Math Sem "ΨDO"
WS 18/19

(1)

Re: Overview & Outlook | 6-2-2019 (last!)

Overview: ΨDO: For $a = a(x,\xi)$, $x \in \mathbb{R}^n$, $\xi \in (\mathbb{R}^n)^* = \mathbb{R}^n$:

$$[a(x,D)u](x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} a(x,\xi)\, \hat{u}(\xi)\, d\xi$$

$$\overset{(!)}{=} \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n}\Big(\int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi} a(x,\xi) d\xi\Big) u(y)\, dy$$

"Programme": (1) Spaces of functions / "objects" (distributions)

- here: $\mathcal{S}(\mathbb{R}^n)$ (Fréchet-space: metric)

$\mathcal{S}'(\mathbb{R}^n)$ temp. distr. (dual).

& Fouriertransform on these

& its interplay with other "operations", in particular derivatives: $i\nabla_x \overset{\mathcal{F}}{\longleftrightarrow}$ mult. with ξ

Last space (normed! – Hilbert!): Sobolev: H^m "$\mathcal{S}'$ + controlled (integral) behaviour of $\partial^\alpha u$"

(2) Symbols & operations etc. "$\int e^{i\varphi(z)} A(z)\, dz$"

symbols ~ amplitudes. Classes here:

S^m: $a = a(x,\xi)$ with

$$\left|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)\right| \le C_{\alpha\beta}\left(1+|\xi|^2\right)^{\frac{m-|\beta|}{2}}$$

$$\left(\sim |\xi|^{m-|\beta|}\right) \;(!)$$

$S^l \subseteq S^m$ for $l \le m$; $S^\infty = \bigcup_m S^m$ (big) , $S^{-\infty} = \bigcap_m S^m$ (small).

($a \in S^m$, $b \in S^l \Rightarrow ab \in S^{m+l}$; $\partial_x^\alpha \partial_\xi^\beta a \in S^{m-|\beta|}$

In particular: $a, b \in S^0 \Rightarrow a+b,\ a \cdot b \in S^0$

$F \in C^\infty(\mathbb{R}^2) \Rightarrow F(a) \in S^0 \quad (a \in S^0)$

Ex's: $\lambda^m(\xi) = \left(1+|\xi|^2\right)^{m/2}$, $\lambda^m \in S^m$

PDO: (non-cst. coeff's) $a(x,\xi) = \sum_{|\alpha| \le m} a_\alpha(x)\, \xi^\alpha$, $a_\alpha \in H^\infty$ (say)

$a \in S^m$!

Asymptotic expansion ($m \in \mathbb{R}$) $\{a_j\}_{j \in \mathbb{N}}$ with $a_j \in S^{m-j}$

$\Rightarrow$ " $\sum_j a_j \in S^m$ " !

$\exists\, a \in S^m$ (unique modulo $S^{-\infty}$):

$$a - \sum_{j<k} a_j \in S^{m-k}$$

(+ info support). (" $a \sim \sum_j a_j$ ")

Operations: $a \in S^m \Rightarrow a^* \in S^m$

$a \in S^m, b \in S^l \Rightarrow a \# b \in S^{m+l}$

(+ asymp. exp. as fct. of a, resp a, b)

(use: see later).

I.e. have algebra of symbols (see also later for op's!)

Inversion: "One can invert elliptic symbols (i.e. get another symbol)" (modulo $S^{-\infty}$)

elliptic (cond. on fct./symbol):

$$\exists \varepsilon > 0 : \quad |a(x,\xi)| \geq \varepsilon (1+|\xi|^2)^{m/2} \quad \text{for } |\xi| \geq \frac{1}{\varepsilon}$$

(cond. for large $|\xi|$: $\quad |a(x,\xi)| \gtrsim |\xi|^m, \quad a \in S^m$)

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3) Operators on $\mathcal{S}$, $\mathcal{S}'$, H^m.

Def. $$[a(x,D)\varphi](x) := \frac{1}{(2\pi)^n} \int e^{ix\cdot\xi} a(x,\xi)\, \hat{\varphi}(\xi)\, d\xi$$

(ok for $\varphi \in \mathcal{S}$ & cont. $\mathcal{S} \to \mathcal{S}$).

On $\mathcal{S}'$: "via duality" (& info on a^*):

$$(a(x,D)u, \varphi) := (u, a^*(x,D)\varphi)$$

$u \in \mathcal{S}', \varphi \in \mathcal{S}$

(cont. on $\mathcal{S}' \to \mathcal{S}'$)

On $H^s(\mathbb{R}^n)$: $a \in S^m \Rightarrow$
$(m \in \mathbb{R})$

$$a(x,D) : H^s \to H^{s-m}$$

cont. (lin. bdd. ops.)

Ex: λ^m & PDO as before!

Important: $$a(x,D)e^{ix\cdot\xi} = a(x,\xi)e^{ix\cdot\xi} \quad (!)$$

(ex. 3.5 p. 51)

Quatizatn (lin. alg.:

LA: ~~Matrix~~ lin. map A; basis $\{e_i\}$

$$x = \sum_i \langle e_i, x\rangle e_i \quad ; \quad \boxed{Ax = \sum_i \langle e_i, x\rangle A e_i}$$

$$A e_i = \sum_j \langle e_j, A e_i\rangle e_j \quad (x = Ae_i)$$

$$\to \quad Ax = \sum_i \langle e_i, x\rangle \sum_j \underbrace{\langle e_j, Ae_i\rangle}_{\equiv a_{ij} \text{ (or } a_{ji}\ldots)} e_j = \sum_{i,j} a_{ij} \langle e_i, x\rangle e_j$$

Quantizatn (ΨDO: "Basis" $\{e_\xi\}_{\xi \in \mathbb{R}^n}$; $e_\xi(x) = e^{ix\xi}$

$$\hat{\varphi}(\xi) = \langle e_\xi, \varphi\rangle_{L^2} = \int e^{ix\xi} \varphi(x)\,dx$$

$$(a(x,D)\varphi)(x) = \int e^{ix\xi} a(x,\xi)\,\hat{\varphi}(\xi)\,d\xi$$

$$\overset{(!)}{=} \int a(x,D)\,e^{ix\xi} \langle e_\xi, \varphi\rangle\,d\xi$$

$$\boxed{(a(x,D)\varphi)(x) = \int \langle e_\xi, \varphi\rangle\, a(x,D)\, e_\xi \, d\xi}$$

"Take scalar prod. with basisvector, multiply with <u>action</u> of op. <u>on</u> basisvector & <u>sum.</u> over basis = action of op."

Can do "other basis" than "Fourier" ($e^{ix\xi}$)

f. ex. "coherent states" / (anti-) Wick quantization

(Gaussians in x, ξ)

Applications (i) Elliptic regularity:

Ex. $a \in S^m$, a elliptic, $f \in C^\infty$ given,

u solution to $a(x,D)u = f$ $(u \in \mathcal{S}')$

– then u also C^∞.

(2) Gårding: $a \in S^{2m}$, $\mathrm{Re}\, a$ elliptic

$(\mathrm{Re}\, a(x,\xi) \geq \varepsilon \lambda^{2m}(\xi), \ |\xi| \geq \frac{1}{\varepsilon})$

$\Rightarrow \forall N \ \exists C_N > 0$:

$$2\,\mathrm{Re}\,(a(x,D)\varphi, \varphi)_{L^2} \geq \varepsilon \|\varphi\|_m^2 - C_N \|\varphi\|_{m-N}^2$$

$(\forall \varphi \in \mathcal{S})$.

Application: Solvability (Chapt. 4.1)

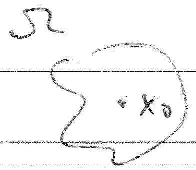
PDO: $a(x,D) = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha$, a_α C^∞.

Principal symb: $p(x,\xi) = \sum_{|\alpha| = m} (\dots)$

Op. $a(x,D)$ is locally solvable at $x_0 \in \mathbb{R}^n$

$:\Longleftrightarrow \exists\, \Omega \ni x_0 : \ \forall f \in C_0^\infty(\Omega) \ \exists$ sol. $u \in \mathcal{D}'(\Omega)$:

$$a(x,D)u = f$$



(Note: Lewy, Mizohata:

$(x,y) \in \mathbb{R}^2$ $\quad \frac{\partial u}{\partial x} + ix \frac{\partial u}{\partial y} = f(x,y)$

sometimes has no sol's!)

Th 1: $a(x,D)$ loc. solv. at x_0

$\iff$ an estimate for $a^*(x,D)$ (the one below!)

Th 2: (--- some 2 ξ cond. on a ---) $(a \in S^m)$

$\Rightarrow \exists \delta > 0 : \forall \varphi \in C_0^\infty(B_\delta)$:

$$\|\varphi\|_{m-1} \leq \|a^*(x,D)\varphi\|_0$$ ($\Rightarrow$ loc. solv. via HB or sim.)

Th. 3: Th. 2 follows from two estimates, one following from each of 2 cond's in Th. 2.

One of these is proved using Gårding (on an auxiliary symbol...)

Philosophy: The 2 cond's in Th. 3:

Find (checkable) ("geometric") conditions on $a = a(x,\xi)$ that ensures Th. 3, hence Th. 2, hence loc. solvability!

2 cond's:

(1) $a(x,D)$ of principle type at x_0

$\iff \nabla_\xi p(x_0,\xi) = 0 \Rightarrow \xi = 0$

("ξ-gradient of principle symbol at x_0 only zero for $\xi = 0$").

(2) $a(x,D)$ principally normal at x_0

$\iff \exists q \in C^\infty(T^*\mathbb{R}^n \setminus \{0\})$, hom. in ξ of degree $m-1$
$(\mathbb{R}^n_x \times \mathbb{R}^n_\xi \setminus \{0\})$

such that

$$\{\bar{p}, p\}(x,\xi) = 2i\,\mathrm{Re}\big(\bar{q}(x,\xi)\,p(x,\xi)\big) \quad \forall \xi \in \mathbb{R}^n \setminus 0$$
$$\forall x \in \underline{B_\delta(x_0)}$$

($\{\bar{p}, p\}$ Poisson bracket of $\bar{p}(x,\xi)$ & $p(x,\xi)$).

For more, see Chap. 4.1.

Generalisation:

$$[a(x,D)u](x) = \int\left(\int e^{i(x-y)\cdot\xi} a(x,\xi)\,d\xi\right) u(y)\,dy$$

(1) generalise symbolclass for a
(or, just different)
– see book Lerner

(2) generalise phase fct. $i(x-y)\cdot\xi$
$\rightsquigarrow i\varphi(x,y,\xi)$.
(FIO). (hyperbolic)