Introduction to the theory of distributions

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Errata and questions - I

Neither completeness nor correctness is claimed for this list. I am, in fact, rather certain that it is neither complete nor correct (i. e., neither does it contain all errors, nor is everything I consider wrong actually wrong).

I have not read the whole text, but I *did* read Sections 1-4 in detail, and caught some things in Sections 5, 7 and 8 and the appendix as well.

I have not met many serious mathematical errors in the book. What I have met are lots of typos (probably introduced at typesetting stage) and some inaccuracies. It is still a very readable and useful book.

Section 1

- **page 10, proof of Theorem 1.3.3:** When you write "The second assertion follows from Theorem 1.3.2", I think you don't mean Theorem 1.3.2 itself, but rather the proof of this theorem.
- **page 11, Theorem 1.4.1:** Replace $\psi_i, ..., \psi_m$ by $\psi_1, ..., \psi_m$.
- **page 12, proof of Theorem 1.4.3:** Replace n by 1 in "and a corresponding set of functions $\psi'_n, ..., \psi'_l$".
- **page 13, note on partitions of unity:** You refer to "the proof of Theorem 1.4.4"; you mean the proof of Theorem 1.4.1 instead.
- **page 13, note on partitions of unity:** Replace $(\psi_i)_{1\leq j<\infty}$ by $(\psi_j)_{1\leq j<\infty}$.

Section 2

- **page 18, between (2.2.2) and (2.2.3):** Replace $a \in R$ by $a \in \mathbb{R}$ (different font!).
- **page 19, the computation that proves (2.2.8):** Replace $-\partial\phi(x)\log|x|\,dx$ by $-\int\partial\phi(x)\log|x|\,dx$.
- **page 24, (2.5.4):** Replace $(\partial_i f)$ by $(\partial_i f)\cdot u$.
- **page 25, between (2.6.1) and (2.6.2):** Replace "at least one $a_\alpha \equiv 0$" by "at least one $a_\alpha \not\equiv 0$".
- **page 27, Lemma 2.7.1:** Replace "$j = 1, ..., m-1$" by "$j = 0, ..., m-1$".
- **page 28, proof of the theorem:** You write: "On the other hand, it is an easy exercise to show that $x^m\partial^j\delta = 0$ for $j < m$. This settles part (i)." I think this rather settles a converse of part (i) (which you never have formulated).
- **page 32, Exercise 2.15 (ii):** A bracket is missing after the exp.

Section 3

- **page 35, proof of Theorem 3.1.2:** You write: "So (3.1.3) extends u to a linear form on $C^\infty(X)$ uniquely." This is correct, but this does not yet prove that there is no other extension of u to a linear form on $C^\infty(X)$. In fact, it could possibly be that some extension of u to a linear form on $C^\infty(X)$ cannot be obtained by (3.1.3) whatever function is chosen in place of ρ. Fortunately, it is trivial to rule this out.

- **page 36, Note (after proof of Theorem 3.1.3):** Why do you require that $X = \mathbb{R}^n$ here?

- **page 37, the inequality after (3.2.3):** I am talking about the inequality

$$\left|\partial^\beta \phi(x)\right| \leq \varepsilon^{N-|\beta|+1} \sum_{|\gamma|=N+1-|\beta|} \sup\{|\partial^\gamma \phi(x)| : |x| < |\} /\gamma! \text{ if } |x| \leq \varepsilon.$$

  I think the $\sup\{|\partial^\gamma \phi(x)| : |x| < |\}$ should be $\sup\left\{\left|\partial^\gamma \partial^\beta \phi(x')\right| : |x'| < 1\right\}$ here. I am not sure about $|x'| < 1$ though; it might also be $|x'| < \varepsilon$ or $|x'| < |x|$.

- **page 37, proof of Lemma 3.2.1:** Replace (3.5.3) by (2.5.3) in "By Leibniz's theorem (cf. (3.5.3))".

- **page 37, proof of Lemma 3.2.1:** Replace $|\partial^\alpha(\phi/x)\psi(x/\varepsilon))|$ by $|\partial^\alpha(\phi(x)\psi(x/\varepsilon))|$. [Note that you seem to be using two different letters (ε and ϵ) for the same thing (probably due to different layers of typography). I write ε for both of them.]

- **page 37, proof of Theorem 3.2.1:** In "Then, clearly, $\phi' \in C_c^\infty(\mathbb{R}^n)$ and $\partial^\alpha \phi'(0)$ if $|\alpha| \leq N$", add a $= 0$ after $\partial^\alpha \phi'(0)$.

- **page 38, proof of Theorem 3.2.2:** I think your $0 < \frac{1}{4}\varepsilon < \delta$ actually should be $0 < 4\varepsilon < \delta$. I am not sure about what bounds on ε are actually needed, though.

- **page 38, proof of Theorem 3.2.2:** Replace $1\langle u, \phi\psi_\varepsilon\rangle\,|$ by $|\langle u, \phi\psi_\varepsilon\rangle|$.

- **page 38:** Both references to (3.2.1) on this page should refer to (3.2.3) instead.

- **page 39:** The left hand side of (3.2.8) should be enclosed in absolute-value brackets.

- **page 39, Exercise 3.1:** Replace x by X here.

Section 4

- **page 40, Theorem 4.1.1:** Replace $Y \subseteq \mathbb{R}^n$ by $Y \subseteq \mathbb{R}^m$.

- **page 41, proof of Theorem 4.1.1:** Replace $\partial(\partial y_j$ by $\partial/\partial y_j$ (this typo appears 2 times in this proof: one time in (4.1.3), another time in the equation preceeding (4.1.3)).

- **page 41, proof of Theorem 4.1.1:** In "the $\partial_x^\alpha \chi(x, y')$ converge uniformly to 0 as $\varepsilon \to 0$", the χ should be a χ_ε.

- **page 43, Definition 4.2.2:** In (4.2.7), the $<$ sign should be a $\langle$ bracket.

- **page 45, Lemma 4.3.1:** This is purely a matter of taste, but I would write $N \geq 1$ rather than $N > 1$. Granted, the $N = 1$ case is completely trivial, but requiring $N > 1$ creates an impression that $N = 1$ wouldn't work.

- **page 45, proof of Lemma 4.3.1:** You write (at the very end of this proof): "these functions, which are of the form (4.3.2), converge to ϕ in $C_c^\infty(I)$". Actually they are not. The function ϕ_m that you construct does not have the form $\sum\limits_{j=1}^{m} \psi_{j1}(z_1) \ldots \psi_{jN}(z_N)$ required by (4.3.2) but rather the form $\sum\limits_{j=1}^{T(m)} \psi_{j1}(z_1) \ldots \psi_{jN}(z_N)$ for some $T(m) \in \mathbb{N}^+$ (concretely, $T(m)$ is the number of all $(g_1, \ldots, g_m)$ satisfying $|g_1| \leq m, \ldots, |g_N| \leq m$). When passing from m to $m + 1$, not *one* summand but *many* summands (namely, $T(m+1) - T(m)$ of them) are added to ϕ_m. Fortunately, this is not a problem because we can add in these summands one by one instead of all of them at the same time, without destroying the convergence (here we really use that $|g|^M \widehat{\phi}_g \to 0$ for every $M \geq 0$).

- **page 47, proof of Theorem 4.3.3 (i):** You write: "We have defined $u \otimes v$ by means of (4.3.7)." What you mean is: "[...] by means of the first part of (4.3.7)".

- **page 47, proof of Theorem 4.3.3 (ii):** Replace $y \in \operatorname{supp} y$ by $y \in \operatorname{supp} v$.

- **page 47, proof of Theorem 4.3.3 (ii):** Replace "one can find $\phi \in C_c^\infty(X)$ and $\phi \in C_c^\infty(Y)$" by "one can find, for any neighbourhoods X' and Y' of x and y, some functions $\phi \in C_c^\infty(X')$ and $\phi \in C_c^\infty(Y')$".

- **page 48, the computation directly below (4.3.14):** In

$$\left\langle u(x', x_n), \chi(x_n) \int \phi(x', t)\, dt, \right\rangle$$

the comma should come after the brackets rather than inside them.

- **page 49, Exercise 4.2:** Replace "of u" by "of $A^* u$".

- **page 49, Exercise 4.4:** Replace $\langle u(x), \phi(x, y)\rangle\rangle$ by $\langle u(x), \phi(x, y)\rangle$.

Section 5

- **page 51, (5.1.1):** The $=$ sign in (5.1.1) should be a $\subseteq$ sign.

- **page 52, proof of Theorem 5.1.2:** Here you show that if $A \subseteq \mathbb{R}^n$ is compact and $B \subseteq \mathbb{R}^n$ is closed, then $A + B$ is closed. This is correct, but it was already used twice on page 51, I think.

- **page 52, (5.1.5):** Replace ∂_i by ∂_j here.

- **page 54, proof of Theorem 5.2.2:** You write $\operatorname{supp}\psi_j \subseteq \operatorname{supp}\psi$ at the beginning of this proof. This is generally wrong; for example, the support of ψ might be a circular ring around 0 (for example, $\overline{B}(0,2) \setminus B(0,1)$). What you mean is that $\operatorname{supp}\psi_j \subseteq$ (convex hull of $\operatorname{supp}\psi$), which is just as good for the proof.

- **page 54, (5.2.6):** Replace $x(\varepsilon_j$ by x/ε_j in (5.2.6).

- **page 55, proof of Theorem 5.2.3:** I have troubles with understanding the "there is a k such that $\langle u, \phi\rangle = \langle u_k, \phi\rangle$ and [...]" part. How do you choose such k ? Fortunately, in the case when K_k is defined as the set $\left\{x \in X \;\mid\; \operatorname{dist}(x, \mathbb{R}^n \setminus X) \geq \dfrac{1}{k}\right\}$ for every k, these troubles disappear, so the proof is okay, but in the general case I don't see it.

- **page 55, proof of Theorem 5.2.3:** You write $\operatorname{supp}\psi_j \subseteq \operatorname{supp}\psi_1$. Again, this is wrong (just as the $\operatorname{supp}\psi_j \subseteq \operatorname{supp}\psi$ on page 54); again this is easy to fix.

- **page 55, Lemma 5.3.1:** Replace $A_1^\varepsilon, ..., A_m^\varepsilon$ by $A_1^\varepsilon \times ... \times A_m^\varepsilon$.

- **page 56, proof of Lemma 5.3.1:** Replace "$x \in A$, $y \in B$" by "$x \in A^\varepsilon$, $y \in B^\varepsilon$".

- **page 56, proof of Lemma 5.3.1:** Replace "$|x| = |x - x'| \leq \delta' + \varepsilon$" by "$|x| = |x - x' + x'| \leq \delta' + \varepsilon$".

- **page 56, proof of Lemma 5.3.1:** Replace "any points in A and B" by "any points in A^ε and B^ε".

- **page 56, shortly before (5.3.2):** I don't see why one can "clearly" choose functions $\rho_1, ..., \rho_m$ in $C_c^\infty(\mathbb{R})$ such tat $\rho\left(x^{(1)}, ..., x^{(m)}\right) = \rho_1\left(x^{(1)}\right) \otimes ... \otimes \rho_m\left(x^{(m)}\right)$ is supported in $K_\varepsilon(\phi)$ and $\rho = 1$ on a neighbourhood of $K_0(\phi)$. As far as I understand, $K_0(\phi)$ needs not be bounded away from the complement of $K_\varepsilon(\phi)$, unless you want to replace $\operatorname{supp}\phi\left(x^{(1)} + ... + x^{(m)}\right)$ by $\left(\operatorname{supp}\phi\left(x^{(1)} + ... + x^{(m)}\right)\right)^\varepsilon$ in the definition of $K_\varepsilon(\phi)$ (or maybe replace "supported in $K_\varepsilon(\phi)$" by "compactly supported"?).
  At this point, let me add that I am not really convinced by the use of (5.3.2) as a definition of the convolution of several distributions (not all of which have compact support). I would personally proceed differently: I would show that whenever a distribution $u \in \mathcal{D}'(X)$ and a closed subset Q of X are given such that $\operatorname{supp} U \subseteq Q$, we can uniquely extend u to a linear form on $\{\phi \in C^\infty(X) \;\mid\; Q \cap \operatorname{supp}\phi \text{ is compact}\}$. This is a kind of generalization of Theorem 3.1.2, and apparently allows us to remove the ρ from (5.3.2), making the theory more transparent. But this looks too simple; I have probably done something wrong here.

- **page 56, one line above Theorem 5.3.1:** You write u_1*...*u_m here; this is the wrong kind of * sign. You want a centered $*$.

- **page 56, Theorem 5.3.2 (ii):** Replace $i \in J$ by $i \in I$.

- **page 57:** You claim that the proofs of all parts of Theorem 5.3.2 are trivial. I would not claim this about (iii); but it might be that I am overthinking this part. At least the proof seems to require the fact that proper maps are closed; maybe it wouldn't hurt to mention this.

- **page 59, between (5.4.1) and (5.4.2):** Replace "Theorem 4.3.1" by "Theorem 4.3.3".

- **page 60, proof of Theorem 5.4.1:** You write: "and set $\rho_\varepsilon(x) = \varepsilon^{-n}(x/\varepsilon)$". This should be "and set $\rho_\varepsilon(x) = \varepsilon^{-n}\rho(x/\varepsilon)$".

- **page 60, proof of Theorem 5.4.1:** You write: "the continuous functions f_ε converge, uniformly when x is in a compact set, to $f(x) = \langle u(y), E_{N+2}(x-y)\rangle$ as $\varepsilon \to 0$". I think the $u(y)$ should be a $\psi u(y)$ here.

- **page 61, the formula for Δ in polar coordinates:** The $\dfrac{h-1}{r}$ here should be $\dfrac{n-1}{r}$.

- **page 62, last computation on this page:** In $-\left\langle \int \dfrac{1}{z}, \dfrac{\partial\phi}{\partial\overline{z}} \right\rangle$, I don't think the $\int$ sign is appropriate. Besides, the last term, $\lim\limits_{\varepsilon\to 0+} \int_{|x|=\varepsilon} \dfrac{\phi}{z}(\mathrm{d}\,x_2 - i\,\mathrm{d}\,x_1)$, should have a $\dfrac{1}{2}$ factor in front of it, unless I am mistaken.

- **page 63, first formula on this page:** I think $\dfrac{\partial}{\partial\overline{z}}\left\langle \dfrac{1}{\overline{z}}, \phi \right\rangle$ should be $\left\langle \dfrac{\partial}{\partial\overline{z}}\dfrac{1}{\overline{z}}, \phi \right\rangle$. Besides, the $\phi(0)$ should be $\pi\phi(0)$.

- **page 64, the first formula on this page (the definition of $C^0\left(\mathbb{R}^n \times \overline{\mathbb{R}}^+\right)$):** This is completely misaligned, and the $\overline{v}$ should be a $\widetilde{v}$. Altogether, the formula should be

  $$C^0\left(\mathbb{R}^n \times \overline{\mathbb{R}}^+\right) = \left\{v \in C^0\left(\mathbb{R}^n \times \mathbb{R}^+\right) \ : \ v = \widetilde{v} \mid \mathbb{R}^n \times \mathbb{R}^+ \text{ for some } \widetilde{v} \in C^0\left(\mathbb{R}^n \times \mathbb{R}\right)\right\},$$

  and if it is necessary to break it in two lines, I would rather break it at the first equality sign.

Section 7

- **page 81, between (7.2.1) and (7.2.2):** There is a noticeable concentration of mistakes here. First, $x = \mathbb{R}^n$ should be $x \in \mathbb{R}^n$. Second, "so that $f(x) = y$" makes no sense (there is no y anywhere near this place). Third, I think a sentence, which should say something like "Let us first assume that f is the function $\mathbb{R}^n \to \mathbb{R}$ which maps every vector to its n-th coordinate.", is missing here. Fourth, it should be made clear that (7.2.2) is not an equality derived from something else, but is supposed to be the definition of f^*u (unless it is me who is mistaken here).

- **page 83, (7.2.7):** This equation is not really formally correct. The left hand side, f^*u, is a distribution on X (or, if you want, on U_y), while the right hand side, $\left|\det g_y'(\xi)\right| 1(\xi') \otimes u(\xi_n)$, is a distribution on $h_y U_y$ (otherwise the tensor product sign does not make sense). I think the correct version would be

$$f^*u = h_y^*\left(1(\xi') \otimes u(\xi_n)\right)$$

(where we already know what h_y^* is, since h_y is a diffeomorphism).

- **page 83, between (7.2.7) and Corollary 7.2.1:** You write: "As it is immediate from (7.2.7)". Is this really (7.2.7), and not (7.2.6)?

- **page 83, two lines beneath (7.2.9):** I do not understand what the "Hence" here means. Where is the bounded convergence $\psi(t/\varepsilon) \to H(t)$ used?

- **page 84, first absatz of this page:** You write: "It has been assumed for simplicity that f is a map $X \to \mathbb{R}$. The reader should have no difficulty in verifying that Theorem 7.2.1 and its consequences hold equally for any $f \in \mathcal{C}_c^\infty(X \to Y)$ when Y is an open subset of $\mathbb{R}$, and (7.2.1) is assumed." At this place, the reference to (7.2.1) might easily be misunderstood. (7.2.1) claims that $f' \neq 0$ on X, while the correct condition is that f' is surjective at every point of X (this correct condition is given in Theorem 7.2.2).

- **page 84, between (7.2.11) and (7.2.12):** Here you write

$$\overline{f}^* u = \left|\det \overline{g}'(\xi)\right| 1(\xi') \otimes u(\xi'') .$$

This formula is inaccurate in the same way as (7.2.7) (see above).

- **page 85, between (7.3.1) and (7.3.2):** The relation $f^*\delta = \phi_f(0)$ should be $(f^*\delta)(\phi) = \phi_f(0)$.

- **page 85, (7.3.2):** Replace $\phi(x', x^4)$ by $\phi(x', x_4)$ on the left hand side of this equation.

- **page 86, between (7.3.8) and (7.3.9):** Here you write

$$\begin{aligned}&\frac{1}{2}t^{-4}\int \phi'(x/t, |x'|/t)\,|x'|^{-1}\,dx' \\ &= \frac{1}{2}t^2 \int \phi'(x, |x'|)\,|x'|^{-1}\,dx' = t^2 \left\langle \delta_+(f), \phi\right\rangle .\end{aligned}$$

I think that both of the two t^2 in this equation should be t^{-2} instead.

- **page 86, (7.3.9):** The C here should be a $\mathbb{C}$.

- **page 87, (7.3.13):** An opening { bracket should be added directly after the first equality sign in this equation.

- **page 87:** At the very bottom of this page, you write:

$$= \frac{1}{4\pi}\int \phi(x)\,\frac{v(y', y_4 - |x'-y'|)}{|x'-y'|}dxdy'.$$

The y_4 in this term should be x_4.

- **page 89, Exercise 7.4:** Something seems to be wrong about the claim that $F = \delta^{(n-3/2)}(t)$ when n is even (do you really want a fractional power of δ? what would that mean?).

Section 8

- **page 91, Theorem 8.1.2:** I think it would be appropriate to add a newline between "(iii) the integral $\int_X f(x, t_0)\,dx$ exists for some $t_0 \in J$." and "Then". Otherwise it looks like the "Then" still belongs to property (iii).

- **page 92, proof of Theorem 8.1.3 (iii):** Replace $\int (f * g)\widehat{}$ by $(f * g)\widehat{}\,(\phi)$.

- **page 93, proof of Theorem 8.1.3 (iii):** Replace x by z in $\int f(y)\,dy \int g(z)\,e^{-ix\cdot\xi} e^{-iy\cdot\xi} dz$.

- **page 94, proof of Lemma 8.2.1:** Replace "(8.3.4)" by "(8.2.4)".

- **page 95, proof of Lemma 8.2.2:** Replace $D^\alpha \phi$ by $D^\alpha \widehat{\phi}$.

- **page 96, proof of Theorem 8.2.2:** The very last equation of this proof,

$$(\tau_{-h}\phi)\widehat{} = \widehat{\phi}(e)\, e^{i\xi\cdot h}, \qquad h \in \mathbb{R}^n,$$

  should clearly have $\widehat{\phi}(\xi)$ instead of $\widehat{\phi}(e)$.

- **page 96, (8.2.10):** I think you forgot to define what $\check{\phi}$ means. (You do give this definition later, however: Between (8.3.11) and (8.3.12), you say that $A^* u = \check{u}$ for $A = -I$.)

- **page 97, three lines beneath (8.3.1):** Replace R by $\mathbb{R}$ in $\mathcal{S}(R^n)$.

- **page 98, proof of Theorem 8.3.1:** I do not understand how you obtain (8.3.3). If we really apply (8.3.1) to $\rho\phi$, we obtain

$$|\langle u, \rho\phi\rangle| \le C \sum_{|\alpha|,|\beta|\le N} \sup \left| x^\alpha D^\beta (\rho\phi)\right|.$$

  Of course, the left hand side of this inequality equals $|\langle u, \phi\rangle|$ because $\rho = 1$ on a neighbourhood of $\operatorname{supp} u$. But why can we estimate the right hand side, $C \sum_{|\alpha|,|\beta|\le N} \sup \left| x^\alpha D^\beta (\rho\phi)\right|$, by a term of the form $C' \sum_{|\alpha|,|\beta|\le N} \sup \left\{ x^\alpha \left| D^\beta \phi(x)\right| : x \in \Omega\right\}$ (with C' being a constant independent of ϕ)? This would be possible if all derivatives of ρ would be bounded on Ω, but who guarantees us this? (Actually, I am pretty sure that there are situations when the derivatives of ρ cannot be bounded. For example, such a situation should be when $\operatorname{supp} u$ is a subset of Ω that comes the closer to $\partial\Omega$ the farther one goes from the origin.)

- **page 99, (8.3.6):** Replace $(\mathbb{R}^n)$ by $\mathcal{S}(\mathbb{R}^n)$ here.

- **page 99, proof of (8.3.9):** Here you write:

$$\tau_{-h}\widehat{\phi}(\xi) = \phi(\xi + h) = \int e^{-ix\cdot(\xi+h)}\phi(x)\,d\xi = \left(\phi e^{-ix\cdot h}\right)^{\widehat{}}.$$

  In this formula, replace $\phi(\xi + h)$ by $\widehat{\phi}(\xi + h)$, and replace $d\xi$ by dx.

- **page 100, computation of** $(A^* u)^{\widehat{}}$ **for invertible matrix** A**:** Three times during this computation, R^n should be replaced by $\mathbb{R}^n$.

- **page 100, computation of** $(A^* u)^{\widehat{}}$ **for invertible matrix** A**:** You seem to be using that $\mathcal{S}(\mathbb{R}^n)$ is dense in $\mathcal{S}'(\mathbb{R}^n)$ here (otherwise, how would you find a sequence $u_j \in \mathcal{S}(\mathbb{R}^n)$ which converges to u in $\mathcal{S}'(\mathbb{R}^n)$?). This might be worth pointing out explicitly (and proving, unless it is the same argument as for Theorem 5.2.2).

- **page 105, between (8.5.4) and (8.5.5):** In

$$C_m \sum_{|\alpha|=m} \sup |\,(1 + |x|)^{n+1}\,|D^\alpha \phi|\,,$$

  there is one | sign too much.

- **page 105, between (8.5.6) and (8.5.7):** In

$$\left(e^{2\pi i x_j} - 1\right)(v\psi) = 0, \qquad j = 1, ..., h,$$

  replace h by n.

- **page 106, proof of Corollary 8.5.1:** Replace v by u here.

- **page 108, (8.6.3):** Replace $\sum\limits_{|\alpha| \leq m}$ by $\sum\limits_{|\alpha|=m}$ here.

Section 9

- **page 115, definition of Cauchy sequences in an inner product space:** You write: "it is called a Cauchy sequence if $||\phi_j - \phi_k|| \to 0$ as $j, k = \infty$." There is an obvious typo here (= should be $\to$).

- **page 116, proof of Theorem 9.1.1:** You write: "Clearly, $||\psi_0|| = d$, whence $\psi_0 \notin \ker u$". Why does $\psi_0 \notin \ker u$ follow from $||\psi_0|| = d$? (And why do you need $\psi_0 \notin \ker u$ anyway?)

- **page 125, Theorem 9.3.5:** Replace $\widehat{\phi}(\eta)\,|d\eta$ by $\widehat{\phi}(\eta))d\eta$ in the formula.

Appendix

- **page 164, (6):** The } bracket is at the wrong place in this formula. It should be after the $p(x) < \varepsilon$, not after the $\varepsilon > 0$.