-\frac{\lambda'}{\kappa}r}f(\frac{\lambda'}{\kappa}) \geq e^{-\frac{\lambda'}{\kappa}} \cdot \frac{3^{n-1}\lambda''}{\kappa^n}.
\]

Thus $f(r) \geq e^{\frac{\lambda'}{\kappa}r - \lambda'} \cdot \frac{3^{n-1}\lambda''}{\kappa^n}$. On the other hand, $f(r) \leq M(\tau) \leq C_\kappa$. As $\kappa > 1$ and $n - 1 \geq 1$, we conclude that $r \leq \frac{\lambda \ln(\kappa)}{\kappa}$ for $\lambda$ depending only on $C$ and $X$.

(4) is similar to [AK+00, Theorem 10.6]. We replicate the proof for the convenience of the reader. Let $x \in spt(\tau) \setminus spt(\partial \tau)$ and let $r_x = d(x, spt(\partial \tau))$. Let $f(r) = M(\tau \cdot B_x(r))$. For $0 < r < r_x$, let
Let \( \alpha \) be a filling of \((\tau, d_x, r)\) such that \( M(\alpha) \leq D(M((\tau, d_x, r)))^{\frac{n}{n-1}} \). Let \( \tau' = \tau - \tau \cup B_x(r) + \alpha \). We deduce from \((F(\tau') - F(\tau)) + \frac{1}{2}d_\kappa(\tau, \tau') \geq 0\) that
\[
\kappa(-M(\tau \cup B_x(r)) + M(\alpha)) + \frac{\kappa}{2}(M(\tau \cup B_x(r)) + M(\alpha)) \geq 0
\]
Thus
\[
-M(\tau \cup B_x(r)) + 3M(\alpha) \geq 0
\]
For a.e. \( 0 < r < r_x \), \( M((\tau, d_x, r)) \leq f'(r) \). Thus \( M(\alpha) \leq D(f'(r))^{\frac{n}{n-1}} \).

Hence
\[
f(r) + 3D(f'(r))^{\frac{n}{n-1}} \geq 0.
\]
Thus \( f(r) \geq \lambda r^n \) for \( 0 < r < r_x \). Since \( f(r) \leq M(\tau) \leq \frac{c}{\kappa} \) by (1), we know \( r_x \leq \left( \frac{c}{\lambda \kappa} \right)^{\frac{1}{n}} \). Now (4) follows from (3). \( \square \)

4.2. Isomorphisms of homologies. Let \( X \) be a complete metric space. Let \( H_*(X), \tilde{H}_L^*(X), \tilde{H}_*^{AK}(X), \tilde{H}_*^{AK}(X) \) be the reduced homology induced by the chain complex of singular chains, Lipschitz chains, compactly supported Ambrosio-Kirchheim (AK) integral currents and finite mass AK integral currents. For subset \( Y \subset X \), the collection elements in \( I_*(X) \) with support contained in \( Y \) is stable under taking boundary. Thus we can define the relative homology groups \( \tilde{H}_*^{AK}(X, Y) \) and \( \tilde{H}_*^{AK}(X, X) \) in the usual way.

There are natural maps \( \tilde{H}_*^{L}(X) \to H_*(X), \tilde{H}_*^{L}(X) \to \tilde{H}_*^{AK}(X) \) and \( \tilde{H}_*^{AK}(X) \to \tilde{H}_*^{AK}(X, X) \) (see \[RS09\], \[Mit13\]).

**Proposition 4.8.** Suppose \( X \) is complete metric space with an \( L \)-Lipschitz bicombing. Let \( K \subset X \) be a compact set. Then

1. \( \tilde{H}_*^{L}(X, X-K) \to \tilde{H}_*(X, X-K), \tilde{H}_*^{AK}(X, X-K) \to \tilde{H}_*^{AK}(X, X-K) \) and \( \tilde{H}_*^{AK}(X, X-K) \to \tilde{H}_*^{AK}(X, X-K) \) are isomorphisms.

2. \( \tilde{H}_*^{L}(X) \to \tilde{H}_*(X), \tilde{H}_*^{L}(X) \to \tilde{H}_*^{AK}(X) \) and \( \tilde{H}_*^{AK}(X) \to \tilde{H}_*^{AK}(X) \) are isomorphisms.

**Proof.** We only prove (1) as (2) is similar. The \( \tilde{H}_*^{L}(X, X-K) \to \tilde{H}_*(X, X-K) \) being an isomorphism follows from a standard argument of straightening singular simplices using the bicombing. The second isomorphism follows from \[RS09\] and \[Mit13\] (the locally strong Lipschitz contractions condition in their paper is implied by the Lipschitz bicombing condition).

Now we look at \( \tilde{H}_*^{AK}(X, X-K) \to \tilde{H}_*^{AK}(X, X-K) \). Note that \( L \)-Lipschitz bicombing on \( X \) implies that \( X \) satisfies (CI\( n \)) for any \( n \),
5. Quasiflats in metric spaces

The main goal of this section is to provide some auxiliary results on building a chain between cycle in the space and its “projection” on a quasiflat. Readers who mainly interests in quasiflats in metric spaces with geodesic bicombing only need to read Section 5.1. Section 5.2 and Section 5.3 are for readers who are interested in more general case where a quasiflats might not be represented by a continuous map.

5.1. Lipschitz quasiflats and quasi-retractions.

Definition 5.1. Let $K \subset X$ be a closed subset. A map $\pi : X \to K$ is called a $\lambda$-quasi-retraction if it is $\lambda$-Lipschitz and the restriction $\pi|_K$ has displacement $\leq \lambda$.

Lemma 5.2. Let $X$ be a length space and let $\Phi : \mathbb{E}^n \to X$ be a $L$-Lipschitz $(L, A)$-quasiflat with image $Q$. Then there exist $L$ depending only on $L$ and $A$, a metric space $\tilde{X}$, an $L$-bilipschitz embedding $X \to \tilde{X}$ and an $L$-Lipschitz retraction $\tilde{X} \to X$ with the following additional properties. $\tilde{X}$ contains a $\tilde{L}$-bilipschitz flat $\tilde{Q}$ such that $d_H(Q, \tilde{Q}) < L$ and $d_H(X, \tilde{X}) < L$.

Proof. We glue $\mathbb{E}^n \times [0, L]$ to $X$ along $\mathbb{E}^n \times \{L\}$ via $\Phi$. Denote the resulting space by $\tilde{X}$, equipped $\tilde{X}$ with the induced length metric.
Let \( \lambda L \) be a constant depending only on \( \lambda, L, A \) and \( n \), and \( \lambda \)-Lipschitz bicombing on \( \mathbb{Q} \). Let \( h \) be as in Corollary 5.3. Suppose \( \rho > \lambda _2 \). Then there exists a constant \( C > 0 \) depending only on \( \lambda L, L, A, n \) such that
\[
M(h_{\ast}([0, 1] \times \sigma)) \leq C \cdot \lambda^n \cdot \rho \cdot M(\tau).
\]

**Corollary 5.5.** Let \( Q, \pi, \lambda _1, \lambda _2 \) be as in Corollary 5.3. Suppose \( X \) has \( \lambda L \)-Lipschitz bicombing. Let \( \sigma \in I_n(X) \) such that \( S = \text{spt} \sigma \subset N_{2\rho}(Q) \setminus N_{\rho}(Q) \). Suppose that \( h : [0, 1] \times S \to X \) is the homotopy from \( \tau \) to \( \pi \circ \tau \) induced by the bicombing on \( X \). Suppose \( \rho > \lambda _2 \). Then there exists a constant \( C > 0 \) depending only on \( \lambda L, L, A \) and \( n \) such that
\[
M(h_{\ast}([0, 1] \times \sigma)) \leq C \cdot \lambda^n \cdot \rho \cdot M(\tau).
\]

**Proof.** By Lemma 5.4 it suffices to show \( d(x, \pi(x)) \leq C \rho \) for any \( x \in S \). Let \( z \) be a point in \( Q \) such that \( \rho \leq d(x, z) = d(x, Q) \leq 2\rho \). Then
\[
d(x, \pi(x)) \leq d(x, z) + d(z, \pi(z)) + d(\pi(z), \pi(x)) \leq 2\rho + \lambda _2 + \lambda _1 d(x, a) \leq 2\rho + \rho + \lambda _1 \cdot (2\rho) = (3 + 2\lambda _1)\rho.
\]

5.2. **Cubulated quasiflats and quasidisks.** We consider a quasiflat $Q$ represented by $\Phi : \mathbb{R}^n \to X$ in a complete metric space satisfying condition $(\text{CI}_{n-1})$. The goal is to define an “push-forward map” sending a current in $\mathbb{R}^n$ to a current in $X$ and define a “projection map” sending a current in $X$ to a current close to $Q$.

Here we say that $\mathcal{C}_R$ is a **regular cubulation at scale** $R$ of $\mathbb{R}^n$ if its zero-skeleton $\mathcal{C}_R^{(0)}$ is given by $v + R \cdot \mathbb{Z}^n$ for some $v \in \mathbb{R}^n$. If $\Phi : \mathbb{R}^n \to X$ is an $(L, A)$-quasi-isometric embedding, then we call a regular cubulation $\mathcal{C}_{R_0}$ admissible (for $q$), if its scale is $R_0 = 2LA$.

**Definition 5.6.** Let $\Phi : \mathbb{R}^n \to X$ be an $(L, A)$-quasi-isometric embedding with image $Q$. Let $\mathcal{C}$ be an admissible regular cubulation of $\mathbb{R}^n$. A **$\lambda$-quasi-retraction** (associated to $q$ and $\mathcal{C}$) is a map $\pi : X \to \mathbb{R}^n$ such that

1. $\pi$ is $\lambda$-Lipschitz;
2. $(\pi \circ q)|_{\mathcal{C}(0)} = \text{id}$ and $d(\pi \circ q(x), x) < \lambda$ for any $x \in \mathbb{R}^n$;
3. $d(q \circ \pi(x), x) < \lambda$ for any $x \in Q$.

Note that since $\mathcal{C}$ is admissible, $q|_{\mathcal{C}(0)}$ is $2L$-bilipschitz. Hence any Lipschitz extension $\pi : X \to \mathbb{R}^n$ of $(q|_{\mathcal{C}(0)})^{-1}$ is a $\lambda$-quasi-retraction associated to $q$ and $\mathcal{C}$. By McShane’s extension result, we can arrange $\lambda$ to depend only on $L$ and $n$.

**Proposition 5.7** (cubulated quasiflats). Let $n \geq 2$, and let $X$ be a complete metric space satisfying condition $(\text{CI}_{n-1})$. Then for all $L, A$ there exist $\lambda, a$ depending only on $L, A, n$ and $X$ such that the following holds. Suppose that $\mathcal{C}_R$ is a regular cubulation of $\mathbb{R}^n$ at scale $R = 2LA$. Let $\mathcal{P}^*_*(\mathbb{R}^n)$ denote the collection of integral currents represented by cubical chains with respect to $\mathcal{C}_R$. If $q : \mathbb{R}^n \to X$ is an $(L, A)$-quasi-isometric embedding, then there exists a chain map $\iota : \mathcal{P}^*_*(\mathbb{R}^n) \to \mathcal{I}^*_{*,c}(X)$ such that

1. $\iota$ maps every vertex $[x_0] \in \mathcal{P}^*_0(\mathbb{R}^n)$ to $[q(x_0)]$ and, for $1 \leq k \leq n$, every oriented cube $B \in \mathcal{P}^*_k(\mathbb{R}^n)$ to a current with support in $N_\lambda(q(\mathcal{C}_R^{(0)}))$;
2. $\mathbf{M}(\iota T) \leq a \cdot \mathbf{M}(T)$ for all $T \in \mathcal{P}^*_*(\mathbb{R}^n)$;
3. For every top-dimensional chain $W \in \mathcal{P}^*_n(\mathbb{R}^n)$, we have $\iota [W] \in \mathcal{I}^*_n(X)$ is $(\lambda, a)$-quasi-minimizing mod $N_\lambda(q((\text{spt } \partial W)^{(0)}))$;
4. For every top-dimensional chain $W \in \mathcal{P}^*_n(\mathbb{R}^n)$, we have

$$d(q(x), \text{spt}(\iota [W])) \leq a$$

for all $x \in \text{spt}(W)$ with $d(x, \text{spt } \partial W) \geq a$. 

MORSE QUASIFLATS 37

\[ \text{ } \]
(5) For every chain \( P \in \mathcal{P}_k(\mathbb{R}^n) \), there exists a homology \( h \in I_{k+1}(\mathbb{R}^n) \) such that

\[
\begin{align*}
\partial h &= \pi_\# \iota P - P; \\
M(h) &\leq a \cdot M(P); \\
spt(h) &\subset N_a(spt P).
\end{align*}
\]

This proposition is essentially [KL18, Proposition 3.7]. The chain map \( \iota \) is constructed skeleton by skeleton, using the condition \((\text{CI}_{n-1})\) and Theorem 3.10. We refer the reader to [KL18] for more details. [KL18, Proposition 3.7] requires \( X \) to be proper, however, the same proof works for Ambrosio-Kirchheim currents in complete metric spaces (images of \( \iota \) having compact supports follows from Theorem 3.10). Assertions (2) and (5) are not in [KL18, Proposition 3.7], however, they are direct consequences of the construction in [KL18] and can be readily justified by induction on dimension.

Lemma 5.8. Let \( n \geq 2 \), and let \( X \) be a complete metric space satisfying condition \((\text{CI}_{n-1})\). Let \( \Phi : \mathbb{R}^n \to X \) be an \((\text{L}, \text{A})\)-quasiflat. Let \( \pi : X \to \mathbb{R}^n \) be as in the beginning of Section 5.2. Let \( \iota \) be as in Proposition 5.7. Then there exists \( a' \) depending only on \( \text{L}, \text{A}, n \) and \( X \) such that for every top-dimensional chain \( W \in \mathcal{P}_n(\mathbb{R}^n) \), we have

\[
\begin{align*}
\bullet &\quad x \in \text{spt}(\pi_\# \circ \iota(W)) \text{ for all } x \in \text{spt}(W) \text{ with } d(x, \text{spt } \partial W) \geq a'; \\
\bullet &\quad \text{spt}(\pi_\# \circ \iota(W)) \subset N_{a'}(\text{spt}(W)).
\end{align*}
\]

The lemma follows by applying Proposition 5.7 (5) to \( P = \partial W \).

Definition 5.9 (Chain projection). Let \( X \) be a complete metric space satisfying condition \((\text{CI}_{n-1})\). Let \( Q \) be an \((\text{L}, \text{A})\)-quasiflat represented by \( \Phi : \mathbb{R}^n \to X \). Let \( \iota \) be as in Proposition 5.7 and let \( \pi : X \to \mathbb{R}^n \) be as in Definition 5.6. We now define a map sending each \( \sigma \in I_n(X) \) to a current supported in a neighborhood of \( Q \). First applying the Federer-Fleming deformation to \( \pi_\# \sigma \) to obtain a cubical chain \( \sigma' \), and then define \( \sigma_Q = \iota(\sigma') \). Note that there exists \( \lambda \) depending only on \( \text{L}, \text{A}, n \) and \( X \) such that \( M(\sigma_Q) \leq \lambda M(\sigma) \) and \( \text{spt}(\sigma_Q) \subset N_\lambda(q \circ \pi(\text{spt}(\sigma))) \).

Remark 5.10. Definition 5.6, Proposition 5.7 and Definition 5.9 also apply to quasidisks. We can take the domain of a quasidisk to be single cube with a suitable cubulation and repeat the previous discussion.

5.3. Homology retract. Using regular cubulations at scale \( R \), one can decompose chains in \( \mathbb{R}^n \) into pieces of size \( R \). More generally, in a metric spaces \( X \) we can decompose according to Lipschitz maps \( \varphi : X \to \mathbb{R}^n \). For integral chains of compact support in \( X \) we are going
to recursively define a weak surrogate of a simplicial decomposition subordinate to (the inverse image of) a given cubulation of $\mathbb{R}^n$.

Let us choose an orientation on $\mathbb{R}^n$. If $C$ is a cubulation of $\mathbb{R}^n$, then each cube $B \in C$ inherits an orientation of $\mathbb{R}^n$. For a cube $B \in C$, we define the face decomposition of $\partial B$ to be $\partial B = \sum_i \epsilon_i B_i$ where each $B_i$ is a codimension 1 face of $B$ and the sign $\epsilon_i = \pm 1$ is determined by our orientation.

**Definition 5.11.** Let $C$ be a regular cubulation of $\mathbb{R}^n$ at scale $R$ and let $\varphi : X \to \mathbb{R}^n$ be a Lipschitz map. For a constant $C > 0$, a (C-controlled) rectifiable stratification (at scale $R$) of an $m$-cycle $S \in Z_{m, c}(X)$ is a collection of currents $(S_B)_{B \in C}$ such that

1. for every $B \in C^{(n-k)}$ holds $S_B \in I_{m-k, c}(X)$ and $\|S_B\|$ is concentrated on $\varphi^{-1}(B^o)$ where $B^o$ denotes the interior of $B$;
2. for every $B \in C^{(n)}$ holds $S_B = S \llcorner \varphi^{-1}(B)$;
3. for $B \in C$, let $\partial B = \sum_i \epsilon_i B_i$ be the face decomposition of $\partial B$, then $\sum_{i=1}^k \epsilon_i S_{B_i}$ is a piece decomposition of $\partial S_B$;
4. $\sum_{B \in C^{(n-k)}} M(S_B) \leq C \cdot \frac{M(S)}{R^n}$.

Note (1) and (2) imply that $S = \sum_{B \in C^{(n)}} S_B$ is a piece decomposition.

If $S \in Z_{m, c}(X)$ has a rectifiable stratification $(S_B)_{B \in C}$, then we call the collection $(S_B)_{B \in C^{(n-k)}}$ the codimension-$k$-skeleton, $S^{(m-k)}$.

Before we turn to the existence of rectifiable stratifications, we provide an auxiliary lemma.

**Lemma 5.12.** Let $\varphi : X \to \mathbb{R}^k$ be a Lipschitz map and let $\sigma \in Z_k(X)$ be a cycle. Denote by $p_i : X \to \mathbb{R}$ be the composition of $\varphi$ with the projection onto the $i$-th coordinate axis. Suppose that

1. $\langle \sigma, p_i, 1 \rangle$ and $\langle \sigma, p_i, -1 \rangle$ exist and are elements in $I_{k-1}(X)$ for all $1 \leq i \leq k$, in particular $\|\sigma\|(p_i^{-1}\{\pm 1\}) = 0$;
2. for any $i \neq j$ hold $\|\langle \sigma, p_i, 1 \rangle + \langle \sigma, p_j, -1 \rangle\|(p_j^{-1}\{\pm 1\}) = 0$.

Let $B = [-1, 1]^k \subset \mathbb{R}^k$ and $B_i^\pm = [-1, 1]^{i-1} \times \{\pm 1\} \times [-1, 1]^{k-i}$ with their interior denoted by $B^o$ and $(B_i^\pm)^o$. Then the face decomposition of $\partial B = \sum_{i=1}^k (B_i^+ - B_i^-)$ induces a piece decomposition

$$\partial(\sigma \llcorner \varphi^{-1}(B^o)) = \sum_{i=1}^k \left( \langle \sigma, p_i, 1 \rangle \llcorner \varphi^{-1}((B_i^+)^o) - \langle \sigma, p_i, -1 \rangle \llcorner \varphi^{-1}((B_i^-)^o) \right).$$
Proof. We will use the abbreviation \( \langle \sigma, p_i, s \rangle \big|_{-1} = \langle \sigma, p_i, 1 \rangle - \langle \sigma, p_i, -1 \rangle \).
We want to prove the following slightly more general statement.

\[
\partial (\sigma \bigcap_{i \in I} p_i^{-1}\{(-1, 1)\}) = \sum_{j \in I} (\langle \sigma, p_j, s \rangle \big|_{-1}) \bigcap_{i \in I \setminus \{j\}} p_i^{-1}\{(-1, 1)\}).
\]

Let us introduce the characteristic functions \( \chi_i := \chi|_{\{1 < p_i < 1\}} \). Then we can write \( \sigma \bigcap_{i \in I} p_i^{-1}\{(-1, 1)\} = \sigma \bigcap_{i \in I} \chi_i \).

We induct on the cardinality of \( I \). The case \(|I| = 1\) is clear.

Suppose \( I = \{i_1, \ldots, i_m\} \). For \( 1 \leq j \leq m \), let \( I_j = I \setminus \{i_j\} \). By induction, \( \sigma \bigcap_{i \in I} \chi_i \) is a normal current.

\[
\partial (\sigma \bigcap_{i \in I} \chi_i) = \langle \sigma \bigcap_{i \in I} \chi_i, p_{i_1}, s \rangle \big|_{-1} + (\partial (\sigma \bigcap_{i \in I} \chi_i)) \bigcap \chi_{i_1}\]

\[
= \langle \sigma \bigcap_{i \in I} \chi_i, p_{i_1}, s \rangle \big|_{-1} + \sum_{j \in I} (\langle \sigma, p_j, s \rangle \big|_{-1} \bigcap_{i \in I \setminus \{j\}} \chi_i) \bigcap \chi_{i_1}\]

\[
= \langle \sigma \bigcap_{i \in I} \chi_i, p_{i_1}, s \rangle \big|_{-1} + \sum_{j \in I} (\langle \sigma, p_j, s \rangle \big|_{-1} \bigcap_{i \in I \setminus \{j\}} \chi_i)
\]

There are \( m \) terms in the above summation, denoted by \( S_1, \ldots, S_m \) from left to right. Note that for \( j > 1 \), \( \|S_j\| \) is concentrated on the set

\[
A_j = (\bigcap_{i \in I_j} p_i^{-1}\{(-1, 1)\}) \cap p_j^{-1}\{\pm 1\}
\]

We claim \( \|S_1\| \) is concentrated on \( A_1 \). Recall \( \|\sigma\|(|\{p_{i_1} = \pm 1\}|) = 0 \), then \( \|\sigma \bigcap_{i \in I} \chi_i\|(|\{p_{i_1} = \pm 1\}|) = 0 \) by \([\text{Lan11}]\) Lemma 4.7. By induction,

\[
\partial (\sigma \bigcap_{i \in I} \chi_i) = \sum_{j \in I} \langle \sigma, p_j, s \rangle \big|_{-1} \bigcap_{i \in I \setminus \{j\}} \chi_i.
\]

Thus by assumption (2), \( \|\partial (\sigma \bigcap_{i \in I} \chi_i)\|(|\{p_{i_1} = \pm 1\}|) = 0 \). Then

\[
\langle \sigma \bigcap_{i \in I} \chi_i, p_{i_1}, 1 \rangle = \langle \sigma \bigcap_{i \in I} \chi_i, p_{i_1}, 1+ \rangle.
\]

As \( \|\sigma \bigcap_{i \in I} \chi_i\| \) is concentrated on \( \bigcap_{i \in I} p_i^{-1}\{(-1, 1)\} \), Lemma 3.4 implies that \( \|\langle \sigma \bigcap_{i \in I} \chi_i, p_{i_1}, 1 \rangle\| \) is concentrated on \( \bigcap_{i \in I} p_i^{-1}\{(-1, 1)\} \cap \{p_{i_1} = 1\} \). Thus \( \|S_1\| \) is concentrated on \( A_1 \) as claimed.

Note that \( A_{j_1} \cap A_{j_2} = \emptyset \) when \( j_1 \neq j_2 \), so the above sum is a piece decomposition.
Now we repeat the discussion with $I_1$ replaced by $I_2$ to obtain another piece decomposition. By comparing these, we conclude $\langle \sigma \prod_{i \in I_1} \chi_i, p_{i_1}, s \rangle |_{-1}^{1} = \langle \sigma, p_{i_1}, 1 \rangle |_{-1}^{1} \prod_{i \in I \setminus \{i_1\}} \chi_i$, which finishes the induction. 

**Proposition 5.13.** Let $\varphi : X \to \mathbb{R}^n$ be an $L$-Lipschitz map. Then there exists $C = C(L, n) > 0$ such that the following holds. For any $S \in \mathbb{Z}_m(X)$ and $R > 0$ there exists a regular cubulation $\mathcal{C}$ of $\mathbb{R}^n$ at scale $R$ such that $S$ has a $C$-controlled rectifiable stratification subordinate to $\mathcal{C}$.

**Proof.** The action of $R : \mathbb{Z}^n$ on $\mathbb{R}^n$ induces a $1$-Lipschitz covering map $h : \mathbb{R}^n \to T$ where $T$ is a product of $n$-circles of length $R$. For a subset $I = \{i_1, i_2, \ldots, i_k\} \subset \{1, 2, \ldots, n\}$, let $\mathbb{R}^I \subset \mathbb{R}^n$ and $T^I \subset T$ be the associated subspaces. Denote by $p_I : X \to \mathbb{R}^I$ the composition of $\varphi$ with the projection $\mathbb{R}^n \to \mathbb{R}^I$. Let $\pi_I = h \circ p_I$.

First we claim there exist a point $(s_1, s_2, \ldots, s_n) \in T$ and a constant $C = C(L, n)$ such that for any collection of mutually disjoint subsets $I_1, I_2, \ldots, I_k$ of $\{1, \ldots, n\}$, we have

(i) $\sigma := \langle \ldots \langle \langle S, \pi_{i_1}, s_{i_1} \rangle, \pi_{i_2}, s_{i_2} \rangle \ldots, \pi_{i_k}, s_{i_k} \rangle \in \mathbf{I}_{n-|I|}(X)$;

(ii) permuting the order of $I_1, \ldots, I_k$ in the definition of $\sigma$ will result in the same current;

(iii) $\sigma = \langle S, \pi_I, s_I \rangle$ where $I = \bigcup_{i=1}^k I_i$;

(iv) $\mathbf{M}(\sigma) \leq C \cdot \frac{\mathbf{M}(S)}{R^{|I|}}$.

Here $|I|$ denotes the cardinality of $I$. To see the claim, note that by repeatedly applying [Lan11, Theorem 6.5] and Fubini, we can find a full measure subset $A_0 \subset T$ such that (i), (ii) and (iii) hold. To arrange (iv), by the coarea formula, for each subset $I$ of $\{1, \ldots, n\}$, we can find a subset $A_I \subset A_0$ such that $\mathcal{L}^n(A_I) \geq (1 - \epsilon)\mathcal{L}^n(A_0)$ and $\mathbf{M}(\sigma) \leq C \cdot \frac{\mathbf{M}(S)}{R^{|I|}}$ whenever $(s_1, \ldots, s_n) \subset A_I$. By choosing $\epsilon$ sufficiently small (depending on $n$), the intersection of all $A_I$ with $I$ ranging over subsets of $\{1, \ldots, n\}$ is non-empty and the claim follows.

Let $\mathcal{C}$ be the cubulation (of scale $R$) with vertex set $h^{-1}((s_1, \ldots, s_n))$. Let $B \in \mathcal{C}^{(n-k)}$ be cube and let $I := I_B$ be the smallest subset of $\{1, \ldots, n\}$ such that $h(B) \subset T^I$, in particular $|I| = n - k$. We define $S_B = \langle S, \pi_I, s_I \rangle h^{-1}(B)$ and $S_{B^0} = \langle S, \pi_I, s_I \rangle h^{-1}(B^0)$. We claim that the collection $(S_B)_{B \in \mathcal{C}}$ satisfies Definition 5.11.

Definition 5.11(2) holds by definition and (4) follows from item (iv). We claim $S_{B} = S_{B^0}$. If $B$ is top-dimensional, then this claim follows from $\|S\|_{\pi_I^{-1}(s_i)} = 0$ for any $i \in \{1, \ldots, n\}$. As we slice at regular point when defining iterated slices, a similar argument implies the lower
JINGYIN HUANG, BRUCE KLEINER, AND STEPHAN STADLER

42 dimensional case of the claim, which implies the second half of Definition 5.11 (1). Now Definition 5.11 (3) follows from Lemma 5.12 and properties (i)-(iii). Note that (iv) implies $S_B$ has finite mass for each $B$, hence $\partial S_B$ has finite mass by (3). The first half of Definition 5.11 (1) follows from (3), (i), (ii) and (iii).

□

Proposition 5.14. There exists a constant $C = C(L, A, n, m, c)$ such that the following holds. Suppose that $X$ is a complete metric space satisfying condition $(\text{SCI}_m)$ with constant $c$ and let $\Phi : \mathbb{R}^n \to X$ be an $(L, A)$-quasiflat with image $Q$. Let $C_{R_0}$ be a regular cubulation of $\mathbb{R}^n$ at scale $R_0 = 2LA$. Denote by $\phi : X \to \mathbb{R}^n$ a quasi-retraction induced by $q$ and $C_{R_0}$ as in Definition 5.6. Let $i : P_*(\mathbb{R}^n) \to I_{*,c}(X)$ be a chain map induced by $q$ and $C_{R_0}$ as in Proposition 5.7. Then for all $R \geq R_0$ the following holds true.

If $S \in \mathbb{Z}_m(X)$ is a cycle with $\text{spt } S \subset N_R(Q)$, then there exists a cubical cycle $P \in P_m(\mathbb{R}^n)$, $\partial P = 0$, and homologies $H \in I_{m+1}(X)$ and $h \in I_{m+1}(\mathbb{R}^n)$ with the following properties.

(1) $\partial H = S - \iota P$ and $\partial h = \phi \# S - P$;
(2) $\text{spt } H \subset N_{CR}(\text{spt } S)$ and $\text{spt } h \subset N_C(\phi(\text{spt } S))$;
(3) $M(H) \leq CR \cdot M(S)$ and $M(h) \leq C \cdot M(S)$;
(4) $M(P) \leq C \cdot M(S)$.

Proof. We are going to use the Federer Fleming deformation, henceforth referred to as (FFD), with respect to the cubulation $C_{R_0}$. By Proposition 5.13, there exists a second regular cubulation $C$ of $\mathbb{R}^n$ at the larger scale $R$ such that $S$ has a rectifiable stratification $(S_B)_{B \in C}$ subordinate to $C$.

We will write $a \lesssim b$ if $a \leq C'b$ for a constant $C'$ depending only on $L, A, m, n$ and $c$.

We only prove the case $m \leq n$ and the $m > n$ is similar.

First we claim that there exists a constant $C_0 = C_0(L, A, n)$ and a family of currents $(P_B)_{B \in C}$ such that

(a) for every $B \in C^{(n-k)}$ holds $P_B \in I_{m-k,c}(\mathbb{R}^n)$ and $\text{spt}(P_B) \subset N_{C_0}(\phi(\text{spt } S_B))$;
(b) each $P_B$ is a cubical chain with respect to $C_{R_0}$;
(c) for $B \in C$, let $\partial B = \sum_{i=1} \epsilon_i B_i$ be the face decomposition of $\partial B$, then $\partial P_B = \sum_{i=1} \epsilon_i P_{B_i}$;
(d) $\sum_{B \in C^{(n-k)}} M(P_B) \leq C_0 \cdot \frac{M(S)}{R}$;
(e) $\text{Fill}(\phi \# S - \sum_{B \in C^{(n-k)}} P_B) \leq C_0 \cdot M(S)$.
We define \((P_B)_{B \in C}\) inductively as follows. Let \(\tau_B = \varphi \# S_B\). Take \(S_B\) with \(B \in C^{(n-m)}\). Then \(S_B\) is a 0-dimensional cycle. Applying \(\text{FFD}\) to \(\tau_B\) with respect to the cubulation \(C_{R_0}\), we obtain a cubical chain \(P_B\) and a homotopy \(h_B\) with \(\partial h_B = P_B - \tau_B\). Note that \(M(P_B) \lesssim M(S_B)\) and \(M(h_B) \lesssim R_0 M(S_B)\). Clearly \(\sum_{B \in C^{(n-m)}} M(P_B) \lesssim \frac{M(S)}{R^m}\) and \(\sum_{B \in C^{(n-m)}} M(h_B) \lesssim R_0 \cdot \frac{M(S)}{R^m} \lesssim \frac{M(S)}{R^{m-t}}\).

Suppose \(P_B\) and \(h_B\) with \(B \in C^{(n-m+k-1)}\) are already defined such that conditions \((a)-(d)\) hold, \(\partial h_B = P_B - \tau_B\)

\[
\sum_{B \in C^{(n-m+k-1)}} M(h_B) \lesssim \frac{M(S)}{R^{m-k}}.
\]

Take \(B\) with \(B \in C^{n-m+k}\). Let \(\partial B = \sum_i \epsilon_i B_i\) be the face decomposition. Define \(P'_B = \tau_B + \sum_i \epsilon_i h_{B_i}\). Then \(\partial P'_B = \sum_i \epsilon_i P_{B_i}\). Applying \(\text{FFD}\) to \(P'_B\) with respect to \(C_{R_0}\), we obtain a cubical chain \(P'_B\) and a homotopy \(h'_B\) with \(\partial h'_B = P'_B - \tau_B\) (note that \(P'_B\) is fixed when applying the radial push-out procedure of \(\text{FFD}\) to \(P'_B\)). Then

\[
\sum_{B \in C^{(n-m+k)}} M(P'_B) \lesssim \sum_{B \in C^{(n-m+k)}} M(\tau_B) + \sum_{B \in C^{(n-m+k-1)}} M(h_B) \lesssim \frac{M(S)}{R^{m-k}}
\]

and

\[
\sum_{B \in C^{(n-m+k)}} M(h_B) \lesssim R_0 \sum_{B \in C^{(n-m+k)}} M(P'_B) \lesssim \frac{M(S)}{R^{m-k}} (\text{as } R \geq R_0)
\]

and

\[
\sum_{B \in C^{(n-m+k)}} M(P_B) \lesssim \sum_{B \in C^{(n-m+k)}} M(P'_B) \lesssim \frac{M(S)}{R^{m-k}}
\]

Then the claim follows.

For each \(P_B\), define \(T_B = \iota(P_B)\) where \(\iota\) is defined with respect to \(C_{R_0}\) (see Proposition 5.7). Define \(P = \sum_{B \in C^{(n)}} P_B\) and \(T = \sum_{B \in C^{(n)}} T_B = \iota(P)\). Now we construct \(H \in I_{m+1}(X)\) such that \(\partial H = S - T\).

We start with \(B\) with \(B \in C^{(n-m)}\). By the estimates in Definition 5.6, \(S_B - T_B\) is a cycle of diameter \(\lesssim R\). We apply the strong cone inequality to obtain \(H_B\) such that \(\text{diam}(\text{spt } H_B) \lesssim R\) and \(M(H_B) \lesssim R(M(S_B) + M(T_B))\). Thus \(\sum_{B \in C^{(n-m)}} H_B \lesssim R \cdot \frac{M(S)}{R^m} \lesssim \frac{M(S)}{R^{m-t}}\).

Suppose \(H_B\) with \(B \in C^{(n-m+k-1)}\) are already defined such that

1. \(\text{diam}(\text{spt } H_B) \lesssim R\);
2. \(\sum_{B \in C^{(n-m+k-1)}} H_B \lesssim \frac{M(S)}{R^{m-k}}\);
Corollary 5.15. Let $\partial H_B = S_B - T_B - \sum_i \epsilon_i H_{B,i}$ where $\partial B = \sum_i B_i$ is the face decomposition.

Take $B$ with $B \in C^{(n-m+k)}$. Consider $\sigma_B = S_B - T_B - \sum_i \epsilon_i H_{B,i}$. Let $\partial B_i = \sum_{ij} \epsilon_{ij} B_{ij}$ be the face decomposition of $B_i$. Then $\partial \sigma = \sum_i \epsilon_i S_{B_i} - \sum_i \epsilon_i T_{B_i} - \sum_i \epsilon_i (S_{B_i} - T_{B_i} - \sum_{ij} \epsilon_{ij} B_{ij}) = 0$ by the sign convention. Applying the strong cone inequality to $\sigma_B$ we obtain a filling $H_B$. As $\text{diam}(H_{B,i}) \lesssim R$ for each $i$, we have $\text{diam}(\sigma_B) \lesssim R$, hence $\text{diam}(H_B) \lesssim R$. Moreover, $M(H_B) \lesssim R \cdot (M(S_B) + M(T_B) + \sum_i M(H_{B,i}))$. Thus by Proposition 5.13 and the previous claim,

$$
\sum_{B \in C^{(n-m+k)}} H_B \lesssim R \cdot \sum_{B \in C^{(n-m+k)}} \sigma_B \lesssim R \cdot \sum_{B \in C^{(n-m+k)}} (M(S_B) + M(T_B)) + R \cdot \sum_{B \in C^{(n-m+k-1)}} M(H_B) \lesssim \frac{M(S)}{R^{m-k-1}}.
$$

All items of the conclusions of the proposition now follow. \hfill \Box

**Corollary 5.15.** Let $X$ be a complete metric space satisfying (SCI$_n$) with constant $c$ and let $\Phi : \mathbb{R}^n \to X$ be an $(L,A)$-quasiflat with image $Q$. Then there exists $C = C(L,A,c,n)$ such that the following holds. If $S \in Z_n(X)$ is a cycle, with $\text{spt} S \subset N_R(Q)$ for some $R \geq 2LA$, then $\text{Fill}(S) \lesssim CR \cdot M(S)$.

**Proof.** Proposition 5.14 provides a controlled homology $H$ between $S$ and $\iota P$ for some cubical cycle $P$ in $\mathbb{R}^n$. However, $P$ is top-dimensional and therefore trivial. It follows that $H$ is a filling of $S$ as required. \hfill \Box

If we make the stronger assumption that $A = 0$, namely that $Q$ is a bilipschitz flat, then instead of using the chain map $\iota$ we can use an actual Lipschitz retraction $\pi : X \to Q$ and push the cycle $S$ to a cycle $S' = \pi#S$. In this case Proposition 5.14 simplifies to

**Lemma 5.16.** There exists a constant $C = C(L,n,m,c)$ such that the following holds. Suppose that $X$ is a metric space satisfying condition (SCI$_m$) constant $c$ and let $Q \subset X$ be an $n$-dimensional $L$-bilipschitz flat. Then for all $R \geq 0$ the following holds true. If $S \in Z_m(X)$ is a cycle with $\text{spt} S \subset N_R(Q)$, then there exists a cycle $S' \in Z_m(Q)$ and a homology $H \in I_{m+1}(X)$ such that

1. $\partial H = S - S'$;
2. $\text{spt} H \subset N_{CR}(\text{spt} S)$;
3. $M(H) \leq CR \cdot M(S)$;
4. $M(S') \leq C \cdot M(S)$.
Definition 5.17. Let $Q : \mathbb{R}^n \to X$ be an $n$-dimensional $(L, A)$-quasiflat and let $\iota$ be the chain mapping in Proposition 5.7. For $a > 0$, $Q$ is an $a$-homology retract if for any $S \in \mathbb{Z}_{n-1}(X)$ such that $\text{spt}(S) \subset N_R(Q)$ with $R > a$, there exist a cubical chain $P \in I_{n-1}(\mathbb{R}^n)$ and $H \in I_n(X)$ such that

1. $\partial H = S - \iota(P)$ and $\text{spt}(\iota(P)) \subset N_a(Q)$;
2. $\text{spt}(H) \subset N_{aR}(\text{spt}(S))$ and $\text{spt}(P) \subset N_a(\pi(\text{spt}(S)))$;
3. $M(H) \leq aRM(S)$ and $\text{Fill}(P - \pi_*(S)) \leq aM(S)$;
4. $M(P) \leq aM(S)$ and $M(\iota(P)) \leq aM(S)$.

It follows from Proposition 5.14 and Proposition 5.7 that an $n$-dimensional $(L, A)$-quasiflat in a metric space $X$ with condition (SCI$_{n-1}$) is an $a$-homology retract for $a = a(L, A, n, X)$.

6. Morse quasiflats

In this section we introduce several possible definitions which characterize the “Morseness” of a quasiflat, some using asymptotic cones and some are asymptotic conditions on the space. And we show quasi-isometric invariance of Morse quasiflats.


Let $X$ be a complete metric space and let $Q \subset X$ be an $n$-dimensional $L$-bilipschitz flat.

Definition 6.1 (Piece property). We say $Q$ has the piece property, if the following holds. Let $\sigma \in \mathbb{Z}_{n-1}(Q)$ be a cycle with canonical filling $\nu \in I_n(Q)$. Then any alternate filling $\tau \in I_n(X)$ of $\sigma$ contains $\nu$ as a piece, i.e. $\|\tau\| = \|\tau - \nu\| + \|\nu\|$ with $\|\tau - \nu\|$ concentrated on $X \setminus Q$.

Remark 6.2. The piece property implies that $Q$ is mass minimizing in the following sense. If $T$ is a piece of $Q$, then $T$ is a minimal filling of $\partial T$.

Lemma 6.3. Suppose that $Q$ has the piece property and $\tau \in I_n(X)$ is a filling of a cycle $\sigma \in \mathbb{Z}_{n-1}(Q)$. Then the filling area of slices $\langle \tau, d_Q, t \rangle$ close to $Q$ becomes small, $\lim_{t \to 0} \text{Fill}(\langle \tau, d_Q, t \rangle) = 0$.

Proof. Denote by $\nu \in I_n(Q)$ the canonical filling of $\sigma$. Consider the filling $\langle \tau - \nu \rangle \subset \{d_Q \leq t\}$ of the slice $\langle \tau, d_Q, t \rangle$. Note that $\tau - \nu$ is a cycle. Since $Q$ has the piece property, $\|\tau - \nu\|$ is concentrated on $X \setminus Q$ and the claim follows. □
Definition 6.4 (Neck property). We say $Q$ has the \textit{neck property}, if there exists a constant $C$ such that the following holds for all $\rho > 0$. Let $\sigma \in \mathbb{Z}_{n-1}(X)$ be a cycle with $\text{spt}(\sigma) \subseteq N_\rho(Q)$ and let $\tau \in I_n(X)$ be a filling, $\partial \tau = \sigma$, with $\text{spt}(\tau) \subset X \setminus Q$. Then

$$\text{Fill}(\langle \tau, d_Q, \rho_0 \rangle) \leq C \cdot \rho \cdot M(\sigma).$$

The name derives from the fact that any chain filling a cycle in $Q$ has to have small necks near $Q$ in the following sense.

Definition 6.5 (Weak neck property). We say $Q$ has the \textit{weak neck property}, if the following holds. Let $\sigma \in \mathbb{Z}_{n-1}(Q)$ be a cycle with a filling $\tau \in I_n(X)$. Then for every $\epsilon > 0$ there exists $\rho_0 \in (0, \epsilon)$ such that the slice $\langle \tau, d_Q, \rho_0 \rangle$ has

$$\text{Fill}(\langle \tau, d_Q, \rho_0 \rangle) < \epsilon.$$

Lemma 6.6. The neck property implies the weak neck property.

Proof. If $\tau$ is supported in $Q$, then there is nothing to show. Otherwise, we set $\tau' = \tau \cap (X \setminus Q)$. For every $\delta > 0$ we choose $\rho > 0$ such that $M(\tau' \cup \{d_Q \leq \rho\}) \leq \delta$. By the pigeonhole principle there exists $k \in \mathbb{N}$ such that $M(\tau_k) \leq \frac{\delta}{2^{k+1}}$ where $\tau_k = \tau' \cup \{\frac{\rho}{2^k} \leq d_Q < \frac{\rho}{2^{k+1}}\}$. The coarea inequality implies $M(\langle \tau_k, d_Q, \rho_0 \rangle) < \frac{\delta}{\rho}$ for some $\rho_0 \in (\frac{\rho}{2^k}, \frac{\rho}{2^{k+1}})$. Since $\langle \tau_k, d_Q, \rho_0 \rangle = \langle \tau, d_Q, \rho_0 \rangle$ we conclude from the neck property $\text{Fill}(\langle \tau, d_Q, \rho_0 \rangle) \leq C \cdot \rho_0 \cdot \frac{\delta}{\rho} \leq C \cdot \delta$ where $C$ is independent of $\delta$. \hfill \Box

Lemma 6.7. Suppose that $X$ satisfies condition $(\text{SCI}_{n-1})$ where $n = \dim Q$. Then the piece property implies the neck property with constant $C$ in Definition 6.4 depending only on $n$, the constants of condition $(\text{SCI}_{n-1})$ and the Lipschitz constant of $Q$.

Proof. Suppose that $Q$ has the piece property. Let $\sigma \in \mathbb{Z}_{n-1}(X)$ be a cycle with $\text{spt}(\sigma) \subset N_\rho(Q)$ and let $\tau \in I_n(X)$ be a filling with $\text{spt}(\tau) \subset X \setminus Q$. By Lemma 5.1(1) there exists a cycle $\sigma' \in \mathbb{Z}_{n-1}(Q)$ and a homology $H \in I_n(X)$ with $\partial H = \sigma - \sigma'$ and $M(H) \leq C \cdot \rho \cdot M(\sigma)$. Denote by $\nu$ the canonical filling of $\sigma'$ inside $Q$. Then $H - \nu$ is a filling of $\sigma$. The piece property implies that $\tau + H = (\tau + H - \nu) + \nu$ is a piece decomposition. Since the support of $\tau$ is disjoint from $Q$, we see that $H = (H - \nu) + \nu$ is a piece decomposition as well. In particular, $M(H - \nu) \leq M(H)$ and the claim follows. \hfill \Box

Definition 6.8 (Full support). Let $n = \dim Q$. We say $Q$ has \textit{full support with respect to a homology theory $h_\ast$}, if the map

$$h_n(Q, Q \setminus \{q\}, \mathbb{Z}) \rightarrow h_n(X, X \setminus \{q\}, \mathbb{Z})$$
is injective for each $q \in Q$.

**Remark 6.9.** If $X$ has a Lipschitz combing, then full support of $Q$ implies the geodesic extension property.

**Lemma 6.10.** Let $n = \dim Q$. Suppose that $X$ satisfies condition $(SC_{n})$. If $Q$ has the weak neck property, then $Q$ has full support with respect to $\tilde{H}^{AK}$.

**Proof.** If $Q$ does not have full support, then we find an embedded top-dimensional ball $B \subset Q$ around a point $x \in Q$ such that a nontrivial multiple of $\partial B$ can be filled by $\tau \in I_{n}(X)$ with $x \not\in \text{spt} \tau$. We may even assume that it avoids the ball $B_{x}(1)$. Let $\nu$ be the canonical filling of $\partial \tau$ in $Q$. Suppose that the weak neck property holds. Then there exists a sequence $\rho_{k} \to 0$ such that the slices $\langle \tau, d_{Q}, \rho_{k} \rangle$ fulfill $\text{Fill}(\langle \tau, d_{Q}, \rho_{k} \rangle) \to 0$. Choose fillings $\mu_{k}$ of $\langle \tau, d_{Q}, \rho_{k} \rangle$ as provided by Theorem 3.10. Set $h_{k} = \tau \downarrow \{d_{Q} \leq \rho_{k}\}$. From Corollary 5.15 and Theorem 3.10 (2), we see that $\text{Fill}(\mu_{k} - h_{k} + \nu) \to 0$ and therefore $h_{k}$ converges to $\nu$ weakly [Wen07, Theorem 1.4]. However, this is impossible since the support of the $h_{k}$ is disjoint from $B_{x}(1)$. \[\Box\]

At last we will close the cycle.

**Lemma 6.11.** Let $n = \dim Q$. Suppose that $X$ satisfies condition $(\text{Cl}_{n-1})$. Suppose that each element in $I_{n+1}(X)$ can be filled by an element in $I_{n+1}(X)$. If $Q$ has full support with respect to $\tilde{H}^{AK}$, then $Q$ has the piece property.

**Proof.** Let $\sigma \in I_{n-1}(Q)$ be a nontrivial cycle with the additional property that $\mathcal{H}^{n}(\text{spt}(\sigma)) = 0$. Denote by $\nu \in I_{n}(Q)$ the canonical filling of $\sigma$ in $Q$ and let $\tau \in I_{n}(X)$ be an arbitrary filling. Consider the piece decomposition $\tau = \tau \downarrow Q + \tau \downarrow (X \setminus Q)$. Let $\tau_{Q} := \tau \downarrow Q$ and $\tau_{Q^{c}} := \tau \downarrow (X \setminus Q)$. Then $\tau_{Q}$ and $\tau_{Q^{c}}$ are integer rectifiable currents ([AK+00, Definition 4.2]). Moreover, $\|\tau_{Q^{c}}\|$ is concentrated on $X \setminus Q$.

We claim that $\tau_{Q} = \nu$. By the proof of [AK+00, Theorem 4.6], $\Theta_{n}(\|\tau_{Q^{c}}\|, x) := \lim_{r \to 0} \frac{\|\tau_{Q^{c}}\|(B(x, r))}{\omega_{n-1} r^{n-1}} = 0$ for $\mathcal{H}^{n}$-a.e. $x \in Q$. We write $\|\tau_{Q} - \nu\| = f \cdot \lceil Q \rceil$ with $f \in L^{1}(Q, \mathbb{Z})$. If our claim fails, then we find a Lebesgue point $p \in Q \setminus \text{spt}(\sigma)$ of $f$ with

$$
\Theta_{n}(\|\tau - \nu\|, p) = \Theta_{n}(\|\tau_{Q} - \nu\|, p) + \Theta_{n}(\|\tau_{Q^{c}}\|, p)
= \Theta_{n}(\|\tau_{Q} - \nu\|, p) = f(p) = k \neq 0.
$$

Then $T = k \cdot \lceil Q \rceil - \tau + \nu$ has density zero at $p$. Hence, for every $\epsilon > 0$ we can find a slice $\langle T, d_{p}, r \rangle$ with $M(\langle T, d_{p}, r \rangle) \leq \epsilon \cdot r^{n-1}$. Let $W_{r}$ be an
almost minimal filling of $\langle T, d_p, r \rangle$ as provided by Theorem 3.10. Then the isoperimetric inequality implies $M(W_r) \lesssim \epsilon \cdot r^n$. By Theorem 3.10 (2), $p$ cannot lie in the support of $W_r$ for $\epsilon$ small enough. Since $W_r$ is homologous to $T \{ d_p \leq r \}$ we see that $[T] = 0$ in $\tilde{H}_n^{AK}(X, X \setminus \{p\})$.

On the other hand, we have $[T] = k \cdot [Q] \in \tilde{H}_n^{AK}(X, X \setminus \{p\})$ since $\tau - \nu$ can be filled by an element in $I_{n+1}(X)$. In particular, $[T] \neq 0 \in \tilde{H}_n^{AK}(X, X \setminus \{p\})$ since $Q$ has full support. Contradiction. Thus the claim follows and the piece property follows from the claim.

In the general case we choose a homology $\alpha \in I_n(Q)$ such that $\sigma' = \partial \alpha + \sigma$ is nontrivial and fulfills $H_n(\text{spt}(\sigma')) = 0$. Then $\tau' = \tau + \alpha$ is a filling of $\sigma'$. Note that $\nu' = \nu + \alpha$ is the canonical filling $\sigma'$ in $Q$. By the case above, we know that $\tau' = (\tau' - \nu') + \nu'$ is a piece decomposition. Since the support of $\alpha$ is contained in $Q$ we obtain the required piece decomposition for $\tau$. □

Let us summarize the results of this subsection.

**Proposition 6.12.** Let $X$ be a metric space and let $Q$ be an $n$-dimensional bilipschitz flat in $X$. Suppose $X$ satisfies condition (SCI$_n$). Then the following conditions on $Q$ are all equivalent:

1. Piece property (cf. Definition 6.1);
2. Neck property (cf. Definition 6.4);
3. Weak neck property (cf. Definition 6.5);
4. Full support with respect to $\tilde{H}_n^{AK}$ (cf. Definition 6.8).

If in addition $X$ has a Lipschitz bicombing, then each of the above condition is equivalent to $Q$ having full support with respect to reduced singular homology.

The last statement follows from Proposition 4.8.

6.2. Asymptotic condition for Morse quasiflats.

**Definition 6.13 (Rigid quasiflat).** Suppose $X$ is a metric space satisfying condition (CI$_{n-1}$). Let $\Phi : \mathbb{R}^n \to X$ be an $(L, A)$-quasiflat. Let $\mathcal{C}_R$ and $\iota$ be as in Proposition 5.7 and let $b > 0$.

We define $\Phi$ to be $(\mu, b)$-rigid, if for every constant $M > 0$ there exists a sublinear function $\mu = \mu_M : [0, \infty) \to [0, \infty)$ with the following property. Let $x \in \mathbb{R}^n$ and $\Phi(x) = p$. Suppose that $\varphi \in \mathbb{Z}_{n-1,c}(B_x(r))$ is a cubical cycle (with respect to $\mathcal{C}_R$) with $M(\varphi) \leq M \cdot r^{n-1}$. Suppose $\tau \in I_{n,c}(B_p(Mr))$ satisfies $\partial \tau = \iota(\varphi)$ and $M(\tau) \leq M \cdot r^n$. Let $\nu \in I_{n,c}(\mathbb{R}^n)$ be the canonical filling of $\tau$ with $\text{spt} \nu = W$. Let $W_b = \{ y \in \mathbb{R}^n : d_y)!^b < r \}$.
W \mid d(y, \partial W) > b$. Then $\Phi(W_b) \subset N_{\mu(r)}(\text{spt}(\tau))$. A quasiflat is rigid if it is $(\mu, b)$-rigid for some choice of $\mu$ and $b$.

We define $\Phi$ is pointed $(\mu, b)$-rigid, if the previous paragraph holds only for a particular base point $x$.

**Remark 6.14.** In the definition of $(\mu, b)$-rigid, the parameters $\mu$ and $b$ are independent of $x \in \mathbb{R}^n$. However, in the pointed version, different $x \in \mathbb{R}^n$ give rise to different $\mu$ and $b$.

**Remark 6.15.** One can set up the above definition slightly differently by using three constants $M_1, M_2, M_3$ and requiring $M(\varphi) \leq M_1 \cdot r^{n-1}$, $\tau \in I_{n,c}(B_p(M_2))$ and $M(\tau) \leq M_3 \cdot r^n$. However, one readily sees that this leads to an equivalent definition.

**Remark 6.16.** Note that the definition of rigid quasiflat depends on the choice of the chain map $\iota$ and its underlying cubulation and therefore also on $X$. However, for a different choice of $\iota$, $\Phi$ will be $(\mu', b')$-rigid for a different choice of $\mu'$ and $b'$.

**Remark 6.17.** Suppose in addition that $X$ has a convex geodesic bicombing. Then we can approximate $\Phi$ such that it is $L'$-Lipschitz and an $(L', A')$-quasi-isometric embedding, with $L', A'$ depending only on $L, A$ and $n$. In this case, we can define $\mu$-rigid by taking $\phi$ to be any element in $Z_{n-1,c}(B_x(r))$ (not necessarily cubical), and using $\Phi_\#$ instead of $\iota$. This gives an equivalent definition, with possibly different $\mu$.

**Remark 6.18.** We can define $(\mu, b)$-rigid for quasidisks in the same way (see Remark 5.10). Of course, every quasidisk is $(\mu, b)$-rigid for some choice of $\mu$ and $b$, however, if one fixes $\mu$ and $b$, then it places a non-trivial geometric condition on large quasidisks.

Now we introduce a related property of quasiflats called *super-Euclidean divergence*. We will supply two equivalent versions of super-Euclidean divergence, one is technically more convenient (Definition 6.19), and one is closer to the definition of super-Euclidean divergence for quasi-geodesics in the literature (Definition 6.21).

**Definition 6.19 (Super-Euclidean divergence I).** Let $X, \Phi$ and $\iota$ be as in Definition 6.13. $\Phi$ has $(\delta)$-super-Euclidean divergence, if for any $D > 1$, there exists a function $\delta = \delta_D : [0, \infty) \to [0, \infty)$ with $\lim_{r \to \infty} \delta(r) = +\infty$ such that the following property holds. Let $r > 0$. Let $x \in \mathbb{R}^n$ be arbitrary and let $\Phi(x) = p$. Suppose that $\varphi \in Z_{n-1,c}(A_x(x, D))$ is a cubical chain representing a nontrivial homology class in $\tilde{H}_{n-1}(\mathbb{R}^n \setminus B_x(D))$. Suppose $M(\varphi) \leq D \cdot r^{n-1}$. Then for any $\tau \in I_{n,c}(A_p(x, D))$
such that \( \partial \tau = \iota(\varphi) \), we have \( \mathbf{M}(\tau) \geq \delta(r) \cdot r^n \) (if no such \( \varphi \) or \( \tau \) exists, then it is understood that this inequality automatically holds).

**Remark 6.20.** Similar to Remark 6.15 one can set up the above definition using several different constants \( D_1, D_2, \ldots \) at different places instead of \( D \), which will lead to an equivalent definition. Moreover, we can repeat Remark 6.16, Remark 6.17 and Remark 6.18 in the context of super-Euclidean divergence.

**Definition 6.21** (Super-Euclidean divergence II). Let \( X, \Phi \) and \( \iota \) be as in Definition 6.13. \( \Phi \) has \(( \delta )\)-super-Euclidean divergence, if for any \( D \geq 1 \), there exists a function \( \delta = \delta_D : [0, \infty) \to [0, \infty) \) with \( \lim_{r \to \infty} \delta(r) = +\infty \) such that the following property holds. Let \( \Phi(0) \) be arbitrary and let \( \Phi(x) = p \). Suppose that \( \varphi \in Z_{n-1,c}(\mathbb{R}^n \setminus B_x(r)) \) be a cubical chain representing a nontrivial homology class in \( \tilde{H}_{n-1}(\mathbb{R}^n \setminus B_x(r)) \). Suppose \( \mathbf{M}(\varphi) \leq D \cdot r^{n-1} \). Then for any \( \tau \in I_{n,c}(X \setminus B_p(\frac{r}{2})) \) such that \( \partial \tau = \iota(\varphi) \), we have \( \mathbf{M}(\tau) \geq \delta(r) \cdot r^n \) (if no such \( \varphi \) or \( \tau \) exists, then it is understood that this inequality automatically holds).

**Lemma 6.22.** Suppose \( X \) satisfies condition \(( CI_{n-1} )\). Then an \( ( L, A )\)-quasiflat \( \Phi : \mathbb{R}^n \to X \) satisfies Definition 6.19 if and only if it satisfies Definition 6.21 (for a possibly different \( \delta \)).

**Proof.** It is clear that Definition 6.21 implies Definition 6.19. Now suppose Definition 6.21 fails for \( \Phi \). Then we find a constant \( D_0 > 1 \), a sequence \( (x_k) \) in \( \mathbb{R}^n \) and a sequence \( r_k \to \infty \) such that the following holds. There exist cubical cycles \( \varphi_k \in Z_{n-1,c}(\mathbb{R}^n \setminus B_{x_k}(r_k)) \) with \( [\varphi_k] \neq 0 \in \tilde{H}_{n-1}(\mathbb{R}^n \setminus B_{x_k}(r_k)) \), \( \mathbf{M}(\varphi_k) \leq D_0 \cdot r_k^{n-1} \) such that \( \iota(\varphi_k) \) can be filled by \( \tau_k \in I_{n,c}(X \setminus B_{p_k}(\frac{r_k}{2})) \) with \( \mathbf{M}(\tau_k) \leq D_0 \cdot r_k^n \) (define \( p_k = \Phi(x_k) \)).

We claim we can assume in addition that \( \text{spt}(\varphi_k) \subset A_{x_k}(r_k, 2r_k) \). Otherwise for each \( \varphi_k \), we find a cubical chain \( \varphi'_k \in Z_{n-1,c}(A_{x_k}(r_k, 2r_k)) \) and \( H_k \in I_{n,c}(\mathbb{R}^n \setminus B_{x_k}(r_k)) \) such that \( \partial H_k = \varphi_k - \varphi'_k \), \( \mathbf{M}(\varphi'_k) \leq D_0 \cdot r_k^{n-1} \) and \( \mathbf{M}(H_k) \leq D_1 \cdot r_k^{n-1} \) (here \( D_1 \) is a constant independent of \( k \)), and then we replace \( \varphi_k \) by \( \varphi'_k \) and \( \tau_k \) by \( \tau_k + \iota(H_k) \).

By the previous claim, we can assume that \( \text{spt}(\partial \tau_k) = \text{spt}(\iota(\varphi_k)) \subset A_{p_k}(\frac{r_k}{2D_0}, r_k D_0) \) up to choosing a larger \( D_0 \). Take \( b > 1 \) whose value will be determined later. We use coarea inequality to find a slice \( \sigma_k = \langle \tau_k, d_{p_k}, r_k' \rangle \) with \( 10b r k D_0 < r_k' < 20b r k D_0 \) such that \( \mathbf{M}(\sigma_k) \leq \frac{1}{10b} \cdot r_k^{n-1} \). Let \( \tau_k' \) be a minimal filling of \( \sigma_k \). It follows from Theorem 3.10 that \( \text{spt}(\tau_k) \subset N_{b_1 r_k}(\partial \sigma_k) \) and \( \mathbf{M}(\tau_k') \leq b_2 \cdot r_k^n \) where \( b_1 \) and \( b_2 \) depend only on \( b \) and \( X \). By choosing \( b \) sufficiently large, \( b_1 \) will be small enough so that \( \text{spt}(\tau_k') \cap B_{p_k}(\frac{r_k}{D_0}) = \emptyset \). Now we replace \( \tau_k \) by \( \tau_k'' = \tau_k \cup \{ d_{p_k} \leq r_k' \} + \tau_k' \).
Note that $\partial \tau_n = \partial \tau_k$. The existence of $\{\tau_n\}$ and $\{\varphi_k\}$ contradicts Definition 6.19. □

**Lemma 6.23.** Let $\Phi : \mathbb{R}^n \to X$ be an $n$-dimensional $(L, A)$-quasiflat in a metric space $X$ satisfying condition (CI$_{n-1}$). Then $\Phi$ is rigid if and only if it has super-Euclidean divergence.

**Proof.** We will use Definition 6.19 in the proof.

If super-Euclidean divergence fails, then we find a constant $D_0 > 1$, a sequence $(x_k)$ in $\mathbb{R}^n$ and a sequence $r_k \to \infty$ such that the following holds.

There exist cubical cycles $\varphi_k \in \mathbf{Z}_{n-1,c}(A_{x_k}(\frac{r_k}{D_0}, D_0 r_k))$ with $[\varphi_k] \neq 0 \in \bar{H}_{n-1}(\mathbb{R}^n \setminus B_{x_k}(\frac{r_k}{D_0})))$, $\mathbf{M}(\varphi_k) \leq D_0 \cdot r_k^{n-1}$ such that $\iota(\varphi_k)$ can be filled by $\tau_k \in I_{n,c}(A_{x_k}(\frac{r_k}{D_0}, D_0 r_k))$ with $\mathbf{M}(\tau_k) \leq D_0 \cdot r_k^n$ (define $p_k = \Phi(x_k)$). Let $\alpha_k \in I_{n,c}(\mathbb{R}^n)$ be the canonical filling of $\varphi_k$. Then $\mathbf{M}(\alpha_k) \leq D' \cdot r_k^n$ with $D'$ depending only on $D_0$ and $n$. Moreover, $B_{x_k}(\frac{r_k}{D_0}) \subset \text{spt}(\varphi_k)$. Let $b > 0$ be arbitrary. Then there exists $D'' = D''(L, A, b, D_0)$ such that $\Phi(B_{x_k}(\frac{r_k}{D_0} - b))$ contains a point which is $D''r_k$ away from $A_{p_k}(\frac{r_k}{D_0}, D_0 r_k) \supset \text{spt} \tau_k$ for all $k$. Thus $Q$ is not rigid.

Now we argue the other direction. If $\Phi$ is not rigid, then for any choice of $b > 0$ (we will determine the value of $b$ later), we find (depending on value of $b$) constants $\epsilon_0 > 0$, $M_0$, a sequence $(x_k)$ in $\mathbb{R}^n$, $r_k \to \infty$, cubical chains $\varphi_k \in I_{n-1,c}(B_{x_k}(r_k))$ and $\tau_k \in I_{n,c}(B_{p_k}(M_0 r_k))$ with $\partial \tau_k = \iota(\varphi_k)$ such that the following holds for all $k$.

1. $\mathbf{M}(\varphi_k) \leq M_0 \cdot r_k^{n-1}$ and $\mathbf{M}(\tau_k) \leq M_0 \cdot r_k^n$.
2. Let $\mu_k$ be the canonical filling of $\varphi_k$. Then there is $y_k \in \{y \in \text{spt}(\mu_k) | d(y, \text{spt}(\varphi_k)) > b\}$ with $B_{\Phi(y_k)}(\epsilon_0 r_k) \cap \text{spt}(\tau_k) = \emptyset$.

In particular,

(6.24) \[ B_{\Phi(y_k)}(\epsilon_0 r_k) \cap \text{spt}(\iota(\varphi_k)) = \emptyset. \]

Let $\varphi'_k = \pi_{\#} \iota(\varphi_k)$, where $\pi : X \to \mathbb{R}^n$ is defined in Section 5.2. As $\text{spt}(\iota(\varphi_k))$ is contained in a fixed neighborhood of $Q$, by applying $\pi$ to (6.24), we can find $\epsilon_1$ independent of $k$ with $B_{y_k}(\epsilon_1 r_k) \cap \text{spt}(\varphi'_k) = \emptyset$. Suppose $\nu'_k$ is the canonical filling of $\varphi'_k$. By choosing $b$ big enough, we have $y_k \in \text{spt}(\nu'_k)$ by Lemma 5.8. Thus

(6.25) \[ B_{y_k}(\epsilon_1 r_k) \subset \text{spt}(\nu'_k) \text{ and } d(y_k, \text{spt}(\varphi'_k)) \geq \epsilon_1 r_k. \]

By applying Federer-Fleming deformation to $\varphi'_k$, we assume both $\varphi'_k$ and $\nu'_k$ are cubical, $B_{y_k}(\epsilon_1 r_k) \subset \text{spt}(\nu'_k)$ and $\mathbf{M}(\varphi'_k) \leq M'_0 \cdot r_k^{n-1}$.

In what follows, $a_1, a_2, \cdots$ will be constants depending only on $L, A, n$ and $X$ (independent of $k$). Let $q_k = \Phi(y_k)$. Proposition 5.7(5) implies
that \( \text{Fill}(\varphi_k - \varphi'_k) \leq a_1 M(\varphi_k) \). Thus \( \text{Fill}(\iota(\varphi_k) - \iota(\varphi'_k)) \leq a_2 M(\varphi_k) \).

Let \( \hat{\tau}_k \) be a minimal filling of \( \iota(\varphi_k) - \iota(\varphi'_k) \). Then \( \text{spt}(\hat{\tau}_k) \) is contained in the \( a_3 r_k^{n-1} \)-neighborhood of \( \text{spt}(\iota(\varphi_k)) \cup \text{spt}(\iota(\varphi'_k)) \) by Theorem 3.10 (2). Note that \( \text{spt}(\iota(\varphi_k)) \subset N_{a_4}(\Phi(\text{spt}(\varphi_k))) \). This together with (6.24) and (6.25) imply that \( d(q_k, \text{spt}(\hat{\tau}_k)) \geq a_5 \epsilon_1 r_k - a_6 r_k^{n-1} \geq a_7 \epsilon_1 r_k \) when \( r_k \) is sufficiently large. Let \( \tau'_k = \hat{\tau}_k + \tau_k \). Then \( \partial \tau'_k = \iota(\varphi'_k) \) and \( d(q_k, \text{spt}(\tau'_k)) \geq a_7 \epsilon_1 r_k \). Thus \( \text{spt}(\tau'_k) \subset A_{q_k}(a_7 \epsilon_1 r_k, (M_0 + a_8) r_k) \). However, \( M(\tau'_k) \leq M(\tau_k) + M(\hat{\tau}_k) \leq (M_0 + a_7) r_k^n \). Thus super-Euclidean divergence fails.

**Definition 6.26.** Suppose \( X \) is a complete metric space satisfying condition (\( C_{n-1} \)). An \( n \)-dimensional quasiflat \( Q \subset X \) is Morse if it has \( \delta \)-super-Euclidean divergence. The Morse parameter of \( Q \) is \( \delta \).

### 6.3. Quasi-isometry invariance.

**Proposition 6.27.** Let \( X \) and \( Y \) be two complete metric spaces satisfying condition (\( C_{n-1} \)) with constant \( c \). Let \( Q \subset X \) be an \( n \)-dimensional Morse \((L, A)\)-quasiflat. Let \( \Phi : X \to Y \) be an \( L' \)-Lipschitz \((L', A')\)-quasi-isometry with an \( L' \)-Lipschitz quasi-isometry inverse. Then \( q(Q) \) is a Morse quasiflat with its Morse parameter depending only on \( L, A, L', A', n, c \) and the Morse parameter of \( Q \).

**Proof.** By Lemma 6.22, it suffices to show super-Euclidean divergence is invariant under quasi-isometry, which is slightly cleaner to work with. Suppose \( Q \) is represented by \( \Phi : \mathbb{R}^n \to X \) such that \( \mathbb{R}^n \) has a cubulation of appropriate scale \( R = R(L, A, L', A') \). Let \( Q' = q(Q) \) represented by \( \Phi' = q \circ \Phi \). Let \( \iota \) and \( \iota' \) be the chain map as in Proposition 5.7 associated with \( Q \) and \( Q' \) respectively.

Let \( q' \) be an \( L' \)-Lipschitz quasi-isometry inverse of \( q \). Let \( Q' = q(Q) \). Suppose \( Q \) has \( \delta \)-super-Euclidean divergence.

Take \( D > 1 \). Take \( x \in \mathbb{R}^n \). Define \( \Phi'(x) = p' \) and \( \Phi(x) = p \). Suppose that \( \varphi \in Z_{n-1, c}(\mathbb{R}^n \setminus B_x(r)) \) is a cubical chain representing a nontrivial class in \( H_{n-1}(\mathbb{R}^n \setminus B_x(r)) \). Suppose \( M(\varphi) \leq D \cdot r^{n-1} \). Let \( \tau' \in I_{n,c}(Y \setminus B_{p'}(\tau D)) \) such that \( \partial \tau = \iota'(\varphi) \).

Let \( \tau = q'_#(\tau') \). Using Theorem 3.10 we can build skeleton by skeleton a homology \( h \) between \( \iota(\varphi) \) and \( q'_#(\iota'(\varphi)) = \iota \) such that \( \delta h = \iota(\varphi) - \iota \), \( M(h) \leq C \cdot M(\varphi) \) and \( \text{spt}(h) \subset N_M(\text{spt}(\iota(\varphi)) \cup \text{spt}(\iota)) \) where \( C \) depends only on \( L, A, L', A, c \) and \( n \).

Let \( \tau_1 = \tau + h \). Then \( \text{spt}(\tau_1) \in X \setminus B_{p'}(\tau D) \) for \( C' > 1 \) depending only on \( C, L', A, L, A \). As \( Q \) has \( \delta \)-super-Euclidean divergence,
Thus \( M(\tau_1) \geq \delta(r) \cdot r^n \). Thus \( M(\tau') \geq \frac{1}{(\ell^n)} M(\tau) \geq \frac{\delta(r) \cdot r^n - CD}{(\ell^n)} \). Thus \( Q' \) has \( \delta' \)-super-Euclidean divergence with \( \delta' = \delta(L, A, L', A', c, n) \).

Corollary 6.28. Suppose \( X \) and \( Y \) are piecewise Euclidean simplicial complexes. Let \( Q \subset X \) be an \( n \)-dimensional Morse \((L, A)\)-quasiflat. Suppose there exists \( f : (0, \infty) \to (0, \infty) \) such that any \( \ell \)-Lipschitz map from the unit sphere \( S^m \) to \( X \) or \( Y \) with \( m \leq n \) can be extended to an \( f(\ell) \)-Lipschitz map from the unit disk to \( X \). Let \( \Phi : X \to Y \) be an \((L', A')\)-quasi-isometry with an \( L' \)-Lipschitz quasi-isometry inverse. Then \( q(Q) \) is a Morse quasiflat with its Morse parameter depending only on \( L, A, L', A', n, f \) and the Morse parameter of \( Q \).

The assumption of Corollary 6.28 enables us to build Lipschitz quasi-isometries \( X^{(n)} \to Y^{(n)} \) and \( Y^{(n)} \to X^{(n)} \) skeleton by skeleton.

If \( X \) and \( Y \) are simplicial complexes with cocompact automorphism groups satisfying condition \((\text{CI}_{n-1})\), then the assumption of Corollary 6.28 is satisfied. Indeed, condition \((\text{CI}_{n-1})\) implies that \( k \)-th reduced homology of \( X \) and \( Y \) are trivial for \( k \leq n - 1 \) (by Federer-Fleming deformation), thus the \( k \)-th homotopy group is also trivial for \( k \leq n - 1 \). The cocompactness guarantees that spheres can be filled with uniform control.

7. Asymptotic conditions on spaces and conditions on asymptotic cones

In this section we discuss conditions concerning asymptotic cones in Section 6.1 are related to asymptotic conditions in space in Section 6.2. For readers whose main interests are in Theorem 1.8 and Theorem 1.9, Lemma 7.6 is redundant as it has some overlap with material in Section 7.1. However, the purpose of Lemma 7.6 is to create a shortcut for readers only interested in the visibility results in Section 1.5 and such readers can skip most of Section 7.1.

7.1. Full support and super-Euclidean divergence.

Lemma 7.1. Let \( X \) be a complete metric space which satisfies \((\text{SCI}_{n-1})\) with constant \( c \). Let \( \Phi : \mathbb{R}^n \to X \) be an \((L, A)\)-quasiflat with image \( Q \). Suppose that \( Q \) has \( \delta \)-super-Euclidean divergence. Then for any asymptotic cone \( X_\omega \), which contains an ultralimit \( Q_\omega \) of \( Q \), holds that \( Q_\omega \) has full support with respect to \( \tilde{H}_\omega^L \).
Proof. Let \((X_\omega,p_\omega) = \omega \lim (\frac{1}{r_k}X,p_k)\) for some sequence \(r_k \to \infty\). Set \(X_k = \frac{1}{r_k}X\). Suppose that \(Q_\omega\) does not have full support with respect to \(H^k\). Denote by \(\Phi_\omega : \mathbb{R}^n \to X_\omega\) the ultralimit of \(\Phi\). There exists a point \(q_\omega = \Phi(x_\omega)\) which does not lie in the support of \(Q_\omega\). After translating \(\mathbb{R}^n\), we may assume \(x_k \equiv 0\). Then we find \(D > 1\) and a Lipschitz cycle \(\varphi\) representing an element in \(Z_{c,n-1}(A_0(\frac{1}{D},D))\) in \(\mathbb{R}^n\) and a Lipschitz chain \(\tau\) representing an element in \(I_{c,n}(A_{q_\omega}(\frac{1}{D},D))\) that fills \(\Phi_{\omega} \# \varphi\), and such that \(M(\tau) + M(\varphi) < D\). Moreover, \(\varphi\) represents a nontrivial element in \(H_{a-1}(\mathbb{R}^n \setminus B_0(\frac{1}{D}))\). Let \(R = R(D,c)\) be a fine scale to be determined later.

By barycentric subdivision, we can write \(\tau\) as a sum of Lipschitz simplices with uniform Lipschitz constant and such that each of them has image in a ball of radius \(R\). Using this representation, we pull-back the 0-skeleton of \(\tau\) to a collection of points \(\tau_k^{(0)}\) in \(X_k\), such that each pair of points in \(\tau_k^{(0)}\) corresponding to the boundary of an edge of \(\tau\) is contained in a ball of radius \(R\). Then we fill in skeleton by skeleton, using the strong coning inequality, to obtain \(\tau_k \in I_{c,n}(X_k)\). Note that there is a constant \(C = C(D,c,n)\) such that \(M(\tau_k) \leq C \cdot M(\tau)\) and \(\text{spt} \tau_k \subset N_{C,R}(\tau_k^{(0)})\). After further subdivision, we may assume that \(\partial \tau_k = \nu \varphi_k\), where \(\varphi_k \in Z_{c,n-1}(A_{x_k}(\frac{1}{D},D))\) is cubical cycle homologous to \(\varphi\) in \(A_0(\frac{1}{D},D)\). Now it is clear that if we choose \(R\) small enough, then the \(\tau_k\) will stay at a uniform positive distance from \(p_k\). Therefore \(Q\) cannot have super-Euclidean divergence in \(X\). \(\square\)

In order to gain information on our space from properties of its asymptotic cones, we will rely on the following weak version of Wenger’s compactness theorem, \([\text{Wen11}]\).

Lemma 7.2. Let \((X_k)\) be a sequence of complete metric spaces satisfying condition \((\text{EII}_{n-1})\). Further, let \(\tau_k \in I_n(X_k)\) be chains such that there exists a constant \(D > 0\) with

\[M(\tau_k) + M(\partial \tau_k) < D \text{ and } \text{diam spt} \tau_k < D.\]

Then, for every \(\epsilon > 0\) there exists a regularized sequence \((\tau'_k)\) with \(\tau'_k \in I_{c,n}(X_k)\) such that

\[M(\tau'_k) + M(\partial \tau'_k) < D \text{ and } \text{diam spt} \tau'_k < D;\]

the family of support sets \((\text{spt} \tau'_k)\) is uniformly totally bounded; and \(\tau_k\) is homologous to \(\tau'_k\) inside \(N_\varepsilon(\text{spt} \tau_k)\). Moreover, there exists a subsequence \(\tau_{k_l}\), a compact metric space \(Z\), a chain \(\tau \in I_{c,n}(Z)\) and isometric embeddings \(\psi : \text{spt} \tau_{k_l} \to Z\) such that \(\psi_{l} \# \tau_{k_l}\) converges to \(\tau\) in the flat distance in \(\ell^\infty(Z)\).
Proof. The first half of the claim follows directly from Proposition 4.2. Hence we find a sequence \((\tau'_k)\) with \(\tau'_k \in I_{c,n}(X_k)\) such that \(M(\tau'_k) + M(\partial \tau'_k) < D\), the family of support sets \((\partial \tau'_k)\) is uniformly totally bounded and \(\tau_k\) is homologous to \(\tau'_k\) inside \(N_\epsilon(spt \tau_k)\). By [Gro81], there exists a compact metric space \(Z\) with a sequence of isometric embeddings \(\psi: spt \tau_{k_1} \to Z\) such that \(\psi_1(spt \tau'_k)\) converges to \(Y \subset Z\) in the Hausdorff distance. By [AK07], Theorem 5.2 and Theorem 8.5, we can assume \(\psi_1\# \tau'_k\) weakly converges to \(\tau' \in I_{c,n}(Z)\) by passing to a subsequence. Moreover, by [Wen11], Proposition 2.2], \(spt(\tau') \subset Y\). Finally, by [Wen07], \(\psi_1\# \tau'_{k_1}\) converges to \(\tau'\) in flat distance, if we replace \(Z\) by \(\ell^\infty(Z)\).

\[\square\]

Lemma 7.3. Let \(X\) be a complete metric space which satisfies (EII\(_{n-1}\)) with constant \(c\). Let \(\Phi: \mathbb{R}^n \to X\) be an \((L, A)\)-quasiflat with image \(Q\). Suppose that in any asymptotic cone \(X_\omega\), which contains an ultralimit \(Q_\omega\) of \(Q\), holds that \(Q_\omega\) has full support with respect to \(H^{\text{AK}}_{s,c}\). Then \(Q\) has \(\delta\)-super-Euclidean divergence in \(X\).

Proof. Suppose that \(Q\) does not have super-Euclidean divergence. Then there exists \(D > 1\) and a sequence \(r_k \to \infty\); cycles \(\varphi_k \in Z_{n-1,c}(A_{r_k}(\frac{r_k}{D}, Dr_k))\) with \(x_k \in \mathbb{R}^n\) and \(M(\varphi_k) \leq D \cdot r_k^{n+1}\); chains \(\tau_k \in I_{n,c}(A_{p_k}(\frac{r_k}{D}, Dr_k))\) with \(p_k = \Phi(x_k)\) and \(M(\tau_k) \leq D \cdot r_k^n\) such that \(\partial \tau_k = \nu_k\). Translating \(\mathbb{R}^n\), we agree on \(x_k = 0\). We set \(X_k = \frac{1}{r_k}X\) and also rescale \(\mathbb{R}^n\) by \(\frac{1}{r_k}\). However, we still use the notation \(\tau_k\) and \(\varphi_k\) to denote the corresponding currents in the rescaled spaces. By passing to a subsequence, we can arrange that \(\varphi_k\) converges in the flat distance to some \(\varphi_\infty \in Z_{n-1,c}(A_0(\frac{1}{D}, D))\). Denote by \(\nu_k \in I_{n,c}(B_0(Dr_k))\) the canonical filling of \(\varphi_k\).

By Lemma 7.2 there exist regularized sequences \((\tau'_k)\) and \(\nu'_k\), a compact metric space \(Z\) with a sequence of isometric embeddings \(\psi_l: spt \tau'_{k_1} \cup \nu_{k_1} \to Z\) such that \(\psi_1\# \tau'_{k_1}\) converges to some \(\tau' \in I_{c,n}(Z)\) and \(\psi_1\# \nu_{k_1}\) converges to some \(\nu \in I_{c,n}(Z)\), both with respect to flat distance in \(\ell^\infty(Z)\). Let \((X_\omega, p_\omega)\) be an ultralimit of \((X_k, p_k)\) and \(Q_\omega \subset X_\omega\) be the ultralimit of the \(Q_k\)'s. By the previous paragraph, \(spt \tau' \cup spt \nu\) with the induced metric from \(Z\) can also be viewed as an isometrically embedded subset of \(X_\omega\). We claim that \(\partial \tau'\) is non-trivial in \(H^{\text{AK}}_{n-1}(Q_\omega \setminus B_{p_\omega}(\frac{D}{10}))\). To see this, we will show that \(\nu\) is non-trivial in \(Z\). Let \(\pi_k : X_k \to \mathbb{R}^n\) be a Lipschitz extension of \((\Phi|_{A_{r_k}^\text{c}(\omega)})^{-1}\) where \(C\) denotes the cubulation underlying \(\nu\). Then the Lipschitz constant of \(\pi_k\) is uniformly bounded. Denote by \(\tilde{\pi}_k : Z \to \mathbb{R}^n\) a Lipschitz extension of \(\pi_k \circ \psi_k\) and let \(\hat{\pi} : Z \to \mathbb{R}^n\) be a limit map. By Lemma 7.2 we see...
that for $k$ large enough $\pi_k \# \nu_k$ contains $B_0(\frac{D}{2})$ in its support. Hence, after passing to a subsequence, $\pi_k \# \nu_k \perp B_0(\frac{D}{2})$ is a constant multiple of $[B_0(\frac{D}{2})]$. So $\pi_\infty \# \nu$ contains $B_0(D)$ in its support. Therefore $\nu$ is nonzero and $Q_\omega$ does not have full support with respect to $\tilde{H}_{*c}^{AK}$.

**Proposition 7.4.** Let $X$ be a complete metric space which satisfies (SCI) and let $Q \subset X$ be a quasiflat. Suppose that in any asymptotic cone $X_\omega$, which contains an ultralimit $Q_\omega$ of $Q$, holds that the homology theories $\tilde{H}_{*c}^{AK}$ and $\tilde{H}_*^L$ are isomorphic. Then $Q$ has $\delta$-super-Euclidean divergence in $X$ if and only if $Q_\omega$ has full support with respect to $\tilde{H}_*^L$.

As a consequence, if $X$ is a complete metric space with a Lipschitz bicombing, then a quasiflat $Q$ in $X$ has $\delta$-super-Euclidean divergence if and only if in any asymptotic cone $X_\omega$ (with base points in $Q$), $Q_\omega$ has full support with respect to $\tilde{H}_*$.

*Proof. The first paragraph follows from Lemma 7.1 and Lemma 7.3. The second paragraph follows from Proposition 6.12. □*

The following is a consequence of Proposition 7.4 and a version of Kunneth formula for relative homology (cf. [Dol12, pp. 190, Proposition 2.6]).

**Corollary 7.5.** Suppose $X_1$ and $X_2$ are metric spaces with Lipschitz bicombing. Let $Q_i \subset X_i$ be Morse quasiflats (cf. Definition 6.26). Then $Q_1 \times Q_2$ is a Morse quasiflat in $X_1 \times X_2$.

### 7.2. Neck property and coarse neck property.

**Lemma 7.6.** Let $(X_k, Q_k, r_k, p_k, \tau_k)$ be a sequence and $D > 0$, $\rho < 1$ constants, such that:

- $X_k$ is a complete metric space with an $L'$-Lipschitz bicombing.
- $Q_k \subset X_k$ is an $n$-dimensional $(L, A)$-quasiflat containing the base point $p_k \in Q_k$.
- $r_k \to \infty$.
- for any asymptotic cone $X_\omega$ arising from the rescaled sequence $\left(\frac{1}{r_k}X_k, p_k\right)$, $Q_\omega$ has the neck property in $X_\omega$ (cf. Definition 6.4) with uniform constant $C'$;
- $\tau_k \in I_n(X_k)$, $\spt \tau_k \subset B(p_k, r_k) \setminus N_{pr_k}(Q_k)$ and $M(\tau_k) \leq Dr_k^n$.
- $\sigma_k := \partial \tau_k$ satisfies $\spt \sigma_k \subset N_{2pr_k}(Q_k)$ and $M(\sigma_k) \leq Dr_k^{n-1}$.

Then there exists a constant $C$ depending only on $C', L', L, A$ and $n$ such that the following holds for all sufficiently large $k$. 

Fill(σ_k) \leq C \rho r_k M(σ_k).

Proof. Since $X_k$ has a Lipschitz combing, we may assume that the quasiflat $Q_k$ is represented by a Lipschitz quasi-isometric embedding. Let $X_k' = (X_k, d_k/r_k)$ and $Q'_k = (Q_k, d_k/r_k)$. We still use $τ_k$ and $σ_k$ to denote the rescaled currents. By Lemma 7.2 we can pass to a regularized sequence $(τ'_k)$ with $τ'_k ∈ I_{c,n}(X'_k)$ such that $τ'_k$ comes with mass bounds and support control and such that $τ'_k$ is homologous to $τ_k$ in a small tubular neighborhood of $spt τ_k$. Moreover, we may assume that there is a compact metric space $Z$, a chain $τ ∈ I_{c,n}(Z)$ and isometric embeddings $ψ_k : spt τ_k → Z$ such that $ψ_k#τ_k$ converges to $τ$ in the flat distance in $ℓ^∞(Z)$. We set $σ = \partial τ$.

Let $X_ω$ be an ultralimit of $X'_k$ and $Q_ω ⊂ X_ω$ be the ultralimit of the $Q'_k$'s. We view $spt τ$ with the induced metric from $Z$ as a subset of $X_ω$. Moreover, we may assume that the $X'_k$'s and $X_ω$ are isometrically embedded in $Z$ by replacing $Z$ by the metric space obtained from gluing $X_ω$ and $X'_k$'s to $Z$ along $spt τ_ω$ and $spt(τ_k)$'s ([BH99, Lemma I.5.24]). Also, we can assume that $Z$ has a convex geodesic bicombing by replacing $Z$ by $ℓ^∞(Z)$. By construction, inside $X_ω ⊂ Z$ we know $τ$ is supported in $X_ω \setminus N_ρ(Q_ω)$ and $σ_ω = \partial τ_ω$ is supported in $N_2ρ(Q_ω)$.

Let $π_k : Z → (Q'_k, d_k/R_k)$ be a Lipschitz quasi-retraction as in Corollary 5.3. Then there exists $L_0$ depending only on $L$ and $A$, such that $π_k$ is $L_0$-Lipschitz. We use Fill(τ) to denote the filling volume inside a space $U$. By assumption we have Fill$_{X_ω}(σ) \leq C'ρ M(σ)$. Since $σ_k$ converges to $σ$ with respect to flat distance in $Z$, we conclude Fill$_Z(σ_k) \leq 2C'ρ M(σ_k)$ for all $k$ large enough. Hence, for all such $k$ holds Fill$_{Q'_k}(π_k#σ_k) \leq 2L_0^n C'ρ M(σ_k)$. By Corollary 5.5 there exists a constant $C_1 = C_1(L', L, A, n)$ such that Fill$_{X'_k}(σ_k−π_k#σ_k) \leq C_1ρ M(σ_k)$ and the proof is complete. □

**Definition 7.7.** An $(L, A)$-quasiflat $Q$ in a metric space has the coarse neck property (CNP), if there exists a constant $C_0 > 0$ such that the following holds. For given constants $C$, $ρ$ and $b_1$, there exists $R = R(C, ρ, b_1, d(p, Q), Q)$ such that for any $R ≥ R$ and any $τ ∈ I_{n−1}(B_p(b_1 R) \setminus N_ρ R(Q))$ satisfying

- $M(τ) ≤ CR^n$;
- $σ := \partial τ ∈ I_{n−1}(X)$, $spt(σ) ⊂ N_2ρ R(Q)$ and $M(σ) ≤ CR^{n−1}$;

we have

\[ \text{Fill}(σ) ≤ C_0ρ R M(σ). \]
Note that the definition of CNP depends on the parameter $C_0$ and the function $R = R(C, \rho, b_1, d(p, Q), Q)$.

**Corollary 7.8.** Suppose $X$ has an $L'$-Lipschitz bicombing and $Q \subset X$ is an $(L, A)$ quasiflat. If for any asymptotic cone $X_{\omega}$ of $X$ (with base points in $Q$) $Q_{\omega}$ has the neck property in $X_{\omega}$, then $Q$ has the coarse neck property in $X$.

### 8. Stability of Morse quasiflats

In this section we prove a version of Morse lemma for Morse quasidisks (Proposition 8.3), and a statement about relative position of two Morse quasiflats (Proposition 8.4). Proposition 8.3 will not be used in the later part of the paper, though the strategy to prove Proposition 8.3 and Proposition 14.2 bares some similarly and Proposition 8.3 is much simpler. Proposition 8.4 will be used in Section 11.

These two propositions are proved in two ways. In Section 8.1 we present the proof under weaker assumption where the ambient metric space has cone inequality. In Section 8.2 we present a much simpler proof under the stronger assumption that the ambient metric space has a Lipschitz bicombing.

#### 8.1. Working without asymptotic cones.

We refer to Definition 6.13 and Remark 6.18 for the definition of $(\mu, b)$-rigid quasidisks. Also we will frequently use the map $\iota$ in Proposition 5.7 and the chain projection in Definition 5.9 in the context of quasidisks.

Throughout this subsection, we will assume the domain of a quasidisk is an $n$-dimensional cube in the Euclidean space $\mathbb{R}^n$. The cube is subdivided into smaller cubes whose size depends on the quasi-isometry constants. Moreover, we will endow the domain with the $\ell^\infty$-metric, instead of the $\ell^2$-metric, so that the level set of the distance function to a point is the boundary of some rectangle.

**Lemma 8.1.** Let $X$ be a complete metric space satisfying the cone inequality condition (CI$_k$). Let $D$ and $D'$ be $n$-dimensional quasidisks in $X$ with $D' \subset N_\epsilon(D)$. Let $\lambda > 0$ be a given constant. Then there exists $C$ depending only on $X, n, \lambda$ and the quasi-isometry constants of $D$ and $D'$ such that the following holds.

Suppose $\mathcal{D}'$ is the domain of $D'$. Take a $k$-dimensional ($k \leq n$) rectangular subcomplex $R$ of $\mathcal{D}'$ such that ratio of the longest side and the shortest side is $\leq \lambda$. Let $\bar{\sigma}'$ be a cubical $k$-chain which represents the relative fundamental class of $R$. Put $\sigma' = \iota(\bar{\sigma}')$. Let $\sigma = (\sigma')_D$
be the chain projection as in Definition 5.9. We assume \( \partial \sigma = (\partial \sigma')_{D} \) (this can always be arranged). Then there exist \( \tau \in I_{k+1,c}(X) \) and \( \beta \in I_{k,c}(X) \) such that

1. \( \partial \tau = \sigma' - \sigma + \beta \);
2. \( \partial \beta = \partial \sigma' - \partial \sigma \);
3. \( M(\tau) \leq CrM(\sigma') \) and \( \text{spt}(\tau) \subset N_{Cr}(\text{spt}(\sigma')) \);
4. \( M(\beta) \leq CrM(\partial \sigma') \) and \( \text{spt}(\beta) \subset N_{Cr}(\text{spt}(\partial \sigma')) \).

We take a new cubical subdivision of \( \mathcal{D}' \) such that the size of cubes are comparable to \( r \). The proof is straightforward by induction on skeletons with respect to this new subdivision. Lemma 8.1 is much simpler than Proposition 5.14 as we already have a skeleton structure on \( \sigma' \) with uniform bounds on the mass of its “cells”. This together with bounding filling radius in terms of mass (cf. Theorem 3.10 (2)) gives the extra estimate on the location of \( \text{spt}(\tau) \) and \( \text{spt}(\beta) \) in (3) and (4).

**Lemma 8.2.** Let \( D \) and \( D' \) be \( n \)-dimensional \( (L,A) \)-quasi-disks in a metric space \( X \). Suppose \( d_{H}(\partial D, \partial D') \leq A_{1} \) and \( D \subset N_{A_{2}}(D') \). Then there exists \( C \) depending only on \( L, A, A_{1}, A_{2} \) and \( n \) such that \( d_{H}(D, D') \leq C \).

**Proof.** Let \( \mathcal{D} \) and \( \mathcal{D}' \) be the domains of \( D \) and \( D' \) respectively. Suppose the domains have radius \( R \gg A, A_{1}, A_{2} \). We can define a map \( f : \mathcal{D} \to \mathcal{D}' \) as follows. Take a point \( p \in \mathcal{D} \), send to its image \( \hat{p} \in X \), and send \( \hat{p} \) to one of its nearest points \( \hat{p}' \) in \( D' \), and send \( \hat{p}' \) to a corresponding point \( p' \in \mathcal{D}' \). Up to perturbing \( f \) a bounded amount comparable to \( \max\{A, A_{1}, A_{2}\} \), we can assume \( f \) is continuous, \( f(\partial \mathcal{D}) \subset \partial \mathcal{D}' \) and \( f|_{\partial \mathcal{D}} \) has degree 1. By a standard homological argument, we know \( f \) is surjective. Hence the lemma follows. \( \square \)

**Proposition 8.3.** Suppose \( X \) is a complete metric space satisfying condition \( (Cl_{n}) \). Given \( (\mu, b) \) as in Definition 6.13 and positive constants \( L, A, A', n \), there exists \( C \) depending only on \( \mu, b, L, A, A', n \) and \( X \) such that the following holds.

Let \( D \) and \( D' \) be two \( n \)-dimensional \( (L, A) \)-quasi-disks in \( X \) such that \( d_{H}(\partial D, \partial D') < A' \) and \( D \) is \( (\mu, b) \)-rigid. Then \( d_{H}(D_{1}, D_{2}) < C \).

Throughout the following proof, \( \lambda_{1}, \lambda_{2}, \ldots \) will be constants whose values depend only on \( X, L, A, A', n \).

**Proof.** Let \( \mathcal{D} \) and \( \mathcal{D}' \) be the domains of \( D \) and \( D' \). Recall that \( \mathcal{D} \) and \( \mathcal{D}' \) are cubes with \( \ell^\infty \) metric. Let \( R = \max\{\text{diam}(\mathcal{D}), \text{diam}(\mathcal{D}')\} \).
Suppose $R \geq 1000A$. Let $\pi : X \to \mathcal{D}$ be the Lipschitz retraction in Definition 5.6. Let $\tau = \iota([\mathcal{D}])$ and $\tau' = \iota([\mathcal{D}'])$ where $[\mathcal{D}]$ and $[\mathcal{D}]'$ denote the fundamental classes. By Proposition 5.7, $d_H(D, \text{spt}(\tau)) \leq \lambda_1$ and $d_H(D', \text{spt}(\tau')) \leq \lambda_1$. Since $d_H(D, D') \leq A$, we have $d_H(\partial \mathcal{D}, \pi(\partial D')) \leq \lambda_2 A$. Thus we assume without loss of generality that $\partial \mathcal{D} = \pi(\partial D')$.

Let $(\partial \tau')_D$ be as in Definition 5.9. We can assume $(\tau')_D = \tau$. By Lemma 8.1, there is $\beta \in I_{n,c}(X)$ such that $\partial \beta = \partial (\tau - \tau')$, $M(\beta) \leq \lambda_3 A M(\partial \tau)$ and $\text{spt}(\beta) \subset N_{\lambda_3 A} \text{spt}(\partial \tau)$.

Let $p \in \mathcal{D}$ be a base point and let $\hat{p} \in X$ be its image. Let $M > 0$ whose value will be determined later (at the moment we need $\lambda$ and $\lambda^2$). By Lemma 8.1, we take $\tau \in \mathcal{D}$ such that the slice $W_\tau := \{ W, c, r_1 \}$ satisfies $M(W_\tau) \leq \lambda_3 \varepsilon r_1$. Let $\tau_1 = \tau \cup \{ c \leq r_1 \}$ and $\tau'_{r_1} = \tau' \cup \{ c \leq r_1 \}$. Then $\partial \tau_1 = \partial (\beta \cup \{ c \leq r_1 \} + W_r + \tau'_{r_1})$.

We claim $\text{spt}(\tau_1) \subset N_{\lambda_1 r_1}(D')$. By Definition 5.6 (2), $\text{spt}(\beta) \cap \{ c \leq r_1 \} \subset N_{\lambda_8 A} \{ \{ d_p \leq r_1 \} \}$. Thus $M(\beta \cup \{ c \leq r_1 \}) \leq \lambda_7 A (r_1)^{n-1}$. Take $\lambda_8$ with $M(\tau_0) \leq \lambda_8 (r_1)^n$ and $\text{spt}(\tau') \subset B_{\beta}(\lambda_8 R)$. Thus if $M \geq \lambda_5 \varepsilon + \lambda_8 + \lambda_8 A$ and $r_1 \geq \varepsilon$, then

$$\text{spt}(\tau_1) \subset N_{\lambda_1 r_1} \text{spt}(\beta \cup \hat{B}(p, r_1) + \tau'_{r_1} + W_\tau).$$

By Definition 5.6 (2), there is $\lambda_9$ such that if $x \in \{ d_p \leq (1 - \lambda_9 \varepsilon) r_1 \}$, then $\varepsilon r_1 \leq d(\hat{x}, \{ c = r_1 \}) \leq d(\hat{x}, \text{spt}(W_\tau))$. Thus $\text{spt}(\tau_1) \subset N_{\lambda_1 r_1} \text{spt}(\tau')$. By Proposition 5.7, $\text{spt}(\tau_1) \subset N_{\lambda_1 r_1} \text{spt}(\tau_1)$, thus the claim follows.

Define $\tau_1 = \tau_r$ and $\tau'_1 = (\tau_1)_D$ (cf. Definition 5.9). As we are using $\ell^\infty$ metric on $\mathcal{D}$, $\tau_1$ is the fundamental class of a rectangular region. By Lemma 8.1, we take $W_1 \in I_{n+1}(X)$ such that

1. $\partial W_1 = \tau_1 - \tau'_1 + \beta;$
2. $\partial \beta_1 = \partial \tau_1 - \partial \tau'_1$;
3. $M(W_1) \leq \lambda_3 \lambda_{11} \varepsilon r_1 M(\tau_1)$ and $\text{spt}(W_1) \subset N_{\lambda_3 \lambda_{11} r_1} \text{spt}(\tau_1)$;
4. $M(\beta_1) \leq \lambda_3 \lambda_{11} \varepsilon r_1 M(\partial \tau_1)$ and $\text{spt}(\beta_1) \subset N_{\lambda_3 \lambda_{11} r_1} \text{spt}(\partial \tau_1)$.
Now slice $W_1$ and repeat the previous procedure. We choose $\epsilon$ sufficiently small to control mass and location of $W_1$ and $\beta_1$ in (3) and (4) so that estimates do not get worse in the further steps.

By cutting $\tau_1$ into appropriate smaller rectangular pieces and apply Lemma 8.1 to each piece, we can assume in addition that $\beta_1$ stays $A$-close to $\partial D$ along part of $\partial \tau_1$ which coincides with the boundary of $D$. Consequently, after choosing $r_2$, we will have $\text{spt}(\beta_1 \cap \{c \leq r_2\}) \subset N_{\lambda_6 A}(\{d_p \leq r_2\})$. This leads to $M(\beta_1 \cap \{c \leq r_2\}) \leq \lambda_7 A(r_2)^{n-1}$ as before and enable the induction process go through when $r_2 \gg A$.

We keep repeating this induction process until we reach the scale $r$. Then $d(\hat{p}, D') \leq r$. As $p$ is arbitrary, $D \subset N_r(D')$ and we are done by Lemma 8.2.

**Proposition 8.4.** Let $X$ be a complete metric space satisfying condition (CI) and let $Q, Q' \subset X$ be $(L, A)$-quasiflats with $\dim Q = n$. Suppose that $Q$ is $(\mu, b)$-rigid. Then there exist $A'$ and $\epsilon$ depending only on $X, L, A, n, b$ and $\mu$ such that either $d_H(Q, Q') \leq A'$, or

\begin{equation}
\limsup_{r \to \infty} \frac{d(B_p(r) \cap Q, Q')}{r} \geq \epsilon
\end{equation}

for some (hence any) $p \in Q$.

**Proof.** By a similar “going down on scale” argument as in Proposition 8.3, we can find $A'$ and $\epsilon$ as required such that either $Q \subset N_{A'}(Q')$ or (8.5) holds. However, in the first case we must have $\dim Q = \dim Q' = n$ as $Q$ is rigid. Thus $d_H(Q, Q') \leq A'$ by Lemma 8.2.

**8.2. Working with asymptotic cones.**

**Proposition 8.6.** Let $X$ be a complete metric space with Lipschitz bicombing and let $Q, Q' \subset X$ be $(L, A)$-quasiflats with $\dim Q = n$. Suppose $Q_\omega$ has full support in $X_\omega$ with respect to the reduced singular homology for any asymptotic cone $X_\omega$ with base points taking in $Q$. Then there exist $A'$ and $\epsilon$ depending only on $X, L, A, n, Q$ such that either $d_H(Q, Q') \leq A'$, or

\begin{equation}
\limsup_{r \to \infty} \frac{d_H(B_p(r) \cap Q, Q')}{r} \geq \epsilon
\end{equation}

for some (hence any) $p \in Q$.

By Proposition 6.12 and Proposition 7.4, the conditions on $Q$ in Proposition 8.6 and Proposition 8.4 are equivalent.
Proof. This follows from Lemma 8.8 and Lemma 8.9 below. □

Lemma 8.8. Let $X$ be a metric space with a Lipschitz bicombing. Let $Q', Q'' \subset X$ be bilipschitz flats with $\dim Q = n$. Suppose $Q$ has full support in $X$ with respect to reduced singular homology. If $Q$ is contained in a tubular neighborhood of $Q'$, then $Q$ is equal to $Q'$. In particular, their dimensions coincide.

Proof. After rescaling we may assume $Q \subset N_1(Q')$. Then $n = \dim(Q) \leq \dim(Q') = n'$. Let $\pi : X \to Q'$ be an $L$-Lipschitz retraction and fix a base point $p \in Q$. For large $r > 0$, choose a nontrivial singular cycle $\sigma_r \in C_{n-1}(Q \setminus B_p(r))$. As $X$ has convex geodesic bicombings, let $\alpha_r$ denote the singular $n$-chain induced by the geodesic homotopy between $\sigma_r$ and $\sigma'_r = \pi_{\#}\sigma_r$. In particular, $\partial \alpha_r = \sigma_r - \sigma'_r$. If $\tau_r$ is an $n$-chain in $Q'$ filling $\sigma'_r$, then $\tau_r + \alpha_r$ fills $\sigma_r$.

Let $\tau_r$ be an $n$-chain filling $\sigma_r$ in $Q$. Let $S_1$ (resp. $S_2$) be the subset of $Q$ (resp. $X$) made of points $q \in B_p(r)$ such that $[\tau_q]$ is non-trivial in $\tilde{H}_n(Q, Q - \{q\})$ (resp. $\tilde{H}_n(X, X - \{q\})$). Then $S_1 = S_2$ by the full support assumption. Note that $S_2 \subset \text{Im}(\tau_r' + \alpha_r)$ as $\tau_r' + \alpha_r - \tau_r$ bounds a singular chain. Thus $S_1 \subset \text{Im}(\tau_r' + \alpha_r)$. Since $d_H(\text{Im} \sigma_r, \text{Im} \sigma'_r) \leq L$, we conclude $Q \subset Q'$ by letting $r \to \infty$. The homology condition on $Q$ implies that $Q = Q'$. □

Lemma 8.9. Let $X, Q, Q'$ be as in Proposition 8.6. There exists a constant $\rho > 0$ depending only on the constants $L, A, n$, the function $\mu$ and the space $X$ with the following property. If $p \in Q$ is such that $d(p, Q') \geq \rho$, then there exists a point $q \in Q \cap B_p(\rho^2)$ with $d(q, Q') \geq d(p, Q') + 1$.

Proof. Suppose for contradiction that there are sequences $(Q_k)$ and $(Q'_k)$ of $(L, A)$-quasiflats in $X$ with the following properties:

- each $Q_k$ is $\mu$-rigid;
- there are points $p_k \in Q_k$ with $\lambda_k = d(p_k, Q'_k) \to \infty$ and such that $d(x_k, Q'_k) \leq \lambda_k + 1$ for all $x_k \in B_{p_k}(\lambda_k^2) \cap Q_k$.

We pass to an asymptotic cone $(X_\omega, p_\omega) = \omega \lim(\frac{1}{\lambda_k} \cdot X, p_k)$. Then $d(p_\omega, Q'_\omega) = 1$ and $Q_\omega \subset N_1(Q'_\omega)$. By Lemma 8.8, $Q_\omega = Q'_\omega$, which yields a contradiction. □

Now we indicate briefly about the asymptotic cone approach for analogue of Proposition 8.3.
Definition 8.10. A family $\mathcal{F} = \{ D_i \subset X_i \}_{i \in I}$ of $n$-quasidisk is Morse if their quasi-isometric constants are uniform, and for any asymptotic cone $Q_\omega \subset X_\omega$, and any $x_\omega \in Q_\omega$, the inclusion of pairs $(Q_\omega, Q_\omega \setminus x_\omega) \rightarrow (X_\omega, X_\omega \setminus \{ x_\omega \})$ induces a monomorphism of homology groups.

Proposition 8.11. Suppose $X$ is a metric space with convex geodesic bicombing. Then

1. A family of $n$-quasidisks is Morse in the sense of Definition 8.10 if and only if they are $(\mu, b)$-rigid for some $\mu$ and $b$ as in Definition 6.13.
2. The analogue of Proposition 8.3 holds for a family of Morse $n$-quasidisks in the sense of Definition 8.10.

Assertion (1) can be proved by repeating the argument in Section 7. For assertion (2), we argue by contradiction and it follows essentially from Lemma 8.8. We leave the details to the reader.

9. Cycles close to Morse quasiflats

The main goal of this section is to prove Proposition 9.1, which is a key step in the proof of Theorem 1.8. However, for readers who prefer to work with the stronger condition that $X$ has convex geodesic bicombing, this section can be skipped (see Remark 9.2). Readers whose main interests lie in the visibility results for Morse quasiflats can also skip this section.

Proposition 9.1. Suppose $X$ is a proper metric space satisfying condition (SC1$_{n-1}$). Let $Q \subset X$ be an $(L, A)$-quasiflat which is an $A_1$-homology retract in the sense of Definition 5.17 and is $(\mu, A_2)$-rigid in the sense of Definition 6.13. There exists a constant $C_0 > 0$ depending only on $L, A, A_1, \dim(Q)$ and $X$ such that the following holds.

For a given $C, \rho$ and $b_1$, there exists $R = R(A_1, A_2, \mu, C, \rho, b_1, d(p, Q), Q)$ such that for any $R \geq R$ and any $\tau \in I_{n-1}(\overline{B}_p(b_1R) \setminus N_{\rho R}(Q))$ satisfying

- $M(\tau) \leq C \cdot R^n$;
- $\sigma := \partial \tau \in I_{n-1}(N_{2\rho R}(Q))$ and $M(\sigma) \leq C \cdot R^{n-1}$;

we have $\text{Fill}(\sigma) \leq C_0 \rho \cdot R \cdot M(\sigma)$.

Remark 9.2. Under a stronger assumption that $X$ has a Lipschitz bicombing, Proposition 9.1 follows from Lemma 6.23, Lemma 7.1, Proposition 6.12 and Lemma 7.6. However, as this route passes through asymptotic cones, we loose the dependence of $R$ on other parameters in Proposition 9.1.
We assume without loss of generality that $\rho \ll 1$, $R \gg 1$, $\rho R \geq \max\{100A, 1\}$ and $A = A_1 = A_2 > 1$.

Let $n = \dim(Q)$. Suppose $Q$ is represented by $\Phi : E^n \to Q$. Without loss of generality, we assume $q$ is $L$-Lipschitz, $p \in Q$ and $b_1 = 1$. Throughout this subsection, $\lambda, \lambda_1, \lambda_2, \ldots$ will be constants which depend only on $L, A, n, \mu$ and $X$. We will also write $k_1 \leq k_2$ if $k_1 \leq \lambda' k_2$ for some $\lambda'$ depending only on $L, A, n, \mu$ and $X$.

Let $\mathcal{C}$ be the collection of elements $\alpha$ in $I_{n,c}(X)$ such that $\partial \alpha = \sigma - \bar{\sigma}$ for some $\bar{\sigma} \in I_{n,c}(X)$ with $\text{spt}(\bar{\sigma}) \subset N_A(Q)$. Let $\{\alpha_i\}$ be a mass minimizing sequence in $\mathcal{C}$. As $Q$ is an $A$-homology retract, $\mathcal{C}$ is a non-empty collection, moreover, we can assume $M(\alpha_i) \leq AM(\sigma)\rho \cdot R$. Let $M = \inf_{i \geq 1} M(\alpha_i)$.

Take $\epsilon_0 = 3A$. For each $\alpha_i$, let $\alpha'_i = \alpha_i \setminus \{d_Q \geq \epsilon_1\}$ such that $2A < \epsilon_1 < 3A$ and

$$M(\partial \alpha'_i) \leq M(\sigma) + M(\sigma)\rho R.$$

We replace $\alpha'_i$ by a minimal filling of $\partial \alpha'_i$. Up to passing to a subsequence, we suppose $\alpha_i$ weakly converges to $\alpha \in I_{n,loc}(X)$, moreover, $\alpha_i$ converges to $\alpha$ in the local flat topology \cite[Theorem 2.3]{KL18}. Then $M(\alpha) \leq M$.

**Lemma 9.3.** There exist constants $\{\lambda_i\}_{i=1}^2$ depending only on $L, A, n$ and $X$ such that the following holds.

1. $\partial \alpha = \sigma - \sigma'$ where $\text{spt}(\sigma') \subset N_{3A}(Q)$;
2. $M(\alpha) \leq \lambda_1 M(\sigma) \rho \cdot R$;
3. for any point $x \in \text{spt} \alpha$ such that $h = d(x, Q)$ satisfies $\frac{\rho R^2}{2} \geq h > 6A$, we have $M(\alpha \cup \bar{B}_x(\frac{h}{2})) \geq \lambda_2 \cdot h^n$;
4. there exists a depending only on $L, A, n, C$ and $X$ such that $\text{spt} \alpha \subset N_{aR}(Q)$.

**Proof.** (1) and (2) are clear. For (3), note that since each $\alpha'_i$ is a minimal filling of $\partial \alpha'_i$ and $\alpha'_i$ converges to $\alpha$ in the local flat topology, we know $\alpha \cup \bar{B}_x(\frac{h}{2})$ is a minimal filling of $\partial(\alpha \cup \bar{B}_x(\frac{h}{2}))$. Thus (3) follows. For (4), let $x \in \text{spt} \alpha$ be a point such that $d(x, Q) = aR \geq R$. Since we are assuming $\rho \ll 1$, $\bar{B}_x(aR - \frac{R}{2}) \cap \text{spt} \partial \alpha = \emptyset$. By the same argument as in (3), we know $a^n R^n \leq M(\alpha \cup \bar{B}_x(\frac{h}{2})) \leq M(\alpha) \leq \lambda_1 M(\sigma) \rho \cdot R \leq \lambda_1 C \rho \cdot R^n$. Thus (4) follows. \qed
Lemma 9.4. The following estimate holds for \( \alpha \). For \( 3A \leq r_1 < r_2 < \rho R \):

\[
\frac{M(\alpha \ll \{d_Q \leq r_2\})}{M(\alpha \ll \{d_Q \leq r_1\})} \geq \left( \frac{r_2}{r_1} \right)^{\kappa}.
\]

where \( 0 < \kappa \leq 1 \) depends only on \( L, A, n \) and \( X \).

Proof. Let \( 3A \leq r < \rho R \). By Definition 5.17, there exists \( \beta_r \) such that

\[
M(\beta_r) \leq ArM(\langle \alpha, d_Q, r \rangle) \quad \text{and} \quad \text{spt}(\partial \beta_r - \langle \alpha, d_Q, r \rangle) \subset NA(Q).
\]

Then

\[
M(\alpha - \alpha \ll \{d_Q \leq r\} + \beta_r) \geq M(\alpha) - M(\beta_r).
\]

Let \( f(r) = M(\alpha \ll \{d_Q \leq r\}) \). By coarea inequality, we have \( M(\langle \alpha, d_Q, r \rangle) \leq f'(r) \) for a.e. \( r \). Then

\[
\ln(f(r_2)) - \ln(f(r_1)) \geq \int_{r_1}^{r_2} \frac{f'(t)}{f(t)} dt \geq \int_{r_1}^{r_2} \frac{1}{At} dt.
\]

Then the lemma follows. \( \square \)

It follows from that for \( 3A \leq h < \rho R \),

\[
M(\alpha \ll \{d_Q \leq h\}) \leq \left( \frac{h}{\rho R} \right)^{\kappa} M(\alpha) \leq \left( \frac{h}{\rho R} \right)^{\kappa} \cdot (\lambda M(\sigma) \rho \cdot R)
\]

where the first inequality follows from Lemma 9.4 and the second inequality follows from Lemma 9.3 (2).

Let \( \sigma_h := \langle \alpha, d_Q, h \rangle \). It follows from the coarea inequality and \( 9.6 \) that for any \( 3A \leq h < \rho R \), there exists \( h' \in [h, 1.01h] \) such that

\[
M(\sigma_{h'}) \leq \left( \frac{1}{h'} \right) \cdot \left( \frac{h'}{\rho R} \right)^{\kappa} \cdot (\lambda M(\sigma) \rho \cdot R).
\]

Let \( Z \) be a maximal \( \frac{h'}{R} \)-separated net in \( \text{spt}(\sigma_{h'}) \). It follows from Lemma 9.3 (4) and \( 9.6 \) that

\[
\#Z \leq \left( \frac{1}{h'} \right)^{\kappa} \cdot \left( \frac{h'}{\rho R} \right)^{\kappa} \cdot (\lambda M(\sigma) \rho \cdot R)
\]

Lemma 9.9. There exist functions \( \phi, \varphi : [0, \infty) \rightarrow [0, \infty) \) depending only on \( L, A, n, C, \rho \) and \( X \) such that

- \( \lim_{x \to \infty} \varphi(x) = 0 \);
- \( \phi \) is decreasing;
- \( \varphi(R)R \leq h < \rho R \), we have \( \text{spt}(\sigma_h) \subset B_p(\phi(\frac{h}{R})R) \).
Proof. Let $\epsilon = \frac{h}{R}$. Take $\delta > \frac{100A}{R}$ whose value will be determined later ($\delta$ is much smaller than $\epsilon$). By (9.6) and the coarea inequality, up to varying $\epsilon$ a bit, we assume $M(\sigma_{\partial R}) \lesssim \left( \frac{1}{\delta R} \right) \cdot \left( \frac{\delta R}{\rho R} \right)^{\kappa} \cdot M(\alpha)$.

Let $\hat{\alpha} = \alpha \setminus \{ dQ \geq \delta R \}$. Define $c : X \to \mathbb{R}$ by $c(\cdot) = d(\cdot, \text{spt}(\sigma))$.

Let $x \in \text{spt}(\sigma)$ and let $r_x = \frac{c(x)}{R}$. Let $r \in (r_x/2, r_x)$ such that $M(\langle \hat{\alpha}, c, r \rangle) \lesssim \frac{M(\delta)}{r x R}$ and $M(\langle \sigma_{\sigma R}, c, r \rangle) \lesssim \frac{M(\sigma_{\partial R})}{r x R}$. Note that

\[
\partial(\hat{\alpha} \setminus \{ c \geq r \}) - \sigma = \langle \hat{\alpha}, c, r \rangle + (\sigma_{\partial R}) \setminus \{ c \geq r \} =: \xi.
\]

Let $\zeta$ be a minimal filling of $\langle \sigma_{\sigma R}, c, r \rangle$. Let $\xi_1 = \langle \hat{\alpha}, c, r \rangle - \zeta$ and $\xi_2 = \xi - \xi_1$. As each $\xi_i$ is a cycle, for $i = 1, 2$, let $\beta_i \in \mathcal{I}_{n,c}(X)$ be as in Definition 5.17 such that $\text{spt}(\beta_i) = N_A(Q)$.

Let $\bar{\alpha} = \hat{\alpha} \setminus \{ c \geq r \} + \beta_1 + \beta_2$. Then $\text{spt}(\partial \bar{\alpha} - \sigma) \subset N_A(Q)$. It follows that $M(\bar{\alpha}) \geq M \geq M(\alpha)$.

However, when $r_x > \epsilon$, $M(\hat{\alpha} \setminus \{ c \geq r \}) \leq M(\bar{\alpha}) - M(\hat{\alpha} \setminus \bar{B}(x, \frac{h}{2}))$, thus by Lemma 9.3 (3),

\[
M(\beta_1) + M(\beta_2) \geq M(\hat{\alpha} \setminus \bar{B}(x, \frac{h}{2})) \geq \lambda_2 \cdot h^n + \lambda_2 \epsilon^n \cdot R^n.
\]

Now we estimate $M(\beta_i)$. By above discussion, $M(\langle \sigma_{\sigma R}, c, r \rangle) \lesssim C \cdot \frac{\delta^{\kappa-1} R^{n-2}}{\rho^{\kappa-1} r_x}$.

By Theorem 3.10 (1), $M(\zeta) \lesssim \left( \frac{C^{\delta^{\kappa-1}}}{{\rho^\kappa-1} r_x} \right)^{\frac{n-1}{n-2}} R^{n-1}$ and $\text{spt}(\zeta)$ is contained in the $\lambda_3 \left( \frac{C^{\delta^{\kappa-1}}}{{\rho^\kappa-1} r_x} \right)^{\frac{n-1}{n-2}} R$-neighborhood of $\text{spt}(\langle \sigma_{\sigma R}, c, r \rangle)$.

Define $r_0$ such that $\lambda_3 \left( \frac{C^{\delta^{\kappa-1}}}{{\rho^\kappa-1} r_0} \right)^{\frac{n-1}{n-2}} = \delta$. If $r_x > r_0$, then $\text{spt}(\beta_2) \subset N_{2\delta R}(Q)$ and $\text{spt}(\beta_1) \subset N_{\alpha R}(Q)$ (by Lemma 9.3 (4)).

In addition $\delta R > 3A$, then

\[
M(\beta_1) \leq A a R M(\xi_1) \leq a A R (M(\langle \hat{\alpha}, c, r \rangle) + M(\zeta)) \lesssim \frac{a M(\alpha)}{r_x} + a R M(\zeta)
\]

and $M(\beta_2) \leq A \cdot (2\delta R) \cdot M(\xi_2) \leq 2 a \delta A R (M(\alpha_{\delta R}) + M(\zeta))$. This together with (9.10) imply

\[
\left[ \frac{1}{r_x} + \left( \frac{\delta^{\kappa-1}}{\rho^{\kappa-1} r_x} \right)^{\frac{n-1}{n-2}} \right] + \left[ \frac{\delta^{\kappa-1}}{\rho^{\kappa-1}} \cdot \delta + \left( \frac{\delta^{\kappa-1}}{\rho^{\kappa-1} r_x} \right)^{\frac{n-1}{n-2}} \cdot \delta \right] \geq \Lambda \epsilon^n
\]

where $\Lambda$ is a constant depending only on $L, A, n, C$ and $X$. Choose $\delta$ such that $\frac{\delta^{\kappa}}{\rho^{\kappa-1}} = \frac{\Lambda \epsilon^n}{2}$. If $\epsilon$ is large enough such that resulting $\delta R \geq 3A$, then above discussion goes through. This gives the function $\varphi$ in the lemma. Let $r_1$ such that

\[
\frac{1}{r_1} + \left( \frac{\delta^{\kappa-1}}{\rho^{\kappa-1} r_1} \right)^{\frac{n-1}{n-2}} + \left( \frac{\delta^{\kappa-1}}{\rho^{\kappa-1} r_1} \right)^{\frac{n-1}{n-2}} \cdot \delta = \frac{1}{2} \cdot \Lambda \epsilon^n
\]
Then \( r_x \leq \max\{r_0, r_1\} \), which gives the function \( \phi \) in the lemma. \( \square \)

**Lemma 9.11.** There exist constants \( \{\lambda_i\}_{i=4}^6 \) depending only on \( L, A, n \) and \( X \) such that the following holds. Suppose \( 3A \leq h \leq \rho R \) and suppose \( \sigma_h \) satisfies (9.7) with \( h' \) replaced by \( h \). Then we can find cubical chain \( \sigma'_h \in I_{n-1}(\mathbb{E}^n) \) such that the following holds:

1. \( M(\sigma'_{h'}) \leq AHM(\sigma_h) \);
2. for any \( \delta > 0 \), there exists \( R = R(Q, C, \delta) \) such that if \( R \geq R \) and \( h \geq \max\{A, \delta R\} \), then \( \sigma'_h \) can be filled inside \( N_{\lambda h}(S') \) where \( S' = \pi(\text{spt}(\sigma_h)) \) and \( \pi : X \to \mathbb{R}^n \) is in Section 5.2;
3. \( L^n(N_{\lambda h}(S')) \leq \left( \frac{h}{\rho R} \right)^{\kappa} \cdot (\lambda_5 M(\sigma) \cdot R) \);
4. for any \( \delta > 0 \), there exists \( R' = R'(\delta) \) such that if \( \delta R \leq h_1 \leq h_2 \leq \rho R \), then \( \text{Fill}(\sigma'_{h_1} - \sigma'_{h_2}) \leq \lambda_6 \left( \frac{h_2}{\rho R} \right)^{\kappa} RM(\sigma) \);
5. \( \text{Fill}(\sigma_h - \iota(\sigma_h')) \leq \left( \frac{h}{\rho R} \right)^{\kappa} \cdot (A\lambda M(\sigma) \rho \cdot R) \).

We use \( L^n \) for \( n \)-dimensional Lebesgue measure.

**Proof.** As \( Q \) is an \( A \)-homology retract, we define \( \sigma_{h'} \) be a cubical chain such that \( \sigma_h - \iota(\sigma_{h'}) = \partial H \) for \( H \in I_{n,c}(X) \) with \( M(H) \leq AhM(\sigma_h) \) and \( \text{spt}(H) \subset N_{Ah}(\text{spt}(\sigma_h)) \). (1) follows from the definition of \( A \)-homology retract.

Now we prove (2). Represent \( Q \) as a map \( \Phi : \mathbb{R}^n \to X \).

Let \( \phi \) and \( \varphi \) be as in Lemma 9.9. Choose \( R_0 \) such that \( \varphi(R) < \delta \) whenever \( R > R_0 \). Let \( \alpha_0 = \alpha \sqcup \{d_Q \geq h_1\} + \tau \). When \( \rho \leq h \leq \max\{A, \delta R\} \) and \( R > R_0 \), we have

- \( M(\sigma'_{h}) \leq \left( \frac{\delta}{\rho} \right)^{\kappa-1} \lambda CA \cdot R^{n-1} \) (by (9.7));
- \( M(H + \alpha_0) \leq (\rho^{1-\kappa} \lambda C + AC + C) \cdot R^n \);
- \( \text{spt}(\sigma_h) \subset B_{p'}(A\phi(\delta) R) \) for some \( p' \in \mathbb{R}^n \) and \( \text{spt}(H + \alpha_0) \subset B_{p}(\phi(\delta) R) \) (by Lemma 9.9).

Let \( M = \max\{\delta^{1-\kappa} \lambda CA, \lambda C + AC + C, A\phi(\delta)\} \) and let \( \mu_M \) be as in Definition 6.13. Choose \( \tilde{R} > R_0 \) such that \( \mu_M(\tilde{R}) < \frac{A}{M} \) whenever \( R \geq \tilde{R} \).

Let \( W \) be the support of the canonical filling of \( \sigma'_h \). Let \( a \) and \( W_a \) as in Definition 6.13. Then for \( R > \tilde{R} \),

\[ \Phi(W_a) \subset N_h(\text{spt}(H + \alpha_0)) \subset N_h(\text{spt}(H)) \subset N_{h+Ah}(\text{spt}(\sigma_h)) \]

where the second \( \subset \) follows from \( N_h(\text{spt}(\alpha_0)) \cap N_A(Q) = \emptyset \). Thus \( \pi(\Phi(W_a)) \subset N_{\lambda'(1+A)h}(S') \) where \( S' = \pi(\text{spt}(\sigma_h)) \) and \( \lambda' = \lambda'(L, A, n) \)
is the Lipschitz constant of \( \pi \). Note that there is \( a' = a'(a, L, A, n, X) \) such that \( W_{a'} \subset \pi(\Phi(W_a)) \). Thus \( W_{a'} \subset N_{\lambda'(1+A)h}(S') \). However, \( W \setminus W_{a'} \subset N_a(\partial W) \subset N_{A+a'}(S') \). Thus (2) follows.

For (3), let \( Z \) be as in (9.8). Then \( S' \subset \pi(N_{\frac{1}{2}}(Z)) \subset N_{\lambda h}(Z) \). Thus

\[
\mathcal{L}^n(N_{\lambda h}(S')) \leq \mathcal{L}^n(N((\lambda + \lambda_h)h)(Z)) \lesssim (\#Z) \cdot h^n \lesssim \frac{1}{h^n} \left( \frac{h}{\rho R} \right) \kappa \text{RM}(\sigma) h^n
\]

by (9.8). Then (3) follows.

Now we prove (4). As \( \text{Fill}(\sigma_{h_1} - \sigma_{h_2}) \leq \text{M}(\alpha \{ d_Q \leq h_2 \}) \leq \left( \frac{h_2}{\rho R} \right) ^\kappa \cdot (\lambda \text{M}(\sigma) \rho \cdot R) \), we know \( \text{Fill}(\pi \# \sigma_{h_1} - \pi \# \sigma_{h_2}) \lesssim \left( \frac{h_2}{\rho R} \right) ^\kappa \text{M}(\sigma) \cdot R \). As \( Q \) is an \( A \)-homology retract, \( \text{Fill}(\pi \# \sigma_{h_1} - \sigma'_{h_1}) \leq \text{AM}(\sigma_{h_1}) \leq A \left( \frac{\kappa}{\rho} \right) ^{\kappa-1} \cdot \lambda \text{M}(\sigma) \lesssim \frac{\kappa}{R h} \cdot \left( \frac{h_2}{\rho R} \right) ^\kappa \text{RM}(\sigma) \) for \( i = 1, 2 \). Thus (4) follows by choosing \( R' \) such that \( R' \delta = 1 \). (5) follows from (9.7) and the definition of homology retract.

Let \( \delta > 0 \) and \( R \geq \max \{ R(Q, C, \delta), R'(\delta) \} \) be as in Lemma 9.11.

For each \( \max \{ A, \gamma R \} \leq hR \leq \rho R \), define \( \sigma''_h = \sigma'_{hR} \) where \( hR \in [hR, 1.1hR] \) is chosen such that \( \sigma_{hR} \) satisfies (9.7).

Let \( \varphi \) be as in Lemma 9.11. By Lemma 9.9, (9.7) and Lemma 9.11, for any \( \max \{ A, \varphi(R) R, \delta R \} \leq hR \leq \rho R \),

- \( \text{diam(spt } \sigma''_h) \leq A \varphi(h) R \) and \( \text{M}(\sigma''_h) \leq \lambda_7 \left( \frac{h}{\rho} \right) ^{\kappa-1} \text{M}(\sigma) \);
- there is an open set \( O_h \subset \mathbb{R}^n \) with \( \mathcal{L}^n(O_h) \leq \lambda_8 \left( \frac{h}{\rho} \right) ^\kappa \text{RM}(\sigma) \) such that \( \sigma''_h \) is the boundary of an element in \( I_{n,c}(O_h) \);
- for \( \max \{ \frac{R}{R}, \delta R \} \leq h_1 \leq h_2 \leq \rho \), \( \text{Fill}(\sigma''_{h_1} - \sigma''_{h_2}) \leq \lambda_9 \left( \frac{h_2}{\rho} \right) ^\kappa \text{RM}(\sigma) \).

\textbf{Proof of Proposition 9.4} Let \( \text{M}(\sigma) = t R^{n-1} \). By the isoperimetric inequality that \( \text{Fill}(\sigma) \leq \text{DM}(\sigma) \frac{1}{n-1} = \text{Dt} \frac{1}{n-1} \cdot R^n \). Suppose

\[
(9.12) \quad t < \rho_0 := \left( \frac{\rho}{D} \right) ^{n-1}.
\]

Then \( \text{Fill}(\sigma) \leq \text{Dt} \frac{1}{n-1} R \cdot t R^{n-1} \leq \rho \text{RM}(\sigma) \). Now we assume \( t \geq \rho_0 \).

We rescale the metric on \( X \) by a factor \( \frac{1}{R} \) and rescale \{ \( \sigma''_h \) \} accordingly. By applying Lemma 9.13 below with \( g(h) = \lambda_7 \left( \frac{h}{\rho} \right) ^{\kappa-1} \),
\[
f(h) = \max\{\lambda_h, \lambda_0\} \left(\frac{h}{\rho}\right) \kappa \text{ and } \phi(h) \text{ as in Lemma 9.9, we choose } \delta = \delta(\rho, \rho, f, g, \phi) \text{ which depends only on } \rho, C, L, A, \mu, X \text{ and } n. \\

\text{Take } R \text{ such that } R \geq \max\{R(Q, C, \delta), R'(\delta)\}. \text{ By Lemma 9.11 the family } \{\sigma''_{\delta, h}\}_{\delta \leq h \leq \rho} \text{ satisfies the assumptions of Lemma 9.13 (after rescaling by } \frac{1}{R}). \text{ By the choice of } \delta, \text{ there exists } \rho \geq h_0 \geq \delta \text{ such that } \text{Fill}(\sigma''_{h_0}) \leq \rho_0 R^m \leq \rho t R^n. \text{ Then}
\]

\[
\text{Fill}(\sigma) \leq \text{Fill}(\sigma - \sigma_{h_0} R) + \text{Fill}(\sigma_{h_0} R - \iota(\sigma_{h_0})) + \text{Fill}(\iota(\sigma''_{h_0})).
\]

Note that \text{Fill}(\sigma - \sigma_{h_0} R) \leq M(\alpha) \leq \Lambda M(\sigma) \rho \cdot R, \text{ Fill}(\sigma_{h_0} R - \iota(\sigma''_{h_0})) \leq \left(\frac{h_0}{\rho}\right)^\kappa \cdot (A \Lambda M(\sigma) \rho \cdot R) \text{ by Lemma 9.11 (5) and Fill}(\iota(\sigma''_{h_0})) \lesssim \text{Fill}(\sigma''_{h_0}) \leq \rho t R^n \lesssim \rho M(\sigma) R. \text{ Thus the proposition follows. } \square

**Lemma 9.13.** Let \( f, g \) and \( \phi \) be functions from \( \mathbb{R}_{\geq 0} \) to \( \mathbb{R}_{\geq 0} \) such that \( \lim_{x \to 0} f(x) = 0 \). Let \( \rho > 0 \). For any \( \epsilon > 0 \), there exists \( \delta = \delta(\epsilon, \rho, f, g, \phi) \) such that the following holds. Suppose \( \{\zeta_h\}_{\delta \leq h \leq \rho} \) is a collection of elements in \( I_{n-1, \epsilon}(\mathbb{E}^n) \) such that

1. \( \text{diam}(\zeta_h) \leq \phi(h) \);
2. \( \text{M}(\zeta_h) \leq g(h) \);
3. \( \text{Fill}(\zeta_{h_1} - \zeta_{h_2}) \leq f(h_2) \) for any \( h_2 \geq h_1 \geq \delta \);
4. \( \zeta_h \) bounds in an open set \( O_h \) with \( \mathcal{L}^n(O_h) \leq f(h) \) for any \( h \geq \delta \).

Then there exists \( \rho \geq h_0 \geq \delta \) such that \( \text{Fill}(\zeta_{h_0}) \leq \epsilon \).

**Proof.** We argue by contradiction and suppose there is a monotone decreasing sequence \( \delta_i \to 0 \) such that for each \( i \), we have a sequence \( \{\zeta_{h_i}\}_{\delta_i \leq h_i \leq \rho} \) satisfying conditions (1) - (4) of the lemma and \( \text{Fill}(\zeta_{h_i}) \geq \epsilon \) for any \( i \) and \( \delta_i \leq h_i \leq \rho \). For \( j \leq i \), let \( \zeta_{ij} = \zeta_{h_j} \). For each fixed \( j \), conditions (1) and (2) imply that \( \zeta_{ij} \) sub-converges in the flat distance as \( i \to \infty \). By condition (3), we can apply a diagonal argument to \( \{\zeta_{ij}\}_{j \geq 1} \) to extract a subsequence \( \{\zeta_{n_j, j}\}_{j \geq 1} \) which is Cauchy in the flat distance. Suppose the canonical filling of \( \zeta_{n_j, j} \) is represented by \( u_j \in L^1(\mathbb{E}^n, \mathbb{R}) \). Then \( \{u_j\} \) is Cauchy sequence in the \( L^1 \)-distance and \( \text{spt}(u_j) \subset O_{h_j} \). Suppose \( u_i \overset{L^1}{\to} u \). Let \( E_i = \text{spt } u_i \) and \( E = \text{spt } u \).

Condition (4) implies that \( \mathcal{L}^n(E_i) \to 0 \). Thus \( u \cdot \chi_{E \setminus E_i} \overset{L^1}{\to} u \). However, \( \|u \cdot \chi_{E \setminus E_i}\|_1 \leq \|u_i - u\|_1 \to 0 \). Thus \( u = 0 \) and \( \|u_i\|_1 \to 0 \), which contradicts that \( \text{Fill}(\zeta_{n_j, j}) \geq \epsilon \) for all \( j \). \( \square \)
10. Neck decomposition and its immediate consequences

In this section we introduce the “coarse neck decomposition” as in Lemma 10.1, which is a key step towards visibility in later sections. We also prove Theorem 1.8 and Theorem 1.9.

**Lemma 10.1.** Suppose $X$ satisfies (SCI$_n$). Let $Q \subset X$ be an $n$-dimensional $(L, A)$-quasiflat with CNP (cf. Definition 7.7). Let $C$ be the constant in Proposition 5.14. Let $p \in X$ be a base point.

For given $\epsilon, b > 0$, there exists $R$ depending only on $b, \epsilon, d(p, Q), L, A, n, X$ and the CNP parameter of $Q$ such that the following holds for any $R \geq R$. Take $T \in I_{n,c}(X)$ spt($\partial T$) $\subset N_C(Q)$, spt($T$) $\subset B_p(bR)$ and $M(T) \leq b \cdot R^n$. Suppose there exists $T' \in I_{n,c}(X)$ such that $\partial T = \partial T'$ with spt($T'$) $\subset N_C(Q)$ and $M(T') \leq bR^n$. Then $T$ admits a piece decomposition $T = U + V$ such that the following holds.

1. Let $\sigma := \partial U - \partial T = -\partial V$. Then Fill($\sigma$) $\leq \epsilon \cdot R^n$.
2. Let $\tau$ be a minimal filling of $\sigma$. Then spt($U + \tau - T'$) $\subset N_{\epsilon R}(Q)$.
3. Fill($U + \tau - T'$) $\leq \epsilon \cdot R^{n+1}.$

$D$ is the lower density bound for minimizing cycles.

**Proof.** Take small constants $\rho$ and $\delta$ whose value will be determined later. Let $T_{x,y} = T \setminus \{x < d_Q < y\}$. We define a piece $\hat{T}$ of $T$ as follows. If $M(T_{x,y}) \leq \delta M(T)$, then we define $\hat{T} = T_{x,y}$. Now assume $M(T_{x,y}) > \delta M(T)$. If $M(T_{x,y}) \leq \delta M(T)$, then we define $\hat{T} = T_{x,y}$. If $M(T_{x,y}) > \delta M(T)$, then we look at $M(T_{x,y})$ and repeat the previous process. This process terminates after at most $\lceil \frac{1}{\delta} \rceil$ steps and we obtain $\hat{T} = T_{x,y}$, with $M(\hat{T}) \leq \delta M(T)$ and $1 \leq n \leq \lceil \frac{1}{\delta} \rceil$. Thus there exists $\frac{\rho}{2^\delta} < \hat{\rho} < \frac{\rho}{2^{n+\delta}}$ such that $\sigma := (\hat{T}, d_Q, \hat{\rho})$ satisfies $M(\sigma) \leq \frac{M(\hat{T})}{\rho/2^n} \leq 2^n \delta M(T)$. Let $U := T \setminus \{d_Q < \hat{\rho}\}$ and $V = T \setminus \{d_Q > \hat{\rho}\}$. Then

- $spt(U) \subset N_{\epsilon R}(Q)$ and spt($V$) $\cap N_{\rho_1 R}(Q) = \emptyset$ with $\rho_1 = \rho \cdot 2^{-\frac{1}{\delta}}$.
- $M(\sigma) \leq 2^{\frac{1}{2}} \rho^{-1} \delta b \cdot R^{n-1}$.

Let $b' = \max\{b, 2^{\frac{1}{2}} \rho^{-1} \delta b\}$. Let

$$R := \sup_{1 \leq n \leq \lceil \frac{1}{\delta} \rceil} \{R(b', \frac{\rho}{2^n}, b_1, d(p, Q), Q)\}.$$
where \( R(b', \frac{\rho}{2}, b, d(p, Q), Q) \) is as in Corollary 7.8. By Corollary 7.8
\[
(10.2) \quad \text{Fill}(\sigma) \leq C_0 \rho \mathcal{M}(\sigma) \leq C_0 \frac{\rho}{2n-1} 2^n \delta M(T) = 2C_0 \delta M(T)
\]
for \( C_0 \) depending only on \( X \) and \( Q \).

Let \( \beta = U + \tau - T' \). Then \( \mathcal{M}(\beta) \leq \mathcal{M}(U) + \mathcal{M}(\tau) + \mathcal{M}(T') \leq \mathcal{M}(U) + \mathcal{M}(V) + \mathcal{M}(T') \leq \mathcal{M}(T) + \mathcal{M}(T') \). By (10.2), \( \text{spt}(\beta) \subset N(\delta_1 + \rho)R(Q) \)
where \( \delta_1 = \left( \frac{2C_0 \delta}{D} \right)^\frac{1}{n} \). By Corollary 5.15
\[
(10.3) \quad \text{Fill}(\beta) \leq C(\delta_1 + \rho)R(M(T) + M(T'))
\]
The lemma follows from (10.2) and (10.3) by choosing \( \rho \) and \( \delta \) small. \( \Box 

**Definition 10.4.** A \( n \)-dimensional quasiflat \( Q \) in a metric space \( X \) has
coarse piece property if there exist \( a > 0 \) and a function \( (\mathbb{R}_{>0})^2 \to \mathbb{R}_{\geq 0} : R = R(\epsilon, b) \) such that for given \( \epsilon, b > 0 \), the following holds for any \( R \geq \bar{R} \). Take \( T \in I_{n,c}(X) \) with \( \text{spt}(\partial T) \subset N_a(Q) \), \( \text{spt}(T) \subset \tilde{B}_R(bR) \) and \( \mathcal{M}(T) \leq b \cdot R^n \). Suppose there exists \( T' \in I_{n,c}(X) \) such that \( \partial T = \partial T' \) with \( \text{spt}(T') \subset N_a(Q) \) and \( \mathcal{M}(T') \leq b \cdot R^n \). Then \( T \) admits a piece decomposition \( T = U + V \) such that the following holds:

1. Let \( \sigma =: \partial U - \partial T = -\partial V \). Then \( \text{Fill}(\sigma) \leq \epsilon \cdot R^n \).
2. Let \( \tau \) be a minimal filling of \( \sigma \). Then \( \text{Fill}(U + \tau - T') \leq \epsilon \cdot R^{n+1} \).

**Theorem 10.5.** Let \( Q : \mathbb{R}^n \to X \) be an \((L, A)\)-quasiflat in a proper metric spaces \( X \). Consider the following conditions:

1. \( Q \) is \((\mu, b)\)-rigid (cf. Definition 6.13).
2. \( Q \) has super-Euclidean divergence (cf. Definition 6.21).
3. \( Q \) has coarse neck property (cf. Definition 7.7).
4. \( Q \) has coarse piece property (cf. Definition 10.4).

Then the following hold:

(a) (1) and (2) are equivalent if \( X \) satisfies \((\text{CI}_{n-1})\);
(b) (1), (2) and (3) are equivalent if \( X \) satisfies \((\text{SCI}_{n-1})\);
(c) (1), (2), (3) and (4) are equivalent if \( X \) satisfies \((\text{SCI}_n)\).

Moreover, all the above equivalence give uniform control on the parameters (e.g. if \( X \) satisfies \((\text{SCI}_{n-1}) \) and \( Q \) has super-Euclidean divergence, then \( Q \) has CNP with parameters of CNP depends only on parameters of super-Euclidean divergence and \( X \)).

In the following proof, \( a_1, a_2, \ldots \) will be constants depending only on \( L, A, n \) and \( X \).
Proof. (a) is Lemma 6.23. For (b), (2) \(\Rightarrow\) (3) is Proposition 5.14 and Proposition 9.1. Now we show (3) \(\Rightarrow\) (1). Let \(\Phi, \varphi, \tau\) and \(\nu\) be as in Definition 6.13. We apply Lemma 10.1 with \(T' = \nu(\nu), T = \tau\) and \(\partial T' = \nu(\varphi)\) to obtain \(\tau = U + V\) and \(\sigma\) as in Lemma 10.1. Note that we only used condition (SCL\(_{n-1}\)) so far as one only needs (SCL\(_n\)) for Lemma 10.1 (3).

Let \(\pi : X \to \mathbb{R}^n\) be the Lipschitz map in Section 5.2. Let \(\alpha = U + \sigma\). As \(\text{Fill}(\partial V) \leq \epsilon R^n, \text{spt}(\sigma) \subset N'_\epsilon(\text{spt}(\partial \sigma))\) where \(\epsilon' = (\frac{1}{\delta})^{\frac{1}{\delta}} R\). Thus
\[
(10.6) \quad \text{spt}(\alpha) \subset N'_\epsilon(\text{spt}(U)) \subset N'_\epsilon(\text{spt}(T)).
\]
Let \(h \in I_{n, c}(\mathbb{R}^n)\) be such that \(\partial h = \partial \pi_\# \alpha - \beta\), \(\beta\) is cubical and \(\text{spt}(h) \subset N'_{\alpha_1}(\text{spt}(\partial \pi_\# \alpha))\). Set \(\alpha_\square = \pi_\# \alpha + h\) and \(T'_\square = \pi_\# T' + h\). Then \(\partial \alpha_\square = \partial T'_\square\) and \(\text{spt}(\alpha_\square) \subset N'_{\alpha_1}(\text{spt}(\pi_\# \alpha))\). Note that \(\alpha_\square = T'_\square\) as they are top dimensional. Let \(\tilde{\alpha} = \nu(\alpha_\square)\) and \(\tilde{T}' = \nu(T'_\square)\). Then
\[
(10.7) \quad \text{spt}(\tilde{T}') = \text{spt}(\tilde{\alpha}) \subset N_{a_2 \epsilon R}(\text{spt} \alpha).
\]
Let \(W_a\) be as in Definition 6.13. Let \(W' = \text{spt}(T'_\square)\) and \(W'_a = \{x \in W' | d(x, \partial W') > a\}\). We find \(a_3\) and \(a_4\) such that
\[
(10.8) \quad W_{a_3} \subset W'_{a_4} \text{ and } Q(W'_{a_4}) \subset N_{a_5}(\text{spt}(T')).
\]
The first \(\subset\) follows from \(T'_\square = \pi_\# \nu(\nu) + h\), and the second \(\subset\) is from Proposition 5.7 (4). Now (10.6), (10.7) and (10.8) imply \(Q\) is \(\mu\)-rigid.

Now we prove (c). (3) \(\Rightarrow\) (4) is Lemma 10.1. It remains to show (4) \(\Rightarrow\) (1). Let \(\varphi, \tau, W\) and \(\nu\) be as in Definition 6.13. Let \(T' = \nu(\nu), T = \tau\) and \(\partial T' = \nu(\varphi)\). Let \(T = U + V\) and \(\tau\) be as in Definition 10.4. By Proposition 5.7, \(T'\) is \((a_6, a_7)\)-quasi-minimizing mod \(N_{a_7}(Q((\text{spt} \partial W)^{(0)})))\). Let \(x \in \text{spt}(T')\) such that
\[
B_x(5a_7) \cap N_{a_7}(Q((\text{spt} \partial W)^{(0)})) = \emptyset
\]
and let \(r_0 = d(x, \text{spt}(U + \sigma))\). Then by Definition 10.4 (2) and the filling density estimate Lemma 3.16, \(a_8(r_0)^{n+1} \leq \epsilon R^{n+1}\). This together with Proposition 5.7 (4) and (10.6) implies that \(Q\) is \(\mu\)-rigid. \(\square\)

Remark 10.9. Theorem 10.5 can be formulated for quasidisks as well. The quasidisk version still holds by the same proof.

The following is a consequence of Proposition 6.12, Proposition 7.3, and Theorem 10.5.

**Theorem 10.10.** Suppose \(X\) is a proper metric space satisfying (SCL\(_n\)). Let \(Q \subset X\) be an \(n\)-dimensional quasiflat. Suppose in addition that any asymptotic cone of \(X\) has a Lipschitz bicombing. Then the following
conditions are equivalent and any of the following condition is equivalent to each of the conditions in Theorem 10.5.

1. $Q_\omega \subset X_\omega$ has piece property (Definition 6.1) for each asymptotic cone $X_\omega$ of $X$.
2. $Q_\omega \subset X_\omega$ has neck property (Definition 6.4) for each asymptotic cone $X_\omega$ of $X$.
3. $Q_\omega \subset X_\omega$ has weak neck property (Definition 6.5) for each asymptotic cone $X_\omega$ of $X$.
4. $Q_\omega \subset X_\omega$ has full support (Definition 6.8) with respect to the reduced singular homology for each asymptotic cone $X_\omega$ of $X$.
5. $Q_\omega \subset X_\omega$ has full support with respect to the reduced homology induced by the Ambrosio-Kirchheim current for each asymptotic cone $X_\omega$ of $X$.

All asymptotic cones here are taken with base points inside $Q$.

Remark 10.11. Similar to Definition 6.13, we can also formulate pointed versions of Definitions 6.21, 7.7, and 10.4. By the same proof we see the analogue Theorem 10.5 and Theorem 10.10 using pointed versions of definitions also hold (in Theorem 10.10 we only consider asymptotic cones with constant base point).

11. Visibility for Morse quasiflats

In this section we work in a proper metric space with a convex geodesic bicombing (cf. Section 3.2). The main goal is to prove that Morse quasiflats are asymptotically conical in such spaces.

11.1. Asymptotic conicality. Throughout this subsection, let $I$ be the isoperimetric constant for $X$ and let $D$ be the lower density bound for minimizing currents in $X$.

Lemma 11.1. Suppose $X$ is a metric space with convex geodesic bicombing and base point $p$. Let $Q \subset X$ be an $n$-dimensional $(L, A)$-quasiflat. Then there exist $\kappa, a, \Theta$ depending only on $L, A, n$ and $d(p, Q)$ such that the following holds. There exists an element $T \in I_{n, \text{loc}}(X)$ such that

1. $d_H(\text{spt}(T), Q) < a$;
2. $M(T \cup B_p(r)) \leq \Theta \cdot r^n$ whenever $r \geq a$.
3. Let $S_{p, r} = (T, d_p, r)$. Then $\text{Fill}(S_{p, r}) \geq \kappa \cdot r^n$ whenever $r \geq a$.  


Proof. We take $T = \iota[\mathbb{R}^n]$ where $\iota$ is defined in Proposition 5.7. (1) follows from Proposition 5.7 (4), (2) follows from Proposition 5.7 (2), (3) follows from Proposition 5.7 (3) and Lemma 3.15. □

In the following we let $X$ be a metric space with convex geodesic bicombing and base point $p$. Further, $Q \subset X$ will be an $n$-dimensional $(L,A)$-quasiflat. If $T$ is a current representing $Q$ as above, then we denote the slice $\langle T,d_p,r \rangle$ by $S_r$. The slice $S_r$ is called $\lambda$-generic for $\lambda > 1$, if $\langle T,d_p,r \rangle \in I_{n-1,c}(X)$ and $M(\langle T,d_p,r \rangle) \leq \hat{\Theta} \cdot r^{n-1}$ with $\hat{\Theta} = \Theta_{\lambda^{-1}}$. By the coarea inequality, for every $r_0 > 0$ there exists a $\lambda$-generic slice in the range $[r_0, \lambda r_0]$.

In the following we will denote the flat distance by $\mathcal{F}$, i.e.

\[ \mathcal{F}(\sigma, \sigma') := \text{Fill}(\sigma - \sigma') \]

for cycles $\sigma$ and $\sigma'$.

**Proposition 11.2.** Suppose $X$ is proper and has a convex geodesic bicombing. Let $p \in X$ be a base point. Suppose $Q \subset X$ is an $n$-dimensional $(L,A)$-Morse quasiflat represented by $T \in I_{n,loc}(X)$ which satisfies Lemma 11.1 with $\kappa, a$ and $\Theta$ defined there ($\kappa,a,\Theta$ depend only on $Q,X,p,L$ and $A$). Let $S_r = \langle T,d_p,r \rangle$ denote the radius $r$ slices of $T$.

Given $\delta > 0$ and $\lambda > 1$, there exists $R = R(\Theta, \lambda, \delta, Q, d(p,Q), X)$ such that the following holds whenever $R > R$.

If $S_{r_0}$ is a $\lambda$-generic slice with $r_0 > R$, then there exists $r \in [R,3R]$ and $\hat{S}_r \in \mathbb{Z}_{n-1,c}(X)$ with spt($\hat{S}_r$) $\subset S_p(r)$ such that

1. $\mathcal{F}(\hat{S}_r, S_r) \leq a_n \cdot r^n$ for a constant $a_n = a_n(D)$ and any minimal filling of $\hat{S}_r - S_r$ is carried by the annulus $A_p(0.9r, 1.1r)$;
2. $\hat{S}_r$ can be written as a sum of a good part and a bad part, $\hat{S}_r = \gamma_r + \beta_r$;
3. the good part $\gamma_r$ is nontrivial and satisfies $0 < M(\gamma_r) \leq \hat{\Theta} \cdot r^{n-1}$ and spt($\gamma_r$) $\subset$ spt($C_p(S_{r_0})$);
4. the bad part $\beta_r$ is small, $M(\beta_r) \leq \delta \cdot r^{n-1}$.

Proof. Our goal is to inductively find a sequence of radii $r_0, r_1, \ldots$ with $r_{j+1} \in [\tfrac{9}{20}r_j, \tfrac{11}{20}r_j]$ and corresponding cycles $\hat{S}_j \in \mathbb{Z}_{n-1,c}(X)$ which satisfy items (1)-(4) on scale $r_j$.

For $j = 0$, we set $\gamma_0 = S_{r_0}$ and $\beta_0 = 0$. 
Now suppose we have already found the scale \( r_j \) and the cycle \( \hat{S}_j \). Let \( M(\gamma_j) = g_j \cdot r_j^{n-1} \) and let \( M(\beta_j) = b_j \cdot r_j^{n-1} \). To get the scale \( r_{j+1} \) we distinguish two cases.

**Case 1.** There exists \( r \in [\frac{9}{20} r_j; \frac{11}{20} r_j] \) such that \( F(h_p, r_j \cdot \hat{S}_j, S_r) \leq a_n \cdot r^n \).

In this case we set \( r_{j+1} := r, \gamma_{j+1} := h_p, r_j \cdot \hat{S}_j \) and \( \beta_{j+1} := h_p, r_j \cdot \hat{S}_j \).

See Section 3.2 for the definition of \( h_p \).

**Case 2.** Negation of Case 1.

Here we are not able to right away to maintain our control on the next lower scale. Instead our estimate on the bad part will get worse. However, this will be compensated by showing that at the same time the upper bound on the good part improves by a definite amount. Our estimates will show that the good part cannot vanish, hence Case 2 occurs only a finite number of times and the upper bound on the bad part increases only a finite number of steps which safes the induction argument. To implement this strategy, we allow the upper bound \( \delta \) on the bad part to depend on \( j \) and introduce an auxiliary threshold \( a_{n-1} \) to be determined later. We then assume \( b_j \leq \delta_j \) and let the argument run as long as \( \delta_j \leq a_{n-1} \).

Let \( \tau_j \) be a filling of \( \hat{S}_j - S_{r_j} \) supported in \( A_p(0.9 r_j, 1.1 r_j) \). Note that

\[(11.3) \quad M(C_p(\hat{S}_j) + \tau_j) \leq (g_j + b_j + a_n) \cdot r_j^n.\]

We may assume that \( r_j \) is large enough, so that we can apply Lemma 10.1 to the chain \( \hat{T}_j := C_p(\hat{S}_j) + \tau_j \) to produce a near/far piece decomposition \( \hat{T}_j = U + V \) where \( U \) denotes the piece supported close to \( Q \). Recall that Lemma 10.1 provides a filling \( \omega \in I_n, c(X) \) of \( \partial V \) with mass \( < \epsilon \cdot r_j^n \) and a filling \( \tilde{W} \) of \( U + \omega - (T \setminus B_p(r_j)) \) with mass \( < \epsilon \cdot r_j^{n+1} \).

For a given radius \( r \), we will now slice \( U, V, W \) and \( \omega \) with respect to \( d(\cdot, p) \) on the range \( [\frac{1}{2} r_j, \frac{11}{20} r_j] \) to obtain controlled slices \( U_r, V_r, W_r \) and \( \omega_r \). Note that since \( \text{spt}(\tau_j) \subset A_p(0.9 r_j, 1.1 r_j) \), we have \( U_r + V_r = \gamma_r + \beta_r \) where \( \gamma_r := h_p, r_j \cdot \hat{S}_j \) and \( \beta_r := h_p, r_j \cdot \hat{S}_j \). Clearly \( M(\gamma_r) \leq g_j \cdot r_j^{n-1} \) and \( M(\beta_r) \leq b_j \cdot r_j^{n-1} \). We need to collect some mass and filling bounds in order to proceed.
Lemma 11.4. There exists an \( r \in \left[ \frac{1}{2} r_j, \frac{11}{20} r_j \right] \) such that \( U_r, V_r, W_r \) and \( \omega_r \) are in \( I_{s,c}(X) \) and the following estimates hold.

\[
M(V_r) \geq \left[ \left( \frac{a_n - 20 \cdot 2^{n+1} \epsilon}{I} \right)^{\frac{n-1}{n}} - 20 \cdot 2^n \epsilon \right] \cdot r^{n-1}
\]

\[
M(U_r) \geq \left[ \left( \frac{\kappa - 20 \cdot 2^{n+1} \epsilon}{I} \right)^{\frac{n-1}{n}} - 20 \cdot 2^n \epsilon \right] \cdot r^{n-1}
\]

\[
\mathcal{F}(U_r + \omega_r, S_r) \leq 20 \cdot 2^{n+1} \epsilon \cdot r^n
\]

\[
M(\omega_r) \leq 20 \cdot 2^n \epsilon \cdot r^{n-1}
\]

Proof. By the coarea inequality, there exists \( r \in \left[ \frac{1}{2} r_j, \frac{11}{20} r_j \right] \) such that

\[
M(W_r) \leq 20 \cdot \frac{M(W)}{r_j} \quad \text{and} \quad M(\omega_r) \leq 20 \cdot \frac{M(\omega)}{r_j}
\]

It follows that \( M(W_r) \leq 20 \cdot 2^{n+1} \epsilon \cdot r^n \) and \( M(\omega_r) \leq 20 \cdot 2^n \epsilon \cdot r^{n-1} \).

Now we estimate \( M(U_r) \). As \( \text{spt}(\tau_j) \subset A_p(0.9 r_j, 1.1 r_j) \), we have \( \partial W_r = U_r + \omega_r - S_r \), hence \( \mathcal{F}(U_r + \omega_r, S_r) \leq M(W_r) \). On the other hand, \( \text{Fill}(S_r) \geq \kappa \cdot r^n \). Thus \( \text{Fill}(U_r + \omega_r) \geq \kappa r^n - M(W_r) \geq (\kappa - 20 \cdot 2^{n+1} \epsilon) \cdot r^n \).

By the isoperimetric inequality, \( M(U_r + \omega_r) \geq (\kappa - 20 \cdot 2^{n+1} \epsilon) \cdot r^{n-1} \).

Thus

\[
M(U_r) \geq \left[ \left( \frac{\kappa - 20 \cdot 2^{n+1} \epsilon}{I} \right)^{\frac{n-1}{n}} - 20 \cdot 2^n \epsilon \right] \cdot r^{n-1}.
\]

Similarly, we can estimate \( M(V_r) \). As \( \mathcal{F}(U_r + \omega_r, S_r) \leq M(W_r) \) and \( \mathcal{F}(\gamma_r + \beta_r, S_r) = \mathcal{F}(U_r + V_r, S_r) > a_n \cdot r^n \), we have \( \text{Fill}(V_r - \omega_r) \geq (a_n - 20 \cdot 2^{n+1} \epsilon) \cdot r^n \). It follows from the isoperimetric inequality that

\[
M(V_r - \gamma_r) \geq \left( \frac{a_n - 20 \cdot 2^{n+1} \epsilon}{I} \right)^{\frac{n-1}{n}} \cdot r^{n-1}.
\]

Thus

\[
M(V_r) \geq \left[ \left( \frac{a_n - 20 \cdot 2^{n+1} \epsilon}{I} \right)^{\frac{n-1}{n}} - 20 \cdot 2^n \epsilon \right] \cdot r^{n-1}.
\]

\[\square\]

Let \( r \) be as in the above lemma. Then we can estimate
(11.5) \( M(U_r) \leq M(\gamma_r) + M(\beta_r) - M(V_r) \)

(11.6) \[
g_j + b_j - \left( \frac{a_n - 20 \cdot 2^{n+1} \epsilon}{I} \right)^{\frac{n-1}{n}} + 20 \cdot 2^n \epsilon \right] \cdot r^{n-1}.
\]

By the construction of Lemma 10.1, the piece decomposition \( U + V \) comes from \( X = \{d_Q > h\} \cup \{d_Q \leq h\} \) for some \( h > 0 \). Let \( \gamma_r^{< h} = \gamma_r \Lambda \{d_Q \leq h\} \) and \( \beta_r^{< h} = \beta_r \Lambda \{d_Q \leq h\} \). Note that \( \gamma_r^{< h} + \beta_r^{< h} = U_r \). By slightly perturbing \( h \) (and at the cost of affecting the previous estimates by a universal constant), we may assume \( \gamma_r^{< h}, \beta_r^{< h} \in I_{s,c}(X) \).

We set \( r_{j+1} := r \) and define
\[
\gamma_{j+1} := \gamma_r^{< h}, \quad \beta_{j+1} := \beta_r^{< h} + \omega_r \quad \text{and} \quad \widehat{S}_{j+1} = \gamma_{j+1} + \beta_{j+1}.
\]

Then \( \widehat{S}_{j+1} = \gamma_{j+1} + \beta_{j+1} = U_r + \omega_r \) is a cycle with
(11.7) \[
\mathcal{F}(\widehat{S}_{j+1}, S_r) = \mathcal{F}(U_r + \omega_r, S_r) \leq 20 \cdot 2^{n+1} \epsilon \cdot r^n
\]
\[
\text{by Lemma } 11.4 \text{ Using } M(\beta_r^{< h}) \leq M(\beta_r) \text{ and } (11.5) \text{ we get}
\]
\[
M(\gamma_{j+1}) \leq M(U_r) + M(\beta_r^{< h}) \leq \left[ g_j + 2b_j - \left( \frac{a_n - 20 \cdot 2^{n+1} \epsilon}{I} \right)^{\frac{n-1}{n}} + 20 \cdot 2^n \epsilon \right] \cdot r_{j+1}^{n-1},
\]

and
(11.8) \[
M(\beta_{j+1}) \leq M(\beta_r) + M(\omega_r) \leq (b_j + 20 \cdot 2^n \epsilon) \cdot r_{j+1}^{n-1}.
\]

Lemma 11.4 also gives rise to a lower bound on \( M(\gamma_{j+1}) \):
(11.9) \[
M(\gamma_{j+1}) \geq M(U_r) - M(\beta_r)
\]
(11.10) \[
\geq \left[ \left( \kappa - 20 \cdot 2^{n+1} \epsilon \right)^{\frac{n-1}{n}} - 20 \cdot 2^n \epsilon - a_{n-1} \right] \cdot r_{j+1}^{n-1}.
\]

The above induction process depends on the choice of \( a_n, a_{n-1}, \delta_j, \epsilon \) and the threshold for \( r \) when we apply Lemma 10.1. Now we indicate the appropriate choices.

First, we choose \( a_n \) such that any minimal filling between \( \widehat{S}_j \) and \( S_r \) is carried by the annulus \( A(p, 0.9 r_j, 1.1 r_j) \). Since \( \mathcal{F}(\widehat{S}_j, S_r) < a_n \cdot r_j^n \), we can take \( a_n = \frac{D}{10r} \). Set \( \chi = \frac{1}{4} \left( \frac{a_n}{r_j} \right)^{\frac{n-1}{n}} \) and \( N = \left\lceil \frac{\phi}{\chi} \right\rceil \).
Second we choose our auxiliary threshold $a_{n-1}$. We want to arrange that whenever Case 2 occurs, the mass of $M(\gamma_r)$ drops by a definite amount (in a scale invariant sense), but still stays positive. We choose $a_{n-1}$ such that
\[
\left( \frac{\kappa - a_{n-1}}{I} \right)^{n-1} - 2a_{n-1} \geq \frac{1}{2} \left( \frac{\kappa}{I} \right)^{n-1}
\]
and
\[3a_{n-1} \leq \chi.\]
Next, define $\delta_j$ inductively such that $\delta_1 = \epsilon$ and $\delta_j = \delta_{j-1} + 20 \cdot 2^n \epsilon$. Then choose $\epsilon$ such that $\delta_N \leq \min\{\delta, a_{n-1}, \chi\}$, $20 \cdot 2^{n+1} \epsilon \leq \min\{a_{n-1}, a_n\}$ and
\[
\left( \frac{a_n - 20 \cdot 2^{n+1} \epsilon}{I} \right)^{n-1} \geq \frac{1}{2} \left( \frac{a_n}{I} \right)^{n-1}.
\]
As there is a bound of mass (11.3) when applying Lemma 10.1 and $\text{spt}(\hat{T}_j) \subset B_p(1.1r_j)$, we choose $R$ as in Lemma 10.1 where the required constant $b$ is chosen $b = \max\{\Theta' + a_n, 1.1, \Theta\}$.

It follows from our choices that as long as $r_j > R$ and the number of times when Case 2 has occurred is less than $N$, we can conclude
1. either $g_{j+1} = g_j$ (Case 1) or $g_{j+1} \leq g_j - \chi$ (Case 2);
2. $g_{j+1} > 0$;
3. either $b_{j+1} = b_j$ (Case 1) or $b_{j+1} \leq \min\{\delta, a_{n-1}, \chi\}$ (Case 2), in particular we can show inductively that $b_i + g_i \leq \Theta'$ for $i \geq j-1$;
4. $F(\hat{S}_{j+1}, S_{r_{j+1}}) < a_n r_{j+1}^n$.

Lemma 11.4 ensures that whenever Case 2 occurs, the piece $U_r$ will have positive mass. Thus Case 2 can occur at most $N$ times. Hence the induction process is actually unlimited and we can continue until the first $j$ with $r_j < R$. Now the proposition follows.

\[\square\]

**Corollary 11.11.** Suppose $Q$ is an $(L, A)$-Morse quasiflat in a metric space $X$ with convex geodesic bicombing represented by a current $T$. Let $p \in Q$ be a base point. Then for given $\lambda > 1$ and $\delta > 0$, there exists $R_0$ depending only on $\delta, \lambda, L, A, d(p, Q)$ and $X$ such that for any $r_0 > R \geq R_0$, there exists $r_0 \in (r_0, \lambda r_0)$ such that any point in $Q \cap B_p(R)$ is at most distance $\delta R$ away from a point in $C_p(\text{spt}(S_{r_0}))$. 
Proof. We choose \( r_0 \) such that \( M(S_{r_0}) \leq \Theta' \cdot r_0^{n-1} \) where \( \Theta' \) depends only on \( \Theta \) and \( \lambda \). Define \( T' = C_p(S_{r_0}) \) as in Proposition 11.2 and let \( R_1 = 10R \) with \( R \) be as in Proposition 11.2. Suppose \( R > R_1 \).

First we consider that case \( R < \frac{r_0}{3} \). Let \( r \leq r_0 \) such that \( \frac{r}{9} \leq R \leq \frac{r}{3} \) and \( \hat{S}_r = G_r + E_r \) satisfies the conclusions of Proposition 11.2. We replace \( E_r \) by a minimal filling \( \hat{E}_r \) with \( \partial E_r = \partial \hat{E}_r \). Then \( M(\hat{E}_r) \leq \delta r^{n-1} \), and by Lemma 3.15, \( \text{spt}(\hat{E}_r) \subset N_{\delta r}(\text{spt}(\partial \hat{E}_r)) \) where \( \delta_1 = (\frac{\delta}{\delta_r})^{\frac{1}{n-1}} \). Since \( M(E_r - \hat{E}_r) \leq 2\delta \cdot r^{n-1} \), by Theorem 3.10 \( E_r - \hat{E}_r \) has a filling with mass \( \leq IM(E_r - \hat{E}_r) \cdot \delta^{\frac{n}{n-1}} \) and such filling is carried by the \( \delta_2 r \)-neighborhood of \( \text{spt}(E_r - \hat{E}_r) \) where \( \delta_2 = (\frac{\delta}{R})^\frac{n}{n-1} \).

Note that \( \text{Fill}(G_r + \hat{E}_r - S_r) \leq \text{Fill}(G_r + E_r - S_r) + \text{Fill}(\hat{E}_r - E_r) \leq a_n r^n + I(\delta) \cdot \delta^{\frac{n}{n-1}} \cdot r^n \) and a minimal such filling \( K_r \) is carried by the annulus \( A(p, (1 - b)r, (1 + b)r) \), where \( b = 0.1 + \delta_1 + \delta_2 \).

Let \( T'_G = C_p(G_r) \) and \( T'_E = C_p(\hat{E}_r) \). Then

\[
M(T'_G + T'_E - K_r) \leq (\Theta' + a_n + I(2\delta) \cdot \delta^{\frac{n}{n-1}}) \cdot r^n.
\]

Moreover, \( \text{spt}(T'_G + T'_E - K_r) \subset B_p((1 + b)r) \). By Lemma 10.1, there exists \( R_2 \) depending only on \( \delta, \lambda, A, p \) and \( X \) such that if \( R > R_2 \), then \( T'_G + T'_E - K_r \) has a piece decomposition \( U + V \), a minimizing \( n \)-chain \( \gamma \) with mass \( < \delta r^n \), and a filling \( W \) of \( U + \gamma - T_r \) with mass \( < \delta r^{n+1} \). Moreover, \( \partial U = \partial \gamma - \partial T_r \).

Let \( z \in T_r \). Recall that \( T \) is \((\Lambda, a)\)-minimizing where \( \Lambda \) and \( a \) are defined in Proposition 5.7 (they depend only on \( L, A, n \) and \( X \)). By Lemma 3.16 whenever \( d(z, \text{spt}(U + \gamma)) > 4a \), we have \( \delta_3 r^{n+1} \geq M(W) \geq c(d(z, \text{spt}(U + \gamma)))^{n+1} \) with \( c \) defined in Lemma 3.16.

As \( r \gg a \), we deduce that

\[
d(z, \text{spt}(U + \gamma)) \leq \delta_3 r \quad \text{where} \quad \delta_3 = \left(\frac{\delta}{c}\right)^{\frac{1}{n+1}}.
\]

As \( \gamma \) is a minimizing chain with \( M(\gamma) \leq \delta \cdot r^n \), by Lemma 3.15 \( \text{spt}(\gamma) \subset N_{\delta r}(\text{spt}(\partial \gamma)) \) where \( \delta_4 = (\frac{\delta}{\delta_r})^{\frac{1}{n}} \). As \( \text{spt}(\partial \gamma) \subset \text{spt}(U) \), we know

\[
\text{spt}(U + \gamma) \subset N_{\delta r}(\text{spt}(U)).
\]

By (11.12) and (11.13), \( \text{spt}(T_r) \subset N_{(\delta_3 + \delta_4)}(\text{spt}(U)) \). It follows from the definition of \( U \) that \( U = U_G + U_E - U_r \), where \( U_G, U_E \) and \( U_r \) are pieces of \( T'_G, T'_E \) and \( K_r \) respectively. As \( K_r \) (hence \( U_r \)) is carried by \( A(p, (1 - b)r, (1 + b)r) \) and \( R \leq \frac{r}{3} \), if \( \delta_1 + \delta_2 < 0.1 \) (this can always be
arranged when δ is small), we have

\begin{equation}
(11.14) \quad \text{spt}(T_R) \subset N_{(\delta_3+\delta_4)r}(\text{spt}(U_G + U_E)).
\end{equation}

As \(\text{spt}(\dot{E}_r) \subset N_{\delta_1 r}(\text{spt}(\partial \dot{E}_r)) \subset N_{\delta_1 r}(\text{spt}(G_r))\), we know \(\text{spt}(T'_G + T'_E)\) (hence \(\text{spt}(U_G + U_E)\)) is contained in the \(\delta_1 r\)-neighborhood of \(\text{spt}(T'_G)\). As \(\text{spt}(T'_G) \subset \text{spt}(T')\), \(\text{spt}(U_G + U_E) \subset N_{\delta_1 r}(\text{spt}(T'))\). This together with (11.14) imply that

\[\text{spt}(T_R) \subset N_{(\delta_1+\delta_3+\delta_4)r}(\text{spt}(T')).\]

In summary, this estimate is obtained under the assumption that

- \(R < \frac{R_0}{3}\);
- \(R > R_0\), where \(R_0 = \max\{R_1, R_2\}\) and \(R_0 \gg a\);
- \(\delta_1 + \delta_2 < 0.1\).

The case \(R \geq \frac{r_0}{3}\) is similar and much easier, where we apply Lemma 10.1 directly to \(T'\). We leave this case to the reader. \(\square\)

**Theorem 11.15.** Suppose \(Q\) is an \((L, A)\)-Morse quasiflat in a metric space \(X\) with convex geodesic bicombing represented by a current \(T\). Let \(p \in X\) be a base point. Then for any given \(\delta > 0\), there exists \(R\) depending only on \(\delta, \lambda, L, A, d(p, Q)\) and \(X\) such that \(d(y, \text{spt}(T)) < \delta d(y, p)\) whenever \(d(y, p) \geq R\) and \(y \in C_p(\text{spt}(T))\).

This has exactly the same proof as [KL18, Theorem 8.6]. The proof of [KL18, Theorem 8.6] relies on [KL18, Theorem 8.1] and [KL18, Lemma 3.3] (lower density bound for quasi-minimizers). However, Corollary 11.11 plays the role of [KL18, Theorem 8.1].

### 11.2. The Tits boundary of a Morse quasiflat

We refer to Section 3.2 for the definition of Tits cone and Tits boundary for a metric space with convex geodesic bicombing.

**Definition 11.16.** Let \(\sigma\) be the given convex geodesic bicombing on \(X\). Let \(Q\) be an \((L, A)\)-quasiflat. We say \(Q\) is **pointed Morse** if for any asymptotic cone \(X_\omega\) of \(X\) with fixed base point, the inclusion \(Q_\omega \to X_\omega\) induces injective map on local homology at each point in \(Q_\omega\).

If we weaken the dependence of \(R\) on \(d(p, Q)\) in Lemma 10.1, Proposition 11.2, Corollary 11.11 and Theorem 11.15 to the dependence of \(R\) on the position of \(p\), then these results hold for pointed Morse quasiflat by the same proof.
Definition 11.17. Let \( \sigma \) be the given convex geodesic bicombing on \( X \). Let \( Q \) be a pointed Morse quasiflat. We consider the collection of all \( \sigma \)-geodesics rays in \( \overline{C}_p(\text{spt}(T)) \) emanating from \( p \), and define the **Tits boundary** of the pointed Morse quasiflat \( Q \), denoted \( \partial_T Q \), to be the subset of \( \partial_T X \) determined by these rays. Note that \( \partial_T Q \) does not depend on the choice of \( p \).

Proposition 11.18. Let \( Q \) be an \((L,A)\) pointed Morse quasiflat. Let \( p \) be a base point in \( X \). Then

1. there exists a function \( \delta : [0, \infty) \to [0, \infty) \) depending only on \( Q, p \) and \( X \) such that \( \lim_{r \to \infty} \frac{\delta(r)}{r} = 0 \) and \( d_H(B_p(R) \cap Q, B_p(R) \cap C_p(\partial_T Q)) \leq \delta(r) R \) whenever \( R \geq r \);
2. \( \partial_T Q \) be also defined as the subset of \( \partial_T X \) represented by geodesic rays \( \ell : [0, \infty) \to X \) which travel sublinearly close to \( Q \), i.e. \( \lim_{t \to \infty} \frac{d(\ell(t),Q)}{t} = 0 \).

If \( Q \) is a Morse quasiflat, then we can strengthen (1) such that \( \delta \) depends on \( d(p,Q) \) rather than \( p \).

Proof. Note that \( \overline{C}_p(\text{spt}(T)) \cap B_p(r) \) is compact for any \( r > 0 \). Thus we can deduce from Corollary 11.11 that the conclusion of this corollary holds with \( C_p(\text{spt}(S_n)) \) replaced by \( C_p(\partial_T Q) \). This together with Theorem 11.15 imply (1). (1) implies that each geodesic ray in \( C_p(\partial_T Q) \) travels sublinearly close to \( Q \). Conversely, given a geodesic ray \( \ell \subset X \) traveling sublinearly close to \( Q \), we find a sequences of points \( \{z_n\}_{n \in \mathbb{Z}^+} \) such that \( \lim_{n \to \infty} \frac{d(\ell(n),z_n)}{n} = 0 \). Since \( \overline{C}_p(\text{spt}(T)) \cap B_p(r) \) is compact for any \( r > 0 \), the collection of geodesic segment \( \{p z_n\}_{n \in \mathbb{Z}^+} \) subconverges to a geodesic ray \( \ell' \subset \overline{C}_p(\text{spt}(T)) \). It is clear that \( \partial_T \ell' \in \partial_T Q \) and \( \partial_T \ell = \partial_T \ell' \), thus (2) follows. \( \square \)

Proposition 11.19. Let \( Q \) be an \( n \)-dimensional \((L,A)\) pointed Morse quasiflat. The Euclidean cone over \( \partial_T Q \) is bilipschitz homeomorphic to \( \mathbb{R}^n \) (here \( \partial_T Q \) is given the induced metric from \( \partial_T X \)). Moreover, the map

\[
H_{n-1}(\partial_T Q, \partial_T Q \setminus \{p\}, Z) \to H_{n-1}(\partial_T X, \partial_T X \setminus \{p\}, Z)
\]

is injective for each point \( p \in \partial_T Q \).

This generalizes that pointed Morse quasi-geodesics give rise to isolated points in the Tits boundary.

Proof. Let \( X_\omega \) be an asymptotic cone of \( X \) with fixed base point \( p \in X \) and let \( Q_\omega \subset X_\omega \) the limit of \( Q \). It follows from the definition of \( X_\omega \)
that the Tits cone $\mathcal{C}_T(X)$ embeds isometrically as a subset of $X_\omega$. Let $Z$ be the cone over $\partial_T Q$ inside $\mathcal{C}_T(X) \subset X_\omega$. Proposition 11.18 implies $Q_\omega = Z$. Thus $Z$ is bilipschitz to $\mathbb{E}^n$, hence the first statement of the proposition holds. Since $Q$ is Morse, $H_n(Z, Z \setminus \{p\}, Z) \rightarrow H_n(X_\omega, X_\omega \setminus \{p\}, Z)$ is injective for each $p \in Z$. Now the second statement of the proposition follows from the Künneth formula (cf. [Dol12, pp. 190, Proposition 2.6]). □

**Theorem 11.20.** Suppose $X$ is a proper metric space with convex geodesic bicombing. Let $Q_1$ and $Q_2$ be two Morse quasiflats in $X$. Suppose $\partial_T Q_1 = \partial_T Q_2$. Then $d_H(Q_1, Q_2) < C$ where $C < \infty$ depends only on $\dim Q_1$ and the Morse parameter of $Q_1$ and $Q_2$.

**Proof.** By Proposition 11.18 $Q$ and $Q'$ are at sublinear distance from each other. Now the theorem follows from Proposition 8.4. □

### 12. Criteria and Examples for Morseness

In this section we give several more characterizations of Morse quasiflats in CAT(0) spaces and present several related examples and non-examples of Morse quasiflats.

#### 12.1. Failure of Morseness and flat half-spaces

Let $X$ be a CAT(0) space. Let $\Sigma_p X$ be the space of directions at $p$. Let $T_p X$ be the tangent cone of $X$ at a point $p \in X$, which is defined to be the Euclidean cone over $\Sigma_p X$. Then $T_p X$ is a CAT(0) space, and there is a 1-Lipschitz map $\log_p : X \rightarrow T_p X$ sending a geodesic emanating from $p$ to its corresponding geodesic emanating from the cone point $o$ of $T_p X$. We refer to [BH99, Chapter II.3] for more background on tangent cones.

Given a scaling sequence $\epsilon_i \rightarrow 0$, we obtain a blow up $Y$ at $p$ which is the ultralimit $\lim_{\omega}(1/\epsilon_i X, p)$. There is an exponential map $\exp : T_p X \rightarrow Y$ which is an isometric embedding (see e.g. [Lyt05, Section 5.2]).

Let $\Phi : \mathbb{R}^n \rightarrow X$ be a bi-Lipschitz embedding. Let $E$ be a countable dense subset of the unit sphere of $\mathbb{R}^n$. By the proof of [Lyt05, Proposition 6.1] (see [Sta18, Lemma 11] as well for explanation), there exists a full measure subset $G \subset \mathbb{R}^n$ such that for any $p \in G$,

(a) $p$ is a point of metric differentiability with non-degenerate metric differential (cf. [Kir94]);

(b) for any vector $\vec{v} \in E$, the points in $\Sigma_{\Phi(p)} X$ represented by the geodesics from $\Phi(p)$ to $\Phi(p + t\vec{v})$ form a Cauchy sequence as

...
Lemma 12.1. Let \( x = \Phi(p) \) and let \( Q = \text{Im} \Phi \). Take a sequence \( \epsilon_i \to 0 \). Let \( \Phi_\omega \) be the ultralimit of \( \{ \frac{1}{\epsilon_i} \Phi : (\frac{1}{\epsilon_i} \mathbb{R}^n, p) \to (\frac{1}{\epsilon_i} X, x) \} \). Let \( Y = \lim_{\omega}(\frac{1}{\epsilon_i} X, p) \) and let \( Y_\epsilon = \exp(T_{\epsilon}X) \subset Y \). Then

\[ t \to 0^+, \text{ hence converges to a point in } \Sigma_{\Phi(p)} X \text{ as } t \to 0^+. \text{ We denote this point by } \partial \Phi(\bar{v}). \]

Proof. (1) \( \text{Im} \Phi_\omega \subset Y_\epsilon; \)

(2) \( F := \text{Im} \Phi_\omega \) is a flat in \( Y; \)

(3) there exists a function \( \delta : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \) with \( \lim_{r \to 0^+} \frac{\delta(r)}{r} = 0 \) such that \( \log_x(\Phi(B_p(r))) \subset N_{\delta(r)}(F); \)

(4) define \( \Phi_i : \frac{1}{\epsilon_i} B_p(\epsilon_i) \to Y_\epsilon \) by \( \Phi_i = \log_x \circ \Phi(\epsilon_i) \), then for any \( \delta > 0 \), there exists \( i_0 \) such that \( d(\Phi_i(x), \Phi_\omega(x)) \leq \epsilon \) for any \( i \geq i_0 \) and \( x \) in the domain of \( \Phi_i \) (we identify the domain of \( \Phi_i \) with an unit ball in the domain \( \Phi_\omega \));

(5) for a non-trivial abelian group \( \mathbb{F} \), if

\[ \tilde{H}_n(Q, Q \setminus \{x\}, \mathbb{F}) \to \tilde{H}_n(X, X \setminus \{x\}, \mathbb{F}) \]

is not injective, then \( \tilde{H}_n(F, F \setminus \{o\}, \mathbb{F}) \to \tilde{H}_n(Y, Y \setminus \{o\}, \mathbb{F}) \) is not injective where \( o = \log_x(x) \).

Proof. (1) follows from property (b) as above. For (2), note that \( \text{Im} \Phi_\omega \) is an isometrically embedded normed vector space in \( Y \) by property (a) above, however, the norm has to be Euclidean as \( Y \) is CAT(0).

Take \( \epsilon > 0 \). Take a finite subset \( E' \subset E \) such that \( E' \) is an \( \epsilon \)-net in the unit sphere of \( \mathbb{R}^n \). Take \( r \) such that

\[ d_{S_{\epsilon}X}(\log_x(\Phi(p)\Phi(p + t\bar{v})), \partial \Phi(\bar{v})) < \epsilon \]

for any \( \bar{v} \in E' \) and \( 0 \leq t \leq r \). As \( \log_x \) is 1-Lipschitz, we deduce that \( \log_x(\Phi(B_p(t))) \subset N_{Ct}(F) \) for any \( 0 \leq t \leq r \) where \( C \) is constant depending only on \( \text{Lip}(\Phi) \). Thus (3) follows. (4) is proved similarly.

Take \( r_0 \) such that \( \delta(r) \leq \frac{r}{1000 \text{Lip}(\Phi)} \) for any \( 0 \leq r \leq r_0 \). The non-injectivity assumption in (5) implies that there is a non-trivial element \( \sigma \in \tilde{H}_{n-1}(\Phi(S_p(r_0)), \mathbb{F}) \) such that \( \sigma = \partial \tau \) for a singular chain \( \tau \) in \( X \) with \( p \notin \text{Im} \tau \). Let \( \tau' = \log_p(\sigma) \) and \( \tau'' = \log_p(\tau) \). Then \( o \notin \text{Im} \tau' \) and \( d(\text{Im} \tau', o) \geq \frac{r_0}{\text{Lip}(\Phi)} \). Denote the CAT(0) projection onto \( F \) by \( \pi_F \). Let \( \sigma'' = \pi_F(\sigma') \) and let \( \tau'' \) be a chain induced by the geodesic homotopy between \( \sigma' \) and \( \sigma'' \). The choice of \( r_0 \) implies that \( o \notin \text{spt}(\tau'') \) and \( o \notin \text{spt}(\sigma'') \). By (4), \( d(\pi_F \circ \Phi_i(x), \Phi_\omega(x)) \leq 2\epsilon \) for any \( x \) in the domain of \( \Phi_i \), thus \( [\sigma''] \) is non-trivial in \( \tilde{H}_n(F, F \setminus \{o\}, \mathbb{F}) \). As \( \sigma'' = \partial(\tau' + \tau) \) with \( o \notin \text{Im}(\tau' + \tau'') \), (5) follows. \( \square \)
Proposition 12.2. Let $Q$ be an $n$-quasiflat in a CAT(0) space $X$. Let $\mathbb{F}$ be a non-trivial abelian group. Consider the following conditions.

1. There exists an asymptotic cone $X_o$ of $X$ and $p_o \in Q_o$ such that the map $\tilde{H}_n(Q_o, Q_o \setminus \{p_o\}, \mathbb{F}) \to \tilde{H}_n(X_o, X_o \setminus \{p_o\}, \mathbb{F})$ is not injective.
2. There exists an asymptotic cone $X_o$ of $X$ such that the limit $Q_o$ of $Q$ is an $n$-flat which bounds a flat half-space in $X_o$.
3. There exists an ultralimit $X_o = \lim_i (X, p_i)$ such that the limit $Q_o$ of $Q$ is an $n$-flat which bounds a flat half-space in $X_o$.

Then (1) and (2) are equivalent. If we assume in addition that $Q$ is a flat, then all three conditions are equivalent.

Note that there is no scaling in condition (3).

Proof. Clearly (2) $\Rightarrow$ (1). Now we prove (1) $\Rightarrow$ (2). We claim it is possible to take an (possibly different) asymptotic cone of $X$ such that $Q_o$ is flat and the non-injectivity condition in (1) still holds. The collection of all $p_o \in Q_o$ such that $\tilde{H}_n(Q_o, Q_o \setminus \{p_o\}, \mathbb{F}) \to \tilde{H}_n(X_o, X_o \setminus \{p_o\}, \mathbb{F})$ is not injective is an open subset of $Q_o$.

Take one such $p_o$, which also satisfies conditions (a) and (b) above. Blowing up at $p_o$, we have $Q'_o = \lim_i (\frac{1}{i} Q_o, p_o)$ sitting inside $Y = \lim_i (\frac{1}{i} X_o, p_o)$. Recall that ultralimits of asymptotic cones are asymptotic cones (cf. [DK17, Chapter 10.7]), thus $Y$ is also an asymptotic cone of $X$. The claim follows from Lemma [12.1]. From now on we assume $Q_o$ is flat in (1).

Let $Y_0 = X_o$ and $Q_0 = Q_o$. Define $Y_1 = T_{p_o} Y_0$ and $Q_1 = \log_{p_o}(Q_0)$. Then $Q_1$ is a flat in $Y_1$. The map $\log_{p_o}$ implies that $\tilde{H}_n(Q_1, Q_1 \setminus \{o_1\}, \mathbb{F}) \to \tilde{H}_n(Y_1, Y_1 \setminus \{o_1\}, \mathbb{F})$ is not injective where $o_1$ is the cone point of $Y_1$. Take $p_1 \in Q_1$ such that $p_1 \neq o_1$. Then $\tilde{H}_n(Q_1, Q_1 \setminus \{p_1\}, \mathbb{F}) \to \tilde{H}_n(Y_1, Y_1 \setminus \{p_1\}, \mathbb{F})$ is not injective. Define $Y_2 = T_{p_1} Y_1$ and $Q_2 = \log_{p_1}(Q_1)$. Then we have non-injectivity of local homology for $Q_2 \to Y_2$ as before. Moreover, there is a line $\ell$ in $Q_2$ such that $Y_2 = \ell \times Y_2'$ and $Q_2 = \ell \times Q_2'$. Then non-injectivity of local homology holds for $Q_2' \to Y_2'$. Repeat the previous process for $Y_2'$. This produces a sequence $(Y_i, Q_i)_{i=1}^n$ such that $Y_i = T_{p_{i-1}} Y_{i-1}$ and $Q_i = \log_{p_{i-1}}(Q_{i-1})$. $Y_i$ splits off more and more line factors and we assume in the end $Y_n = Q_n \times Q_n$. Non-injectivity of local homology holds for $Q_n \to Y_n$. Hence $Q_n$ bounds an isometrically embedded copy of $Q_n \times [0, a]$. 

We claim for any pair of compact sets \((C,K)\) with \(C \subset K\), \(C \subset Q\) and \(K \subset X\), there exist a sequence of pairs of compact subsets \((C_i,K_i)\) and a scaling sequence \(\lambda_i \to 0\) such that

1. \(C_i \subset K_i\);
2. \(C_i \subset Q_0 = Q_\omega\) and \(K_i \subset X_\omega\);
3. \(\lim_i \frac{1}{\lambda_i}(C_i,K_i) = (C,K)\) in the sense of Gromov-Hausdorff.

We induct on \(n\). The base case \(n = 0\) is trivial. In general, by [Kle99, Lemma 2.1], we can always take \((C_i',K_i')\) satisfying all the above conditions except that \(C_i' \subset Q_{n-1}\) and \(K_i' \subset X_{n-1}\). The claim follows by applying the induction assumption to each pair \((C_i',K_i')\) and running a diagonal argument. Now we apply the claim to the case where \(C\) is a top-dimensional in \(Q\) and \(K = C \times [0,a]\). It gives an ultralimit \(Y'\) of \(X_\omega\) such that the limit \(Q'\) of \(Q_\omega\) in \(Y'\) has a top-dimensional cube \(C' \subset Q'\) which bounds \(C' \times [0,a]\). \(Y'\) is also an asymptotic cone of \(X\) by [DK17, Chapter 10.7]. Now (2) follows by blowing up \(Y'\) again.

Now we assume \(Q\) is a flat. (3) \(\Rightarrow\) (2) is clear. (2) \(\Rightarrow\) (3) follows essentially from the argument in [FL10]. Asymptotic cones in [FL10] are defined using fixed base point, and (3) \(\Rightarrow\) (2) can be seen as a version of [FL10, Theorem B] with varying base point (we do not need local compactness). The idea is to consider compact sets \(K_i = C_\omega \times [0,a_i]\) in \(X_\omega\) with \(C_\omega\) being a top-dimensional unit cube in \(Q_\omega\). Let \(D_i = C_\omega \times \{a_i\}\). We approximate each \(D_i \to X_\omega\) by a sequence of continuous maps \(f_{ij}\) into \(X\). Take \(a_i \to \infty\) and study the geometry of \(\text{Im } f_{ij}\) and their projections on \(Q\) will give the required half-flat. We refer to [FL10, Section 3] for more details.

**Corollary 12.3.** Suppose \(X\) is a proper CAT(0) space. Suppose \(F \subset X\) is a flat such that the stabilizer of \(F\) in \(\text{Isom}(X)\) acts cocompactly on \(F\). Then \(F\) is Morse if and only if \(F\) does not bound an isometrically embedded half-flat.

For an abelian group \(F\), we say an \(n\)-dimensional quasiflat \(Q\) in \(X\) is \(F\)-Morse if for any asymptotic cone \(X_\omega\) (with base points in \(Q\)), the inclusion \(Q_\omega \to X_\omega\) induces injective map on the \(n\)-th local homology with coefficient \(F\) at each point in \(Q_\omega\).

**Corollary 12.4.** Suppose \(X\) is a CAT(0) space. Let \(Q\) be a quasiflat and let \(F\) be a non-trivial abelian group. Then \(Q\) is \(\mathbb{Z}\)-Morse if and only if \(Q\) is \(F\)-Morse.

12.2. Metrics on half-planes.
Proposition 12.5. There exists a Riemannian metric $d$ on the upper half plane $\mathbb{R}^2_{\geq 0}$ such that

1. $X = (\mathbb{R}^2_{\geq 0}, d)$ satisfies condition $(\text{CI}_n)$ for any $n$;
2. the boundary $\ell$ of $\mathbb{R}^2_{\geq 0}$ is a quasi-geodesic which is not a Morse quasi-geodesic (it violates the super-Euclidean divergence condition);
3. there does not exists an asymptotic cone $X_\omega$ of $X$ (with base points in $\ell$) such that the limit $\ell_\omega$ of $\ell$ bounds a bilipschitzly embedded flat half plane.

We will construct such metric on $\mathbb{R}^2_{\geq 0}$ in several steps (Definition 12.6, Definition 12.9 and Definition 12.11).

Definition 12.6. Pick $L \in [1, \infty)$. An $L$-bad ball is a smooth Riemannian metric $g_L$ on the ball $B(0, R_L) \subset \mathbb{R}^2$ with the polar coordinate form

$$g_L = dr^2 + (f(r)r)^2 d\theta^2,$$

where:

1. $f : [0, R_L] \to [1 - \frac{1}{L}, L]$ is smooth function.
2. $\max f = L$.
3. $f|_{[0,L]} \equiv 1$ and $f|_{[R_L/10, R_L]} \equiv 1$.
4. (Slow change on annuli) $|\partial_r (\log f(r))| < \frac{1}{Lr}$.
5. (Standard area) $|B(0, R_L)|_{g_L} = \pi R_L^2$.

Lemma 12.7. There is a constant $A < \infty$ such that every L-bad ball satisfies a coning inequality with constant $A$, i.e. every Lipschitz 1-cycle $\sigma$ contained in an $R$-ball has a filling $\tau$ with $M(\tau) \leq ARM(\sigma)$.

Proof. Let $B_L$ be an $L$-bad ball and suppose $L \geq 2$. Let $R > 0$ and denote by $\tilde{A}_o(R, 2R)$ the annulus in the flat cone $C_\alpha$ with with tip $o$ and cone angle $\alpha = f(R)$. From the slow change of annuli, we see that for all $r \in [R, 2R]$ holds $\frac{1}{2} \leq \frac{f(r)}{f(R)} \leq 2$.

Let $\psi : A_0(R, 4R) \to \tilde{A}_o(R, 4R)$ be the natural radial isometric homeomorphism, which is arclength preserving on the inner boundary circles. The control on $f$ ensures that $\psi$ is locally 4-bilipschitz. Now let $\gamma$ be a Jordan curve in $B_L$. Let $R$ be the smallest radius such that $\gamma$ is contained in $B_0(2R)$. We will distinguish two cases. Let $S \subset \mathbb{R}^2$ be a flat sector with angle $\frac{\pi}{4}$ and tip at the origin. Choose $C > 0$ such that there exists a point $p \in \mathbb{R}^2$ with $B_r(CR) \subset S$ and $\|p\| = 2R$. 

Suppose the diameter of $\gamma$ is less than $CR^2$. Note that the angle $\alpha$ is at least $\frac{1}{2}$ by our assumption on $L$ and the definition of $f$. Then $\psi(\gamma)$ is contained in a ball of radius $CR$ centered at a point on the outer boundary of $\tilde{A}_0(R,4R)$. This concludes the first case by the cone inequality in $C_\alpha$ (note that by taking $C$ sufficiently large, we can assume any geodesic between two points in $\gamma$ does not escape $A_0(R,4R)$, thus the diameter of $\gamma$ can be computed inside $A_0(R,4R)$, similarly, we assume the diameter of $\varphi(\gamma)$ is computed inside $\tilde{A}_o(R,4R)$).

So let us assume the diameter of $\gamma$ is at least $CR^2$. Then we obtain a filling by coning off $\gamma$ at the origin. Since $f(\lambda r)\lambda r \leq 2f(r)\lambda r$ holds for all $r \geq 0$ and $\lambda \in [\frac{1}{2},1]$, we see that the map $B_0(r) \to B_0(\lambda r)$ which multiplies by $\lambda$ is $2\lambda$-Lipschitz. This implies that the area of the cone is at most $R \cdot \mathcal{H}^1(\gamma)$ and the proof is complete. \qed

Remark 12.8. The comparison to annuli in flat cones also shows that the doubling constant of an $L$-bad ball becomes unbounded as $L \to \infty$.

Definition 12.9. Choose some $r_0 > 1$. The very bad ball of scale $r_0$ is obtained from the Euclidean ball $B_0(r_0) \subset \mathbb{R}^2$ as follows. Setting $a_k := 2^{-k}r_0$, $p_k := (a_k,0) \in \mathbb{R}^2$, for we replace the Euclidean ball $B_{p_k}(\frac{a_k}{10})$ with a $k$-bad ball rescaled to have radius $\frac{a_k}{10}$, provided the rescaled ball is Euclidean up to the radius $k$, or equivalently, $\frac{k}{R_k} \cdot \frac{a_k}{10} \geq k$ using the notation of Definition 12.6. There are only finitely many $k$ such that $a_k \geq 10R_k$ so we are replacing only finitely many balls.

An expanded very bad ball of scale $r_0$ is obtained by gluing the boundary of a very bad ball of scale $r_0$ to the inner boundary of the Euclidean annulus $A_0(r_0,2r_0) \subset \mathbb{R}^2$.

Lemma 12.10. Let $r_k \to \infty$ and let $\{B_k\}$ be a sequence of bad balls of scale $r_k$. Let $Z$ be an ultralimit of the rescaled sequence $\{\frac{1}{r_k}B_k\}$. Then $Z$ is not doubling.

Proof. For any integer $k > 0$, we see ultralimit of (properly scaled) $k$-bad balls in $Z$. Thus $Z$ contains isometrically embedded rescaled $k$-bad balls for any $k > 0$. Hence $Z$ is not doubling by Remark 12.8 \qed

Definition 12.11. Modify the flat metric on the half-plane $\mathbb{R}^2_+$ as follows. For every pair of positive integers $j,k$, let $p_{jk}$ be the point of $\mathbb{R}^2_+$ with coordinate $(j2^k, 2^k)$. We replace the ball $B_{p_{jk}}(10^{-1}2^k)$ with a very bad ball of scale $10^{-1}2^k$ for each pair $j,k \geq 0$. Let $X$ denote the resulting Riemannian manifold.

Remark 12.12. The key point for such arrangement is that the size of the very bad ball is comparable to the height of the very bad ball, and
is also comparable to the size of “Euclidean regions” between this very 
bad balls and other very bad balls.

There are two metrics on $X$, one is induced by the Riemannian metric 
defined in Definition 12.11, denoted by $d_X$, and one is the original 
Euclidean metric, denoted by $d_{\text{euc}}$.

We claim that $X$ has the following properties. It satisfies the coning 
inequality for 1-cycles. However, for any sequence $(x_k)$ on $\partial X$ and any 
sequence $\lambda_k \to \infty$, the asymptotic cone $\omega \lim(\frac{1}{\lambda_k} X, x_k)$ is not doubling 
and therefore not bilipschitz to the Euclidean halfplane.

We will establish the claims on $X$ in the following.

**Lemma 12.13.** The boundary $\partial X$ is a quasigeodesic.

**Proof.** The metric on $X$ is flat in a neighborhood of $\partial X$ and therefore 
the canonical parametrization of $\partial X$ is 1-Lipschitz. On the other hand, 
the metric $g_L$ on an $L$-bad ball fulfills 

\[
(12.14) \quad g_L \geq (1 - \frac{1}{L})g_{\text{euc}} 
\]

by definition. Hence $\partial X$ is a bilipschitz line. \qed

**Lemma 12.15.** The distance to the boundary in $X$ is comparable to 
the Euclidean distance to the boundary 

\[
\frac{1}{C}d_{\text{euc}}(\cdot, \partial X) \leq d_X(\cdot, \partial X) \leq Cd_{\text{euc}}(\cdot, \partial X). 
\]

**Proof.** The left inequality follows from (12.14). To see the right in-
equality, we choose a point $p$ in $X$ and consider the vertical segment 
$\gamma$ joining $p$ to a point $q$ on $\partial X$ which realizes the Euclidean distance. 
Since $q$ lies on the boundray, it is not contained in a bad ball. Recall 
that the Euclidean metric on an $L$-bad ball is only altered on an inner 
core of radius $\frac{R_L}{10}$. If $\gamma$ intersects the core of a bad ball disjoint from 
p, then we replace the segment inside the core by the shortest path 
along the boundary of the core. This increases the Euclidean length 
of $\gamma$ only by factor $\frac{\pi}{2}$. Now if $p$ itself lies in the core of an $L$-bad ball 
$B_L$, then we change the segment of $\gamma$ inside $B_L$ to a piecewise radial 
geodesic. We first join $p$ to the center of $B_L$ and then join the center 
to the first intersection point of $\gamma$ with $B_L$. The Euclidean length of 
the original segment is at least $\frac{9}{10}R_L$ whereas the length of the new 
path is at most $\frac{11}{10}R_L$. This concludes the proof as the length of the 
modified path in $X$ is the same as its Euclidean length. \qed
Lemma 12.16. Suppose that expanded very bad balls satisfy \((C_{i}n)\) for any \(n\) with constants independent of the scale of the very bad ball. Then \((X, d_{X})\) satisfies \((C_{i}n)\) for any \(n\).

Proof. Let \(\gamma\) be a smooth Jordan curve in \(X\). We distinguish two cases according to the size of \(\gamma\) relative to \(\partial X\).

Suppose \(d_{X}(\gamma, \partial X) \leq 100 \text{diam}_{X}(\gamma)\). Set \(D_{0} = \text{diam}_{X}(\gamma)\) and \(D = \text{diam}_{\text{euc}}(\gamma)\). By Lemma 12.15 and 12.14, there exists \(C\) independent of \(\gamma\) such that \(D \leq CD_{0}\) and \(d_{\text{euc}}(\gamma, \partial X) \leq 100CD_{0}\). Choose the minimal natural number \(k\) such that \(CD_{0} \leq 2^{k}\). Now we choose the smallest Euclidean rectangle \(P\) containing \(\gamma\) with two vertices on \(\partial X\) and the other two at centers of very bad balls at height \(2^{k}\). By Definition 12.6 (5) and rotation symmetry of the metric on a bad ball, we know that Area\((P)\) and length\((\partial P)\) measured with respect to \(d_{\text{euc}}\) and \(d_{X}\) result the same value. The height and width of \(P\) is \(\leq 200CD_{0}\) by the distribution of very bad balls, thus we obtain Area\((P) \leq (200CD_{0})^{2}\). Since the diameter of \(\gamma\) is always less than twice its length, we found the required filling.

Suppose \(d_{X}(\gamma, \partial X) > 100 \text{diam}_{X}(\gamma)\). Note that \(\gamma\) lies either in an entirely Euclidean region or intersects at least one very bad ball \(B\). In the latter case, \(\gamma\) is the expanded very bad ball \(B'\) around \(B\) (if \(\gamma\) escapes \(B'\), then it travels through Euclidean regions whose size is comparable to the size of \(B\), which is comparable to \(d_{X}(\gamma, \partial X)\) by Lemma 12.15, this gives a lower bound for \(\text{diam}_{X}(\gamma)\) which contradicts \(\text{diam}_{X}(\gamma) < \frac{1}{100}d_{X}(\gamma, \partial X))\). As \(B'\) is surrounded by Euclidean regions, \(\text{diam}_{B}(\gamma) = \text{diam}_{X}(\gamma)\). This finishes the proof by our assumption. \(\square\)

The following is a consequence of estimates regarding the rectangle \(P\) in the proof of Lemma 12.16 and Lemma 12.15.

Corollary 12.17. The quasi-geodesic \(\partial X\) does not have super-Euclidean divergence.

Lemma 12.18. There is a constant \(A < \infty\) such that every expanded very bad ball satisfies a coning inequality with constant \(A\).

Proof. The proof is quite similar to Lemma 12.16 as it can be viewed as a “polar coordinate” version of Lemma 12.16. Let \(x\) be the center of the very bad ball \(B\) and \(\gamma\) be a smooth Jordan curve in \(B\). Again we consider two cases \(d(p, \gamma) \geq 100 \text{diam}(\gamma)\) and \(d(p, \gamma) < 100 \text{diam}(\gamma)\). The details are left to the reader. \(\square\)
Proof of Proposition 12.5. It remains to prove (3). Let \( \lim_{k} (X, x_k) = X_\omega \) be an asymptotic cone of \( X \) where \( (x_k) \) is a sequence on \( \partial X \) and \( \lambda_k \to \infty \). Let \( \ell_\omega \) (resp. \( x_\omega \)) be the limit of \( \partial X \) (resp. \( x_k \)). We argue by contradiction and suppose \( \ell_\omega \) bounds a bilipschitz half plane \( H_\omega \).

Let \( K \) be the closed upper half of the unit disk in \( \mathbb{R}^2 \). Suppose \( \partial K = s \cup s' \) where \( s \) is straight and \( s' \) is an arc on the unit circle. Let \( s \cap s' = \{v_1, v_2\} \). Let \( f_\infty : K \to X_\omega \) be a bilipschitz embedding such that \( f_\infty(0) = x_\omega, f_\infty(K) \subset H_\omega \) and \( f_\infty(s) \subset \partial H_\omega \). Then there is a sequence of Lipschitz maps \( f_k : K \to \frac{1}{\lambda_k} X \) such that \( \lim_{\omega} f_k = f_\infty, f_k(s) \subset \partial X \) and \( f_k(0) = x_k \). To construct such \( f_k \)'s, we take a fine enough triangulation of \( K \), approximating \( f_\infty \) on the 0-skeleton, then extending skeleton by skeleton using the coning inequality in \( X \).

By Lemma 12.13 and Lemma 12.15, there exists \( \delta > 0 \) such that for \( k \) sufficiently large, \( \frac{1}{\lambda_k} d_{\text{euc}}(f_k(s'), f_k(0)) > \delta \) and \( f_k(v_1), f_k(v_2) \) are in different components of \( \partial X \setminus B_k \) where \( B_k = \{x \in X \mid d_{\text{euc}}(x, f_k(0)) \leq \lambda_k \delta \} \). Thus \( B_k \subset \text{Im} f_k \). Thus \( \text{Im} f_k \) contains very bad balls of size comparable to \( \text{diam}(\text{Im} f_k) \). As \( \text{Im} f_\infty = \lim_{\omega} \text{Im} f_k \), we know \( \text{Im} f_\infty \) contains an isometrically embedded copy of an ultralimit of rescaled very bad balls. Thus \( \text{Im} f_\infty \) is not doubling by Lemma 12.10, which yields a contradiction. \( \square \)

Remark 12.19. We point out the following stronger result. Let \( Z \) and \( B_k \) be as in Lemma 12.10. We define the center of \( Z \) to be the limit of centers of the \( B_k \)'s. Then a point \( p \in X_\omega \setminus \ell_\omega \) either has an open neighborhood bilipschitz a small disk in \( \mathbb{R}^2 \), or has a neighborhood isometric to a neighborhood of the center of \( Z \), which is homeomorphic to \( \mathbb{R}^2 \). The proof is left to the interested reader.

Remark 12.20. It is also true that the ultralimit of the boundary \( \partial X \) above does not have full support. Pulling back Lipschitz maps from asymptotic cones as in the proof of Proposition 12.5, one can show that the coning inequality of \( X \) passes on to its asymptotic cones. This is enough to conclude that the definition of full support does not depend on the choice of homology theory. Hence we can conclude from Corollary 12.17 and Proposition 7.4.

12.3. An example. A truncated hyperbolic space of dimension \( \geq 3 \) provides an example of a CAT(0) space which contains Morse-geodesics and flats of dimension \( \geq 2 \). Taking products we obtain examples of CAT(0) spaces which contain Morse-flats which are not top-dimensional.

We are going to twist this example a bit to produce a smooth irreducible example.
Let $\bar{M}$ be a finite volume cusped hyperbolic manifold of dimension $n \geq 3$. Suppose that $\bar{M}$ contains a separating closed hypersurface $N$ which is totally geodesic \cite{RT98}, \cite{LR01}. We remove the cusps of $\bar{M}$ and deform the metric conformally near the boundary, leaving the metric unchanged in a neighborhood of $\bar{N}$, to obtain $\bar{M}'$, which is negatively curved in the interior and such that each component of its boundary is a totally geodesic flat torus. The double $\tilde{M}$ of $\bar{M}'$ contains a finite family of totally geodesic flat hypersurfaces and a closed totally geodesic hyperbolic hypersurface $\bar{N}$. Then there exist finite coverings $\beta : V \rightarrow M \times M$

of any degree, branched along $N \times N$. The pull-back metric on $V$ is locally CAT(0) \cite[Section 4.4]{Gro87} and can be smoothed near $\beta^{-1}(N \times N)$ to a metric of nonpositive sectional curvature \cite{FS90}.

Denote by $\pi_V : \tilde{V} \rightarrow V$ and $\pi_M : \tilde{M} \rightarrow M$ the universal covers of $V$ respectively $\bar{M}$. We obtain an induced covering map between universal covers $\tilde{\beta} : \tilde{V} \rightarrow \tilde{M} \times \tilde{M}$

which branches along $(\pi_M \times \pi_M)^{-1}(N \times N)$.

Since $\tilde{M}$ is a space with isolated $(n - 1)$-flats, the product space $\tilde{M} \times \tilde{M}$ has isolated $(2n - 2)$-flats. Further, a pair of geodesic in $\tilde{M}$ yields a 2-flat in $\tilde{M} \times \tilde{M}$. By the criterion in Proposition \ref{prop:2-flat-criterion} we see that if none of these geodesics lies in one of the isolated flats, then the associated 2-flat is Morse.

Now we turn to $\tilde{V}$. Since the top-dimensional tori in $M \times M$ are disjoint from $N \times N$, the corresponding top-dimensional flats in $\tilde{M} \times \tilde{M}$ lift to flats in $\tilde{V}$. In particular, $\tilde{V}$ contains flats of dimension $\geq 4$. Similarly, if we take two closed geodesics in $M$, both of them disjoint from $N$ and the flat hypersurfaces, then their product yields an immersed flat torus in $M \times M$. The inverse image of this torus under $(\pi_M \times \pi_M) \circ \tilde{\beta}$ is a discrete family of 2-flats, and each of these has to be Morse by Corollary \ref{cor:2-flat-criterion}.

13. Questions and Further Directions

There are many interesting aspects of Morse quasiflats yet to be explored. Some obvious further directions come from properties already known in the context of Morse quasi-geodesics or top rank quasiflats, but remained untouched for Morse quasiflats.
13.1. **Topology on the set of all Morse quasiflats.**

There are several topologies on the set of Morse quasi-geodesics [CS14, Cor17, CM19]. On the other hand, one can put a visual metric on the set of top rank quasiflats when the underlying metric space satisfies appropriate conditions [KL18, Section 6]. Thus we ask:

(1) How much of these discussions generalize to the collection of Morse quasiflats in a metric space, so that one has a nice topology on this collection which is quasi-isometry invariant and metrizable?
(2) Can one describe such topology for some interesting examples more concretely in terms of the specific combinatorial data of the underlying spaces/groups?
(3) Can one say something interesting on the dynamics of a group acting on its collection of Morse quasiflats (with interesting topology)? Can one construct natural invariant measures?

13.2. **Characterization of Morse quasiflats.** As discussed before, one can define $n$-dimensional Morse quasiflats by requiring that the corresponding bilipschitz flats in asymptotic cones have full support with respect to the reduced local homology. However, besides the case of CAT(0) space in Proposition 1.11, we do not know whether choosing different coefficients for homology gives equivalent definitions, though we expect this to hold in combable groups, as well as in metric spaces with convex geodesic bicombing, or more generally metric spaces with coning inequalities. We also conjecture that the “halfspace criterion” in Proposition 1.11 holds in the context of metric spaces with convex geodesic bicombing, in which case we replace flat by isometrically embedded normed vector spaces, and flat halfspace by halfspaces in normed vector spaces. However, there is a limit on how far we can push this “halfspace criterion”, see Section 12.2.

One can characterize Morse quasi-geodesics by looking at nearest point projections onto the quasi-geodesic and enforcing a “sublinear contraction” property for such projections [CS14, Cor17, ACGH17]. A question is to formulate a higher dimensional version of “sublinear contraction” and show that it is equivalent to other characterizations of Morse quasiflats mentioned in this paper. This is related to Gromov’s definitions of rank [Gro93, Definition VI, pp.86].

Morse quasi-geodesics were characterized by “middle third recurrent” in [DMS10, Proposition 3.24 (3)]. Another question is to weaken the
$(\mu, b)$-rigid condition (Definition 6.13) in the spirit of “middle third recurrent”, and show equivalence with the $(\mu, b)$-rigid condition.

Many characterizations of Morse quasi-geodesics also characterizes more general “stable” subsets of the metric space without much modification. Recall that a quasi–convex subspace $Y$ of a geodesic metric space $X$ is $N$-stable if every pair of points in $Y$ can be connected by a geodesic which is $N$-Morse in $X$. A different notion of stability was introduced in [DT15]. We expect the notion of stability and its different characterizations to have higher rank analogues, which generalizes Morse quasiflats. Maximal higher rank submanifolds like $H^2 \times \mathbb{R}$ in $SL(3, \mathbb{R})/SO(3, \mathbb{R})$ are natural candidates for “stability” in a higher rank sense, see also [Sch89].

References


[Ol'] Ol'shanskiĭ. Periodic factor groups of hyperbolic groups.


