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This is an overview of material needed for the course. It consists of three parts, material from MEASURE AND INTEGRATION THEORY (Ana3), from FUNCTIONAL ANALYSIS (FA1), and from COMPLEX ANALYSIS (Fkt). We assume knowledge of metric, and more generally, topological spaces. A special case, also assumed known, is that of (semi-)normed vector spaces.

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MEASURE AND INTEGRATION THEORY

Let X be a non-empty set (for example $X = \mathbb{R}^d$ or any subset of \mathbb{R}^d), and let $\mathcal{P}(X)$ be the family of all subsets of X (its *power set*).

Definition 1 (σ -algebra). A family of subsets of X , $\mathcal{A} \subset \mathcal{P}(X)$, is called a σ -algebra (on X) if and only if ('iff')

- (i) $X \in \mathcal{A}$.
- (ii) $A \in \mathcal{A} \Rightarrow X \setminus A \in \mathcal{A}$.
- (iii) $(A_j \in \mathcal{A} \text{ for all } j \in \mathbb{N}) \Rightarrow \bigcup_{j=1}^{\infty} A_j \in \mathcal{A}$.

The pair (X, \mathcal{A}) is called a measurable space, and $A \subset X$ is measurable $:\Leftrightarrow A \in \mathcal{A}$.

Proposition 1 (Generated σ -algebras; Borel-(σ -)algebra).

- (i) For any family $\mathcal{B} \subset \mathcal{P}(X)$ there exists a smallest σ -algebra $\sigma(\mathcal{B})$ containing \mathcal{B} (that is, $\sigma(\mathcal{B}) \supset \mathcal{B}$, and if \mathcal{C} is a σ -algebra with $\mathcal{C} \supset \mathcal{B}$, then $\mathcal{C} \supset \sigma(\mathcal{B})$), given by

$$\sigma(\mathcal{B}) := \bigcap_{\mathcal{A} \subset \mathcal{P}(X), \mathcal{A} \text{ } \sigma\text{-algebra}, \mathcal{A} \supset \mathcal{B}} \mathcal{A}. \quad (1)$$

We call $\sigma(\mathcal{B})$ the σ -algebra generated by \mathcal{B} .

- (ii) Let (X, \mathcal{T}) be a topological space (for example, a metric space (X, d) with the topology \mathcal{T}_d generated by the metric d). The σ -algebra $\sigma(\mathcal{T})$ is called the Borel- σ -algebra (or Borel-algebra) (on (X, \mathcal{T})), denoted $\mathcal{B}(X)$ (more correct would be: $\mathcal{B}(X, \mathcal{T})$), and $B \subset X$ is a Borel-set (or Borel or Borel-measurable) $:\Leftrightarrow B \in \mathcal{B}(X)$.
- (iii) For a measurable space (X, \mathcal{A}) and a subset $B \subset X$ (not necessarily measurable), the induced σ -algebra (or trace- σ -algebra) on B is defined by $\mathcal{A}_B := \{B \cap A \mid A \in \mathcal{A}\}$. If $B \in \mathcal{A}$, then $\mathcal{A}_B \subset \mathcal{A}$.

Example 1 (Borel-algebra on $\mathbb{R}^d, \mathbb{R}_{\geq 0}, \overline{\mathbb{R}}, \overline{\mathbb{R}}_{\geq 0}$). Let $X := \mathbb{R}^d$ with the usual topology $\mathcal{T}_{\text{Eucl}}$, generated by the Euclidean metric $|\cdot|$. We denote $\mathcal{B}^d := \mathcal{B}(\mathbb{R}^d) := \sigma(\mathcal{T}_{\text{Eucl}})$ the Borel-algebra on \mathbb{R}^d , and write $\mathcal{B} := \mathcal{B}^1$ when there is no risk of confusion. We denote $\mathcal{B}_{\geq 0} := \mathcal{B}^1_{\mathbb{R}_{\geq 0}} := \{\mathbb{R}_{\geq 0} \cap A \mid A \in \mathcal{B}^1\}$. It is the Borel-algebra of $\mathbb{R}_{\geq 0}$ (with the topology on $\mathbb{R}_{\geq 0}$ the one induced from \mathbb{R}). For $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$, we denote $\mathcal{B}(\overline{\mathbb{R}}) := \sigma(\mathcal{B}^1 \cup \{\{-\infty\}\} \cup \{\{+\infty\}\})$. It is the Borel-algebra on $\overline{\mathbb{R}}$ for the usual topology on $\overline{\mathbb{R}}$. Finally, $\overline{\mathcal{B}}_{\geq 0} := \{\overline{\mathbb{R}}_{\geq 0} \cap A \mid A \in \mathcal{B}(\overline{\mathbb{R}})\} (= \mathcal{B}(\overline{\mathbb{R}}_{\geq 0}))$.

Definition 2 ((Positive) measure).

(i) Let \mathcal{A} be a σ -algebra on X . A map $\mu : \mathcal{A} \rightarrow [0, \infty]$ is called a (positive) measure (on X , or on (X, \mathcal{A})) iff

$$(i) \quad \mu(\emptyset) = 0.$$

(ii) For all $A_j \in \mathcal{A}$, $j \in \mathbb{N}$, with $A_j \cap A_k = \emptyset$ for $j \neq k$:

$$\mu\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} \mu(A_j) \quad (\sigma\text{-additivity}). \quad (2)$$

The triple (X, \mathcal{A}, μ) is called a measure space.

(ii) A measure μ (or, more correctly, a measure space (X, \mathcal{A}, μ)) is called finite (or bounded) iff $\mu(X) < \infty$, and σ -finite iff there exists $(X_j)_{j \in \mathbb{N}}$, $X_j \in \mathcal{A}$, $X = \bigcup_{j=1}^{\infty} X_j$, with $\mu(X_j) < \infty$ for all $j \in \mathbb{N}$.

Example 2 (Lebesgue-Borel measure on \mathbb{R}^d). There exists a unique measure, called the Lebesgue-Borel measure λ^d , on \mathcal{B}^d so that for all rectangles $Q := \times_{j=1}^d [a_j, b_j] \subset \mathbb{R}^d$, $-\infty \leq a_j \leq b_j \leq \infty$, holds

$$\lambda^d(Q) = \prod_{j=1}^d (b_j - a_j). \quad (3)$$

Furthermore, λ^d is translation and rotation invariant, and σ -finite.

Definition 3 ((μ -)Null sets; complete measure). Let (X, \mathcal{A}, μ) be a measure space.

(i) A subset $N \subset X$ is called a (μ -)null set iff $N \in \mathcal{A}$ and $\mu(N) = 0$.

(ii) (X, \mathcal{A}, μ) (or just μ) is called complete iff all subsets of null sets are null sets.

Theorem 1 (Completion of measure). Let (X, \mathcal{A}, μ) be a measure space. Then there exists a smallest complete measure space $(X, \overline{\mathcal{A}}, \overline{\mu})$ (called the completion of (X, \mathcal{A}, μ)) containing (X, \mathcal{A}, μ) (that is, $\overline{\mathcal{A}} \supset \mathcal{A}$, $\overline{\mu}|_{\mathcal{A}} = \mu$, and $\overline{\mu}$ is complete).

Example 3 (Lebesgue measure on \mathbb{R}^d). The completion of $(\mathbb{R}^d, \mathcal{B}^d, \lambda^d)$ (which is not in itself complete) is denoted $(\mathbb{R}^d, \overline{\mathcal{B}}^d, \overline{\lambda}^d)$. Elements of $\overline{\mathcal{B}}^d$ are called Lebesgue-measurable (subsets of \mathbb{R}^d), and $\overline{\lambda}^d$ is called (d -dimensional) Lebesgue measure. One has

$$\overline{\mathcal{B}}^d = \left\{ B \cup \widetilde{N} \mid B \in \mathcal{B}^d, \exists N \in \mathcal{B}^d \text{ with } \lambda^d(N) = 0, \widetilde{N} \subset N \right\}. \quad (4)$$

Furthermore, $A \in \overline{\mathcal{B}}^d$ iff for all $\varepsilon > 0$ there exists $U \subset \mathbb{R}^d$ open and $C \subset \mathbb{R}^d$ closed, with $C \subset A \subset U$, such that $\lambda^d(U \setminus C) < \varepsilon$.

Note, in particular, that $\mathcal{B}^d \subsetneq \overline{\mathcal{B}}^d \subsetneq \mathcal{P}(\mathbb{R}^d)$.

Definition 4 (Measurable maps and functions).

- (i) Let $(X, \mathcal{A}), (Y, \mathcal{C})$ be measurable spaces. A map $f : X \rightarrow Y$ is called $(\mathcal{A}\text{-}\mathcal{C}\text{-})$ measurable iff $f^{-1}(C) \in \mathcal{A}$ for all $C \in \mathcal{C}$. We denote by $\mathcal{M}(X, Y)$ the set of all $\mathcal{A}\text{-}\mathcal{C}$ -measurable maps. (More correct would be $\mathcal{M}((X, \mathcal{A}), (Y, \mathcal{C}))$.)
- (ii) In the special case $(Y, \mathcal{C}) = (\mathbb{R}, \mathcal{B}^1)$, we denote $\mathcal{M}(X) := \mathcal{M}(X, \mathbb{R})$ the set of all measurable functions $f : X \rightarrow \mathbb{R}$, and $\mathcal{M}_+(X) := \{f \in \mathcal{M}(X) \mid f \geq 0\}$. Note that $\mathcal{M}_+(X) = \mathcal{M}(X, \mathbb{R}_{\geq 0}) = \mathcal{M}((X, \mathcal{A}), (\mathbb{R}_{\geq 0}, \mathcal{B}_{\geq 0}))$.
- (iii) In the special case $(Y, \mathcal{C}) = (\overline{\mathbb{R}}, \mathcal{B}(\overline{\mathbb{R}}))$, we denote $\overline{\mathcal{M}}(X) := \mathcal{M}(X, \overline{\mathbb{R}})$ the set of all measurable numerical (or extended real-valued) functions $f : X \rightarrow \overline{\mathbb{R}}$, and $\overline{\mathcal{M}}_+(X) := \{f \in \overline{\mathcal{M}}(X) \mid f \geq 0\}$. Again, $\overline{\mathcal{M}}_+(X) = \mathcal{M}(X, \overline{\mathbb{R}}_{\geq 0})$.
- (iv) We denote by $\mathcal{M}(X, \mathbb{C})$ the set of all complex functions $f : X \rightarrow \mathbb{C}$ such that $\Re(f), \Im(f) \in \mathcal{M}(X, \mathbb{R})$.

Remark 1. One has

$$\mathcal{M}(X) = \left\{ f : X \rightarrow \mathbb{R} \mid f^{-1}\left((-\infty, a)\right) \in \mathcal{A} \text{ for all } a \in \mathbb{R} \right\}, \quad (5)$$

and similarly with $(-\infty, a)$ replaced with $(-\infty, a], (a, \infty)$, or $[a, \infty)$. Analogous statements hold for $\overline{\mathcal{M}}(X)$.

We denote, for $a \in \overline{\mathbb{R}}$,

$$\{f < a\} := f^{-1}\left((-\infty, a)\right) = \{x \in X \mid f(x) \in (-\infty, a)\}, \quad (6)$$

and similarly for other types of intervals.

Definition 5 (Distribution function).

Let (X, \mathcal{A}, μ) be a measure space. For $f \in \overline{\mathcal{M}}(X)$, we call the function $\mu_f : \mathbb{R} \rightarrow \mathbb{R}$ given by

$$\mu_f(t) := \mu(\{f > t\}) = \mu(\{x \in X \mid f(x) > t\}) \quad (7)$$

the distribution function of f (relative to μ).

Definition 6 (Almost everywhere (a.e.); $f = g$ a.e.).

Let (X, \mathcal{A}, μ) be a measure space.

- (i) A mathematical statement $Q = Q(x)$ (which is assumed to make sense for all $x \in X$) is said to hold $(\mu\text{-})$ almost everywhere (a.e., or $\mu\text{-a.e.}$) iff there exists a $(\mu\text{-})$ null set N such that $Q(x)$ is true/holds for all $x \in X \setminus N$.
- (ii) Let $f, g : X \rightarrow \mathbb{M}$ ($\mathbb{M} \in \{\mathbb{R}, \mathbb{R}_{\geq 0}, \overline{\mathbb{R}}, \overline{\mathbb{R}}_{\geq 0}, \mathbb{C}\}$) be measurable, then $f = g$ a.e. iff $f(x) = g(x)$ a.e. This defines an equivalence relation $\sim_{\text{a.e.}}$ on $\mathcal{M}(X, \mathbb{M})$.

Definition 7 (Step functions).

Let (X, \mathcal{A}) be a measurable space. A function $f : X \rightarrow \mathbb{R}$ is called a step function iff there exists $N \in \mathbb{N}$, $A_1, \dots, A_N \in \mathcal{A}$, and $a_1, \dots, a_N \in \mathbb{R}$ such that

$$f = \sum_{n=1}^N a_n \mathbb{1}_{A_n}. \quad (8)$$

Here, $\mathbb{1}_B$ is the characteristic (or indicator) function of (the set) B , given by $\mathbb{1}_B(x) = 1$ if $x \in B$, and equal 0 otherwise ($x \notin B$).

Note that step functions are measurable by definition. We denote the set of all non-negative step functions by

$$E_+ := \left\{ f : X \rightarrow \mathbb{R} \mid f \geq 0, f \text{ step function} \right\}. \quad (9)$$

Theorem 2 (Approximating measurable functions by step functions). *Let (X, \mathcal{A}) be a measurable space. Then $f \in \mathcal{M}(X, \overline{\mathbb{R}})$ iff there exists a sequence $(f_n)_{n \in \mathbb{N}}$ of step functions $f_n : X \rightarrow \mathbb{R}$ with $f = \lim_{n \rightarrow \infty} f_n$ (pointwise on X). If $f \in \mathcal{M}_+(X)$, then the sequence can be chosen monotone ($f_n \nearrow f$), and if f is a bounded function, then the sequence can be chosen such that the convergence is uniform on X .*

Definition 8 (Definition and properties of Lebesgue integral).

Let (X, \mathcal{A}, μ) be a measure space.

1. Let $f \in E_+$ ($f \geq 0$, f step function), with $f = \sum_{n=1}^N a_n \mathbb{1}_{A_n}$, $A_n \in \mathcal{A}$, $a_n \in \mathbb{R}$. Then

$$\int_X f \, d\mu := \int_X f(x) \, d\mu(x) := \sum_{n=1}^N a_n \mu(A_n) \in [0, \infty] \quad (10)$$

is the (μ) -integral of f over X . It is independent of the representation in (8).

2. Let $f \in \overline{\mathcal{M}}_+(X)$ ($f : X \rightarrow [0, \infty]$, measurable), and let $(f_n)_{n \in \mathbb{N}} \subset E_+$ be an approximating sequence as in Theorem 2. Then

$$\int_X f \, d\mu := \int_X f(x) \, d\mu(x) := \lim_{n \rightarrow \infty} \left(\int_X f_n(x) \, d\mu(x) \right) \in [0, \infty] \quad (11)$$

is the (μ) -integral of f over X . The limit is well-defined, since the sequence $(\int_X f_n \, d\mu)_{n \in \mathbb{N}} \subset [0, \infty]$ is non-decreasing. The limit is independent of the chosen sequence $(f_n)_{n \in \mathbb{N}}$.

3. For $f : X \rightarrow \overline{\mathbb{R}}$, let $f_{\pm} := \max\{\pm f, 0\}$ (so $f = f_+ - f_-$, $|f| = f_+ + f_-$). Then f is (μ) -integrable over X $\Leftrightarrow f \in \overline{\mathcal{M}}(X)$ and $\int_X f_+ \, d\mu < \infty$, $\int_X f_- \, d\mu < \infty$. In this case,

$$\int_X f \, d\mu := \int_X f(x) \, d\mu(x) := \int_X f_+ \, d\mu - \int_X f_- \, d\mu \in \mathbb{R} \quad (12)$$

is the (μ) -integral of f over X . We denote the set of integrable functions by

$$\begin{aligned} \mathcal{L}^1 &:= \mathcal{L}^1(X) := \mathcal{L}^1(\mu) := \mathcal{L}^1(X, \mu) := \mathcal{L}^1(X, \mathcal{A}, \mu) := \left\{ f : X \rightarrow \mathbb{R} \mid f \text{ } \mu\text{-integrable} \right\}, \\ \overline{\mathcal{L}}^1 &:= \overline{\mathcal{L}}^1(X) := \overline{\mathcal{L}}^1(\mu) := \overline{\mathcal{L}}^1(X, \mu) := \overline{\mathcal{L}}^1(X, \mathcal{A}, \mu) := \left\{ f : X \rightarrow \overline{\mathbb{R}} \mid f \text{ } \mu\text{-integrable} \right\}. \end{aligned}$$

4. For $A \in \mathcal{A}$, and $f \in \mathcal{L}^1$ (or $f \in \overline{\mathcal{L}}^1$), let $\int_A f \, d\mu := \int_X f \mathbb{1}_A \, d\mu$.

5. Properties of the integral:

- (a) For $f \in \overline{\mathcal{L}}^1$, $f \geq 0$, one has: $\int_X f \, d\mu = 0 \Leftrightarrow f = 0$ μ -a.e.
- (b) The map $f \mapsto \int_X f \, d\mu$ from $\overline{\mathcal{L}}^1$ to \mathbb{R} is linear and monotone.

(c) For $f \in \overline{\mathcal{L}^1}$,

$$\left| \int_X f \, d\mu \right| \leq \int_X |f| \, d\mu \quad (\text{triangle inequality}). \quad (13)$$

(d) For $f \in \overline{\mathcal{L}^1}$, $f \geq 0$, and all $\varepsilon > 0$,

$$\mu(\{f \geq \varepsilon\}) \leq \frac{1}{\varepsilon} \int_X f \, d\mu \quad (\text{Chebyshev's inequality}). \quad (14)$$

Proposition 2 (Riemann versus Lebesgue interal in \mathbb{R}).

For $f : [a, b] \rightarrow \mathbb{R}$ ($a, b \in \mathbb{R}, a < b$) Riemann-integrable, denote $\int_a^b f(x) \, dx$ the Riemann-integral of f over $[a, b]$.

1. For $f : [a, b] \rightarrow \mathbb{R}$ Riemann-integrable there exists $g : \mathbb{R} \rightarrow \mathbb{R}$ measurable, with $f = g$ a.e. on $[a, b]$ such that

$$\int_a^b f(x) \, dx = \int_{[a,b]} g(x) \, d\lambda^1(x). \quad (15)$$

2. Let $f : [0, \infty) \rightarrow \mathbb{R}$ be measurable, and continuous on $(0, \infty)$. Then

$$\int_{\mathbb{R}} f \mathbb{1}_{[1, \infty)} \, d\lambda^1 = \lim_{n \rightarrow \infty} \int_1^n f(x) \, dx, \quad (16)$$

$$\int_{\mathbb{R}} f \mathbb{1}_{[0, 1]} \, d\lambda^1 = \lim_{n \rightarrow \infty} \int_{1/n}^1 f(x) \, dx. \quad (17)$$

In particular,

$$\int_{[0,1]} x^a \, d\lambda^1(x) < \infty \Leftrightarrow a > -1, \quad (18)$$

$$\int_{[1, \infty)} x^b \, d\lambda^1(x) < \infty \Leftrightarrow b < -1. \quad (19)$$

Also,

$$\int_0^\infty \frac{\sin x}{x} \, dx = \frac{\pi}{2} = \lim_{R \rightarrow \infty} \left(\int_{[0,R]} \frac{\sin x}{x} \, d\lambda^1(x) \right), \quad (20)$$

$$\int_{\mathbb{R}} e^{-x^2} \, d\lambda^1(x) = \sqrt{\pi}. \quad (21)$$

In what follows, let (X, \mathcal{A}, μ) be any measure space.

Definition 9 (Essential supremum). For a measurable function $f : X \rightarrow \overline{\mathbb{R}}$ the essential supremum of f is

$$\begin{aligned} \text{ess sup } f &= \text{ess sup}_X f = \inf \{ s \in \overline{\mathbb{R}} \mid f(x) \leq s \text{ } \mu\text{-a.e.} \} \\ &= \inf \left\{ \sup_{x \in X \setminus N} f(x) \mid N \subset X, N \text{ } \mu\text{-null set} \right\}. \end{aligned} \quad (22)$$

Definition 10 (The semi-normed spaces $\mathcal{L}^p(X)$, $p \in [1, \infty]$).

(i) For $p \in [1, \infty)$, let

$$\begin{aligned} \mathcal{L}^p &:= \mathcal{L}^p(X) := \mathcal{L}^p(\mu) := \mathcal{L}^p(X, \mu) := \mathcal{L}^p(X, \mathcal{A}, \mu) \\ &:= \left\{ f : X \rightarrow \mathbb{C} \mid f \in \mathcal{M}(X, \mathbb{C}), \int_X |f|^p d\mu < \infty \right\} \end{aligned} \quad (23)$$

and, for $f \in \mathcal{L}^p(X)$, let

$$\|f\|_p := \left(\int_X |f|^p d\mu < \infty \right)^{1/p}. \quad (24)$$

(ii) For $p = \infty$, let

$$\begin{aligned} \mathcal{L}^\infty &:= \mathcal{L}^\infty(X) := \mathcal{L}^\infty(\mu) := \mathcal{L}^\infty(X, \mu) := \mathcal{L}^\infty(X, \mathcal{A}, \mu) \\ &:= \left\{ f : X \rightarrow \mathbb{C} \mid f \in \mathcal{M}(X, \mathbb{C}), \operatorname{ess\,sup}_X |f| < \infty \right\}, \end{aligned} \quad (25)$$

and, for $f \in \mathcal{L}^\infty(X)$, let

$$\|f\|_\infty := \operatorname{ess\,sup}_X |f|. \quad (26)$$

Then, for all $p \in [1, \infty]$, $\|\cdot\|_p$ is a semi-norm on $\mathcal{L}^p(X)$: $\|f\|_p = 0 \Leftrightarrow f \sim_{a.e.} 0$ (which does not mean $f = 0$).

Theorem 3 (Minkowski and (generalised) Hölder inequalities).

(i) (Minkowski) Let $p \in [1, \infty]$, then $\|f + g\|_p \leq \|f\|_p + \|g\|_p$ for all $f, g \in \mathcal{L}^p(X)$.

(ii) (Hölder) Let $p, q \in [1, \infty]$, with $\frac{1}{p} + \frac{1}{q} = 1$. Then, for all $f \in \mathcal{L}^p(X), g \in \mathcal{L}^q(X)$,

$$\int_X |fg| d\mu \leq \|f\|_p \|g\|_q. \quad (27)$$

(iii) (Generalised Hölder) Let $n \in \mathbb{N}$ ($n \geq 2$), and let $p_1, \dots, p_n \in [1, \infty]$, and let $p \in [1, \infty]$ satisfy $\frac{1}{p} = \sum_{j=1}^n \frac{1}{p_j}$. Then, for all $f_j \in \mathcal{L}^{p_j}(X)$, $j = 1, \dots, n$,

$$\left\| \prod_{j=1}^n f_j \right\|_p \leq \prod_{j=1}^n \|f_j\|_{p_j}. \quad (28)$$

(iv) (Interpolation in \mathcal{L}^p -spaces). Let $1 \leq p < r < q \leq \infty$, $f \in \mathcal{L}^p(X) \cap \mathcal{L}^q(X)$. Let $\theta \in (0, 1)$ with $\frac{1}{r} = \frac{\theta}{p} + \frac{1-\theta}{q}$. Then $f \in \mathcal{L}^r(X)$, and

$$\|f\|_r \leq \|f\|_p^\theta \|f\|_q^{1-\theta}. \quad (29)$$

Hence, for $f : X \rightarrow \mathbb{C}$ measurable, the set

$$\Gamma_f := \{p \in [1, \infty] \mid f \in \mathcal{L}^p(X)\} \subset \mathbb{R} \quad (30)$$

is an interval.

(v) Let $p \in [1, \infty]$, $f \in \mathcal{L}^p(X) \cap \mathcal{L}^\infty(X)$. Then $f \in \cap_{q \geq p} \mathcal{L}^q(X)$, and $\lim_{q \rightarrow \infty} \|f\|_q = \|f\|_\infty$.

Theorem 4 (The normed spaces $L^p(X)$, $p \in [1, \infty]$).

For $p \in [1, \infty]$, the relation $\sim_{a.e.}$ defines an equivalence relation on $\mathcal{L}^p(X)$, and $\|\cdot\|_p$ defines a norm on the quotient vector space $L^p(X)$, which makes $(L^p(X), \|\cdot\|_p)$ a Banach space. For $p = 2$, $L^2(X)$ is a Hilbert space, with inner/scalar product $\langle f, g \rangle := \int_X \overline{f(x)}g(x) \, d\mu(x)$.

Remark 2. By abuse of notation we will call $f \in L^p(X)$ functions when we should really be talking about equivalence classes (this abuse of notation/language is well established).

Theorem 5 (a.e. convergent subsequences).

Let $p \in [1, \infty]$, and assume $(f_j)_{j \in \mathbb{N}} \subset L^p(X)$, $f \in L^p(X)$, satisfy $\lim_{j \rightarrow \infty} \|f_j - f\|_p = 0$. Then there exists a subsequence $(f_{j_k})_{k \in \mathbb{N}}$ with $\lim_{k \rightarrow \infty} f_{j_k}(x) = f(x)$ a.e., that is, the subsequence $(f_{j_k})_{k \in \mathbb{N}}$ converges pointwise to f for μ -almost every $x \in X$.

Theorem 6 (Denseness of step functions in $L^p(X)$).

Let $p \in [1, \infty)$, then the linear subspace of step functions,

$$\begin{aligned} E &:= \text{Span}\{\mathbb{1}_A \mid A \in \mathcal{A}, \mu(A) < \infty\} \\ &= \left\{g : X \rightarrow \mathbb{C} \mid g = \sum_{n=1}^N a_n \mathbb{1}_{A_n}, N \in \mathbb{N}, A_1, \dots, A_N \in \mathcal{A}, \mu(A_j) < \infty, a_1, \dots, a_N \in \mathbb{C}\right\} \end{aligned} \quad (31)$$

is dense in $L^p(X)$: For all $f \in L^p(X)$ and all $\varepsilon > 0$, there exists $g \in E$ such that $\|f - g\|_p < \varepsilon$.

Definition 11 (Locally integrable functions). Let (X, \mathcal{T}) be a topological space, and let μ be a measure on $(X, \sigma(\mathcal{T}))$. (Example: \mathbb{R}^d with Lebesgue(-Borel) measure.) For $p \in [1, \infty]$, we denote

$$L^p_{\text{loc}}(X) := \left\{f : X \rightarrow \mathbb{C} \mid f \in \mathcal{M}(X, \mathbb{C}), f \in L^p(K) \text{ for all } K \subset X \text{ compact}\right\}. \quad (32)$$

Theorem 7 (Monotone convergence / Beppo Levi). Let $(f_j)_{j \in \mathbb{N}}$, $f_j : X \rightarrow \mathbb{R}$, be a sequence of measurable functions with

$$0 \leq f_1 \leq f_2 \leq f_3 \leq \dots \quad (33)$$

Then, with $f(x) := \lim_{j \rightarrow \infty} f_j(x)$,

$$\lim_{j \rightarrow \infty} \int_X f_j \, d\mu = \int_X f \, d\mu. \quad (34)$$

The possibility that both sides are $+\infty$ is included.

Theorem 8 ((Lebesgue) Dominated convergence). Let $(f_j)_{j \in \mathbb{N}}$, $f_j : X \rightarrow \mathbb{R}$, be a sequence of measurable functions. Assume there exists $g \in L^1(X)$ such that $|f_j(x)| \leq g(x)$ for a.e. $x \in X$ and all $j \in \mathbb{N}$, and that $f(x) := \lim_{j \rightarrow \infty} f_j(x)$ exists a.e. on X .

Then

$$\lim_{j \rightarrow \infty} \int_X f_j \, d\mu = \int_X f \, d\mu. \quad (35)$$

In this case both sides are finite.

Theorem 9 (Fatou's Lemma). Let $(f_j)_{j \in \mathbb{N}}$, $f_j : X \rightarrow \mathbb{R}$, be a sequence of measurable functions, with $f_j(x) \geq 0$ a.e. on X for all $j \in \mathbb{N}$. Then

$$\int_X \left(\liminf_{j \in \mathbb{N}} f_j \right) \, d\mu \leq \liminf_{j \in \mathbb{N}} \left(\int_X f_j \, d\mu \right). \quad (36)$$

Theorem 10 (Continuity and differentiability of parameter-dependent integrals). Let (M, d) be a metric space, (X, \mathcal{A}, μ) a measure space, and $f : M \times X \rightarrow \mathbb{R}$ a map satisfying

(i) The map $x \mapsto f(t, x)$ is integrable for all $t \in M$.
Let $F : M \rightarrow \mathbb{R}$ be given by $F(t) := \int_X f(t, x) d\mu(x)$.

1. Let $t_0 \in M$, and assume furthermore:

(ii) The map $t \mapsto f(t, x)$ is continuous at t_0 for all $x \in X$.

(iii) There exists integrable an function $g : X \rightarrow [0, \infty]$ such that $|f(t, x)| \leq g(x)$ for all $t \in M$ and $x \in X$.

Then F is continuous at t_0 :

$$\begin{aligned} \lim_{t \rightarrow t_0} F(t) &= \lim_{t \rightarrow t_0} \left(\int_X f(t, x) d\mu(x) \right) = \int_X \left(\lim_{t \rightarrow t_0} f(t, x) \right) d\mu(x) \\ &= \int_X f(t_0, x) d\mu(x) = F(t_0). \end{aligned} \quad (37)$$

2. Let $M = I \subset \mathbb{R}$ be an open interval, and assume (i) holds. Assume furthermore that

(ii') The map $t \mapsto f(t, x)$ is differentiable on I for all $x \in X$.

(iii') There exists an integrable function $g : X \rightarrow [0, \infty]$ such that $|\frac{\partial f}{\partial t}(t, x)| \leq g(x)$ for all $t \in M$ and $x \in X$.

Then F is differentiable on I , the map $x \mapsto \frac{\partial f}{\partial t}(t, x)$ is integrable for all $t \in I$, and

$$\frac{d}{dt} \left(\int_X f(t, x) d\mu(x) \right) = F'(t) = \frac{dF}{dt}(t) = \int_X \frac{\partial f}{\partial t}(t, x) d\mu(x). \quad (38)$$

Definition 12 (Product- σ -algebra).

Let (X_j, \mathcal{A}_j) , $j = 1, \dots, n$, be measurable spaces. The product- σ -algebra

$$\bigotimes_{j=1}^n \mathcal{A}_j := \mathcal{A}_1 \otimes \dots \otimes \mathcal{A}_n := \sigma(p_1, \dots, p_n) \quad (39)$$

(on $X := \times_{j=1}^n X_j$) is the smallest σ -algebra such that the projections $p_j : X \rightarrow X_j$, $x = (x_1, \dots, x_n) \mapsto x_j$, are all measurable.

Example 4. $\mathcal{B}^d = \mathcal{B}^1 \otimes \dots \otimes \mathcal{B}^1$ (d times). However, $\overline{\mathcal{B}^d} \supsetneq \overline{\mathcal{B}^1} \otimes \dots \otimes \overline{\mathcal{B}^1}$.

Theorem 11 (Product measure).

Let $(X_j, \mathcal{A}_j, \mu_j)$, $j = 1, \dots, n$, be σ -finite (!) measure spaces, and let $X := \times_{j=1}^n X_j$. Then there exists a unique measure $\mu := \otimes_{j=1}^n \mu_j := \mu_1 \otimes \dots \otimes \mu_n$ (called the product measure (of μ_1, \dots, μ_n)) on $\mathcal{A} := \otimes_{j=1}^n \mathcal{A}_j$ such that

$$\left(\bigotimes_{j=1}^n \mu_j \right) \left(\bigtimes_{j=1}^n A_j \right) = \prod_{j=1}^n \mu_j(A_j) \quad \text{for all } A_j \in \mathcal{A}_j, j = 1, \dots, n. \quad (40)$$

Furthermore, (X, \mathcal{A}, μ) is σ -finite.

Theorem 12 (Fubini-Tonelli).

Let $(X_j, \mathcal{A}_j, \mu_j)$, $j = 1, 2$, be σ -finite (!) measure spaces, and let $(X, \mathcal{A}, \mu) := (X_1 \times X_2, \mathcal{A}_1 \otimes \mathcal{A}_2, \mu_1 \otimes \mu_2)$. Let $f : X \rightarrow \overline{\mathbb{R}}$ (or \mathbb{C}) be \mathcal{A} -measurable. Then is, for all $g \in \{\Re(f_+), \Re(f_-), \Im(f_+), \Im(f_-)\}$, the functions

$$X_1 \rightarrow [0, \infty], \quad x_1 \mapsto \int_{X_2} g(x_1, x_2) \, d\mu_2(x_2), \quad (41)$$

$$X_2 \rightarrow [0, \infty], \quad x_2 \mapsto \int_{X_1} g(x_1, x_2) \, d\mu_1(x_1) \quad (42)$$

\mathcal{A}_1 -measurable, respectively, \mathcal{A}_2 -measurable. Furthermore,

1. (Tonelli) If $f \geq 0$ a.e. (that is, $f(X \setminus N) \subset [0, \infty], \mu(N) = 0$), then

$$\begin{aligned} \int_X f(x) \, d\mu(x) &= \int_{X_1} \left(\int_{X_2} f(x_1, x_2) \, d\mu_2(x_2) \right) d\mu_1(x_1) \\ &= \int_{X_2} \left(\int_{X_1} f(x_1, x_2) \, d\mu_1(x_1) \right) d\mu_2(x_2). \end{aligned} \quad (43)$$

Note: It is possible that all three integrals are $+\infty$.

2. (Fubini) If one of the three integrals

$$\begin{aligned} \int_X |f(x)| \, d\mu(x), \quad \int_{X_1} \left(\int_{X_2} |f(x_1, x_2)| \, d\mu_2(x_2) \right) d\mu_1(x_1), \\ \int_{X_2} \left(\int_{X_1} |f(x_1, x_2)| \, d\mu_1(x_1) \right) d\mu_2(x_2) \end{aligned} \quad (44)$$

is finite, then they are all finite, and (43) holds.

Theorem 13 (Layer Cake Principle). Let (X, \mathcal{A}, μ) be a σ -finite measure space, and let $\mathcal{B}_{\geq 0}$ be the Borel-algebra of $\mathbb{R}_{\geq 0}$. Let ν be a measure on $\mathcal{B}_{\geq 0}$ such that $\phi(t) := \nu([0, t])$ is finite for all $t > 0$, and let $f : X \rightarrow \mathbb{R}_{\geq 0}$ be \mathcal{A} - $\mathcal{B}_{\geq 0}$ -measurable. Then

$$\int_X \phi(f(x)) \, d\mu(x) = \int_{\mathbb{R}_{\geq 0}} \mu(\{x \in X \mid f(x) > t\}) \, d\nu(t). \quad (45)$$

Recall that $\mu_f(t) = \mu(\{f > t\})$ is the distribution function of f relative to μ .

In particular, if $f \in \mathcal{L}^p(X)$, then (by choosing $d\nu(t) = pt^{p-1}d\lambda^1(t)$)

$$\int_X |f|^p \, d\mu = p \int_{\mathbb{R}_{\geq 0}} t^{p-1} \mu(\{|f| > t\}) \, d\lambda^1(t), \quad (46)$$

and if $f \in \mathcal{L}^1(X)$ with $f \geq 0$, then (by choosing $p = 1$)

$$\int_X f \, d\mu = \int_{\mathbb{R}_{\geq 0}} \mu(\{f > t\}) \, d\lambda^1(t). \quad (47)$$

Also (by choosing μ the Dirac measure δ_x at $x \in X$, and $p = 1$),

$$f(x) = \int_{\mathbb{R}_{\geq 0}} \mathbb{1}_{\{f > t\}}(x) \, d\lambda^1(t) \quad (\text{Layer Cake Representation of } f). \quad (48)$$

Theorem 14 (Transformation formula for λ^d).

Let $U \subset \mathbb{R}^d$ be open, and $\varphi : U \rightarrow \varphi(U) \subset \mathbb{R}^d$ a diffeomorphism. Then, for all $f \in \mathcal{L}^1(\varphi(U), \lambda^d)$,

$$\int_{\varphi(U)} f(y) d\lambda^d(y) = \int_U f(\varphi(x)) |\det(D\varphi(x))| d\lambda^d(x). \quad (49)$$

Lemma 1 (Notation and certain concrete integrals in \mathbb{R}^d).

1. For $x \in \mathbb{R}^d$, $r > 0$, we denote $B_r^d(x) := B_r(x) := \{y \in \mathbb{R}^d \mid |x - y| < r\}$, and $\omega_d := \lambda^d(B_1(0)) = \lambda^d(\overline{B_1(0)})$. One has $\omega_d = \frac{\pi^{d/2}}{\Gamma(d/2+1)}$, with $\Gamma(z) := \int_0^\infty t^{z-1} e^{-t} dt$, $z > 0$, the Gamma-function.

2. One has

$$\int_{B_1(0)} |x|^\alpha d\lambda^d(x) < \infty \Leftrightarrow \alpha > -d, \quad (50)$$

$$\int_{\mathbb{R}^d \setminus B_1(0)} |x|^\alpha d\lambda^d(x) < \infty \Leftrightarrow \alpha < -d, \quad (51)$$

$$\int_{\mathbb{R}^d} \frac{1}{(1 + |x|)^\alpha} d\lambda^d(x) < \infty \Leftrightarrow \alpha > -d. \quad (52)$$

Definition 13 (Spaces of differentiable functions on \mathbb{R}^d). Denote, for $k \in \mathbb{N}$,

$$C^0(\mathbb{R}^d) := C(\mathbb{R}^d) := C(\mathbb{R}^d, \mathbb{C}) := \{f : \mathbb{R}^d \rightarrow \mathbb{C} \mid f \text{ is continuous}\}, \quad (53)$$

$$C^k(\mathbb{R}^d) := \{f : \mathbb{R}^d \rightarrow \mathbb{C} \mid f \text{ is } k \text{ times continuous differentiable}\}, \quad (54)$$

$$C^\infty(\mathbb{R}^d) := \bigcap_{k \in \mathbb{N}} C^k(\mathbb{R}^d), \quad (55)$$

and define, for $f \in C(\mathbb{R}^d)$, the support of f by $\text{supp}(f) := \overline{\{x \in \mathbb{R}^d \mid f(x) \neq 0\}}$. Denote, for $k \in \mathbb{N} \cup \{\infty\}$,

$$C_c^k(\mathbb{R}^d) := \{f \in C^k(\mathbb{R}^d) \mid \text{supp}(f) \subset \mathbb{R}^d \text{ is compact}\}. \quad (56)$$

Theorem 15 (Denseness of $C_c^k(\mathbb{R}^d)$ in $L^p(\mathbb{R}^d)$).

1. The set $C_c(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d)$ with respect to $\|\cdot\|_p$ for $1 \leq p < \infty$.

More precisely: For all $f \in L^p(\mathbb{R}^d)$, $1 \leq p < \infty$, and all $\varepsilon > 0$ there exists $\phi \in C_c(\mathbb{R}^d)$ with $\|\phi - f\|_p < \varepsilon$. Note: The result fails in $L^\infty(\mathbb{R}^d)$.

2. The set $C_c^\infty(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d)$ with respect to $\|\cdot\|_p$ for $1 \leq p < \infty$.

Again, the result fails in $L^\infty(\mathbb{R}^d)$.

3. As a consequence, $C_c^k(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d)$ with respect to $\|\cdot\|_p$ for $1 \leq p < \infty$, and all $k \in \mathbb{N} \cup \{\infty\}$.

Again, the result fails in $L^\infty(\mathbb{R}^d)$.

Remark 3 (Notation in \mathbb{R}^d). We will most often write $\int_{\mathbb{R}^d} f(x) dx$ or $\int f(x) dx$ or simply $\int f dx$ instead of $\int_{\mathbb{R}^d} f(x) d\lambda^d(x)$ from now on. Also, we will often use the notation $|A| := \lambda^d(A)$ for the Lebesgue(-Borel) measure of a (measurable) set $A \subset \mathbb{R}^d$. This way, for the distribution function of $f : \mathbb{R}^d \rightarrow \mathbb{R}$ (relative to Lebesgue measure λ^d) we have

$$(\lambda^d)_f(t) = \lambda^d(\{f > t\}) = \lambda^d(\{x \in \mathbb{R}^d \mid f(x) > t\}) = |\{f > t\}|. \quad (57)$$

We assume knowledge of metric, and more generally, topological spaces. A special case, also assumed known, is that of (semi-)normed vector spaces. All vector spaces in the sequel will be over \mathbb{K} , $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Finally, we assume knowledge of scalar (or inner) product spaces. Note that, in complex vector spaces, our scalar products are linear in the *second* entry, and conjugate linear in the *first* one.

Theorem 16 (Banach's fixed point theorem). *Let (X, d) be a complete metric space, and let $T : X \rightarrow X$ be a contraction, that is, there exists $K \in (0, 1)$ such that*

$$d(T(x), T(y)) \leq K d(x, y) \quad \text{for all } x, y \in X. \quad (58)$$

Then T has a unique fixed point: There exists a unique $x_0 \in X$ such that $T(x_0) = x_0$.

Definition 14 (Dense subspaces).

Let $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ be a normed (vector) space. A (linear) subspace $V \subset \mathcal{X}$ is called dense (in \mathcal{X} ; more correctly, in $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$) iff $\bar{V} = \mathcal{X}$ (closure in the topology generated by the (metric generated by the) norm $\|\cdot\|_{\mathcal{X}}$).

Definition 15 (Banach and Hilbert spaces).

1. (Banach space) *A normed vector space $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ such that the metric space (\mathcal{X}, d) with $d(x, y) := \|x - y\|_{\mathcal{X}}$, $x, y \in \mathcal{X}$, is a complete metric space is called a Banach space.*
2. (Hilbert space) *A Banach space $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ for which the norm is generated by a scalar product (that is, there exists a scalar product $\langle \cdot, \cdot \rangle$ on \mathcal{X} such that $\|x\|^2 = \langle x, x \rangle$ for all $x \in \mathcal{X}$) is called a Hilbert space.*

Definition 16 ((Bounded) linear operators).

1. *Let \mathcal{X}, \mathcal{Y} be two vector spaces; a linear map $T : \mathcal{X} \rightarrow \mathcal{Y}$ is called a linear operator. We denote by $L(\mathcal{X}, \mathcal{Y})$ the set of all linear operators from \mathcal{X} to \mathcal{Y} . It is a vector space. If $\mathcal{X} = \mathcal{Y}$, we write $L(\mathcal{X})$.*
2. *If $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$, $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$ are both normed spaces, and if there exists a constant $C > 0$ such that $(Tx := T(x))$*

$$\|Tx\|_{\mathcal{Y}} \leq C \|x\|_{\mathcal{X}} \quad \text{for all } x \in \mathcal{X}, \quad (59)$$

then T is called a bounded (linear) operator (or simply bounded). We denote the vector space of all bounded operators from \mathcal{X} to \mathcal{Y} (more correctly, from $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ to $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$) by $B(\mathcal{X}, \mathcal{Y})$. If $\mathcal{X} = \mathcal{Y}$ (and $\|\cdot\|_{\mathcal{X}} = \|\cdot\|_{\mathcal{Y}}$ (!)), we write $B(\mathcal{X})$.

3. *We denote by*

$$\|T\| := \|T\|_{B(\mathcal{X}, \mathcal{Y})} := \sup_{x \in \mathcal{X} \setminus \{0\}} \frac{\|Tx\|_{\mathcal{Y}}}{\|x\|_{\mathcal{X}}} = \sup_{x \in \mathcal{X}, \|x\|_{\mathcal{X}}=1} \|Tx\|_{\mathcal{Y}} (< \infty) \quad (60)$$

the operator norm of $T \in B(\mathcal{X}, \mathcal{Y})$.

4. *A linear operator $T \in L(\mathcal{X}, \mathcal{Y})$ is called an isometry iff $\|Tx\|_{\mathcal{Y}} = \|x\|_{\mathcal{X}}$ for all $x \in \mathcal{X}$. Then $T \in B(\mathcal{X}, \mathcal{Y})$, and $\|T\| = 1$.*

Theorem 17 (Continuous linear maps and the space $(B(\mathcal{X}, \mathcal{Y}), \|\cdot\|_{B(\mathcal{X}, \mathcal{Y})})$).

Let $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$, $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$ be normed spaces.

1. A linear operator $T \in L(\mathcal{X}, \mathcal{Y})$ is continuous iff it is a bounded linear operator (that is, iff $T \in B(\mathcal{X}, \mathcal{Y})$).
2. The operator norm $\|\cdot\|_{B(\mathcal{X}, \mathcal{Y})}$ is a norm on $B(\mathcal{X}, \mathcal{Y})$.
3. If $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$ is a Banach space, then so is $(B(\mathcal{X}, \mathcal{Y}), \|\cdot\|_{B(\mathcal{X}, \mathcal{Y})})$.
4. In particular, the (topological) dual of \mathcal{X} (more correctly, of $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$), defined by $\mathcal{X}' := B(\mathcal{X}, \mathbb{K})$, $\|\cdot\|_{\mathcal{X}'} := \|\cdot\|_{B(\mathcal{X}, \mathbb{K})}$, is always a Banach space.

Definition 17 (Bi-dual and reflexive space).

Let $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ be a normed space.

1. The bi-dual (of \mathcal{X}) is the space $\mathcal{X}'' := (\mathcal{X}')'$ (with the norm defined accordingly). It is always a Banach space.
2. The natural injection is the (linear) map $J : \mathcal{X} \rightarrow \mathcal{X}''$, $x \mapsto J(x)$, given by $J(x)(\varphi) := \varphi(x)$, $\varphi \in \mathcal{X}'$. (By Hahn-Banach below, J is injective.)
3. If the natural injection $J : \mathcal{X} \rightarrow \mathcal{X}''$ is surjective, then \mathcal{X} is called reflexive. Note that any reflexive space is necessarily a Banach space.

Example 5 (Dual of $L^p(X)$). Let (X, \mathcal{A}, μ) be a measure space.

1. The spaces $(L^p(X), \|\cdot\|_p)$ are reflexive for $p \in (1, \infty)$: $(L^p(X))' = L^{p'}(X)$ (or rather, isometrically isomorphic) with $1/p + 1/p' = 1$.
2. If μ is σ -finite, then $(L^1(X))' = L^\infty(X)$ (again, isometrically isomorphic).
3. The dual of $L^\infty(X)$ is more complicated to express.

Theorem 18 (Hahn-Banach and consequences).

1. Let $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ be a normed space, and $U \subset \mathcal{X}$ a linear subspace. If $\varphi : U \rightarrow \mathbb{K}$ is linear and continuous (that is, bounded on U), then there exists an extension (not necessarily unique!) $\tilde{\varphi} : \mathcal{X} \rightarrow \mathbb{K}$ of φ which is also continuous and linear (hence, $\tilde{\varphi} \in \mathcal{X}'$), and which has the same (operator/dual) norm as φ , $\|\tilde{\varphi}\|_{\mathcal{X}'} = \|\varphi\|_{B(U, \mathbb{K})}$.
2. Let $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ be a normed space, and $U \subset \mathcal{X}$ a linear subspace. Assume $z \in \mathcal{X} \setminus \overline{U}$. Then there exists $\psi \in \mathcal{X}'$ with $\psi(x) = 0$ for all $x \in U$, $\psi(z) = 1$, and $\|\psi\|_{\mathcal{X}'} = \text{dist}(z, U)^{-1}$.
3. In particular, let $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ be a normed space and $z \in \mathcal{X}$. Then there exists $\psi \in \mathcal{X}'$ with $\psi(z) = \|z\|_{\mathcal{X}}$, and $\|\psi\|_{\mathcal{X}'} \leq 1$. In particular, for all $x \in \mathcal{X}$,

$$\|x\|_{\mathcal{X}} = \max \left\{ |\varphi(x)| \mid \varphi \in \mathcal{X}', \|\varphi\|_{\mathcal{X}'} = 1 \right\}. \quad (61)$$

As a result, the natural injection $J : \mathcal{X} \rightarrow \mathcal{X}''$ is an isometry: $\|J(x)\|_{\mathcal{X}''} = \|x\|_{\mathcal{X}}$ for all $x \in \mathcal{X}$.

Example 6 (Bounded operators between L^p -spaces).

Let (X, \mathcal{A}, μ) be a measure space.

1. One has, by (61) and $(L^p(X))' = L^{p'}(X)$, that for all $p \in [1, \infty)$,

$$\|f\|_p = \max \left\{ |\langle g, f \rangle| \mid g \in L^{p'}(X), \|g\|_{p'} = 1 \right\}, \quad (62)$$

with the notation that, for $f, g \in \mathcal{M}(X, \mathbb{C})$,

$$\langle g, f \rangle := \int_X g(x)f(x) \, d\mu(x). \quad (63)$$

With the convention that $1' := \infty$, (62) also holds for $p = \infty$, using that $f \in L^\infty(X) = (L^1(X))'$ and the definition of the norm in the dual.

2. It follows that, for $T : L^p(X) \rightarrow L^q(X)$ bounded, $p, q \in [1, \infty]$, one has

$$\begin{aligned} \|T\| &= \|T\|_{B(L^p(X), L^q(X))} = \|T\|_{B(L^p, L^q)} = \sup_{f \in L^p(X), \|f\|_p=1} \|Tf\|_{L^q(X)} \\ &= \sup \left\{ |\langle Tf, g \rangle| \mid f \in L^p(X), \|f\|_p = 1, g \in L^{q'}(X), \|g\|_{L^{q'}(X)} = 1 \right\}. \end{aligned} \quad (64)$$

Definition 18 (Orthogonality (Hilbert space)).

Let $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ be a Hilbert space.

1. Let $x, y \in \mathcal{H}$, then $x \perp y$ (x and y are orthogonal) iff $\langle x, y \rangle = 0$.
2. For $U \subset \mathcal{H}$, we define

$$U^\perp := \{x \in \mathcal{H} \mid x \perp y \text{ for all } y \in U\}. \quad (65)$$

Theorem 19 (Orthogonal complement (Hilbert space)).

Let $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ be a Hilbert space.

1. For all $U \subset \mathcal{H}$, the subset $U^\perp \subset \mathcal{H}$ is a closed (!) linear subspace.
2. For any closed (!) linear subspace $V \subset \mathcal{H}$, one has $\mathcal{H} = V \oplus V^\perp$ (that is, $x \perp y$ for all $x \in V, y \in V^\perp$, and $\mathcal{H} = V + V^\perp$ as (algebraic) direct sum).

Theorem 20 (Riesz' representation theorem (Hilbert space)). Let $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ be a Hilbert space, and let $\varphi \in \mathcal{H}'$. Then there exists a unique $x = x(\varphi) \in \mathcal{H}$ such that $\varphi(y) = \langle x, y \rangle$ for all $y \in \mathcal{H}$. One has $\|x\|_{\mathcal{H}} = \|\varphi\|_{\mathcal{H}'}$. The map $\Phi : \mathcal{H}' \rightarrow \mathcal{H}, \varphi \mapsto \Phi(\varphi) := x(\varphi)$ is linear (and hence, an isomorphism) if \mathcal{H} is real, and a conjugate linear surjective isometry if \mathcal{H} is complex.

Definition 19 (Adjoint operator).

1. Let \mathcal{X}, \mathcal{Y} be normed spaces. For $T \in B(\mathcal{X}, \mathcal{Y})$, define $T' : \mathcal{Y}' \rightarrow \mathcal{X}'$ by

$$(T'\varphi)(x) := (\varphi \circ T)(x) = \varphi(Tx) \quad (\in \mathbb{K}). \quad (66)$$

That is, $T'\varphi := \varphi \circ T : \mathcal{X} \rightarrow \mathbb{K}$; $T'\varphi \in \mathcal{X}'$ since $T \in B(\mathcal{X}, \mathcal{Y}), \varphi \in \mathcal{Y}'$ are both continuous.

Then $T' \in B(\mathcal{Y}', \mathcal{X}')$, and $\|T'\|_{B(\mathcal{Y}', \mathcal{X}')} = \|T\|_{B(\mathcal{X}, \mathcal{Y})}$. The operator T' is called the (Banach space) adjoint (operator) of T .

2. Let $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ be a Hilbert space, and let $T \in B(\mathcal{H})$. The map $T^* : \mathcal{H} \rightarrow \mathcal{H}$ defined by $T^* := \Phi \circ T' \circ \Phi^{-1}$ (with $\Phi : \mathcal{H}' \rightarrow \mathcal{H}$ the map from Theorem 20) is called the (Hilbert space) adjoint (operator) of T . The map $*$: $B(\mathcal{H}) \rightarrow B(\mathcal{H}), T \mapsto T^*$ is a conjugate linear surjective isometry, and an involution $((T^*)^* = T)$. It satisfies $(TS)^* = S^*T^*$ for all $T, S \in B(\mathcal{H})$.
3. An operator $T \in B(\mathcal{H})$ is called self adjoint (or self-adjoint or selfadjoint) iff $T^* = T$. It is called unitary iff $T^*T = TT^* = \mathbb{1}_{\mathcal{H}}$.

Theorem 21 (The BLT-Theorem).

Let $(\mathcal{X}, \|\cdot\|_{\mathcal{X}}), (\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$ be normed spaces, let $V \subset \mathcal{X}$ be a dense (!) subspace, and assume $T : V \rightarrow \mathcal{Y}$ is a bounded operator (with $\|\cdot\|_{\mathcal{X}}$ restricted to V the norm on V). If $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$ is a Banach space (!), then there exists a unique (!) bounded operator $\tilde{T} \in B(\mathcal{X}, \mathcal{Y})$ extending T (that is, $\tilde{T}|_V = T$).

Theorem 22 (Open mapping theorem).

Let \mathcal{X} and \mathcal{Y} be Banach spaces, and assume $T \in B(\mathcal{X}, \mathcal{Y})$ is surjective. Then T is an open mapping, that is, if $U \subset \mathcal{X}$ is open, then $T(U) \subset \mathcal{Y}$ is open.

Corollary 1 (Bounded inverse theorem). Let \mathcal{X} and \mathcal{Y} be Banach spaces, and assume $T \in B(\mathcal{X}, \mathcal{Y})$ is bijective. Then the inverse operator $T^{-1} : \mathcal{Y} \rightarrow \mathcal{X}$ is also bounded (hence, continuous).

Corollary 2 (Closed graph theorem). Let \mathcal{X} and \mathcal{Y} be Banach spaces, and assume $T \in L(\mathcal{X}, \mathcal{Y})$ (!) satisfies: For every sequence $(x_n)_{n \in \mathbb{N}} \subset \mathcal{X}$ with both $x_n \rightarrow 0$ ($\in \mathcal{X}$), $n \rightarrow \infty$, and $Tx_n \rightarrow y$ ($\in \mathcal{Y}$), $n \rightarrow \infty$, follows that $y = 0$. Then $T \in B(\mathcal{X}, \mathcal{Y})$ (that is, T is continuous/bounded).

Theorem 23 (Uniform boundedness principle / Banach-Steinhaus). Let \mathcal{X} be a Banach space (!) and \mathcal{Y} a normed (!) vector space. Suppose that $F \subset B(\mathcal{X}, \mathcal{Y})$ is a collection of continuous linear operators (that is, bounded operators) from \mathcal{X} to \mathcal{Y} . If for all $x \in \mathcal{X}$ one has

$$\sup_{T \in F} \|Tx\|_{\mathcal{Y}} < \infty, \tag{67}$$

then

$$\sup_{T \in F, \|x\|_{\mathcal{X}}=1} \|Tx\|_{\mathcal{Y}} = \sup_{T \in F} \|T\|_{B(\mathcal{X}, \mathcal{Y})} < \infty. \tag{68}$$

Corollary 3 (Useful consequence of Banach-Steinhaus). Let \mathcal{X} be a Banach space (!) and \mathcal{Y} a normed (!) vector space, and let $(T_n)_{n \in \mathbb{N}} \subset B(\mathcal{X}, \mathcal{Y})$. Assume $\lim_{n \rightarrow \infty} T_n x =: Tx$ exists for all $x \in \mathcal{X}$. Then $T \in B(\mathcal{X}, \mathcal{Y})$. (Importance: pointwise limits of continuous maps are not, in general, continuous.)

COMPLEX ANALYSIS

Let $U \subset \mathbb{C}$ be open.

Definition 20 (Holomorphic function).

1. A function $f : U \rightarrow \mathbb{C}$ is called complex differentiable at $z_0 \in U$ iff

$$\lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0} (=: f'(z_0) \in \mathbb{C}) \quad (69)$$

exists. The (complex) number $f'(z_0)$ is called the (complex) derivative of f at z_0 .

2. The function f is called holomorphic (in U) iff it is complex differentiable at z for all $z \in U$.

Theorem 24 (Cauchy-Riemann equations).

Identify the function $f : U \rightarrow \mathbb{C}$ with $F = (u, v) : U \rightarrow \mathbb{R}^2$ (that is, $f(z) = u(x, y) + iv(x, y)$ ($z = x + iy$), $u = \Re(f)$, $v = \Im(f)$).

Then f is holomorphic at $z_0 \in U$ iff F is differentiable at (x_0, y_0) and satisfies the Cauchy-Riemann equations,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{at} \quad (x_0, y_0). \quad (70)$$

Definition 21 ((Complex) analytic function).

1. A function $f : U \rightarrow \mathbb{C}$ is said to have a (convergent) power series expansion at $z_0 \in U$ iff there exist a sequence $(a_n)_{n \in \mathbb{N}} \subset \mathbb{C}$ and $r > 0$ such that

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad \text{for all } z \in B_r(z_0) \subset U. \quad (71)$$

2. The function f is called (complex) analytic (in U) iff it has a (convergent) power series expansion at z for all $z \in U$.

Theorem 25 (Unique continuation).

Let $D \subset \mathbb{C}$ be connected and open, let $f, g : D \rightarrow \mathbb{C}$ be analytic, and assume there exists a convergent sequence $(z_k)_{k \in \mathbb{N}} \subset D$ of distinct points with limit in D such that $f(z_k) = g(z_k)$ for all $k \in \mathbb{N}$. Then $f(z) = g(z)$ for all $z \in D$.

Theorem 26 (Holomorphic equals (complex) analytic). A function $f : U \rightarrow \mathbb{C}$ is holomorphic iff it is (complex) analytic. In this case, for all $z_0 \in U$ there exists $r = r(z_0) > 0$ such that

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n, \quad z \in B_r(z_0) \subset U \quad (\text{Taylor-series}), \quad (72)$$

with

$$f^{(0)}(z) := f(z), \quad f^{(1)}(z) := f'(z), \quad f^{(n+1)}(z) := (f^{(n)})'(z), \quad n \geq 1. \quad (73)$$

Let $D \subset \mathbb{C}$ be connected and open.

Theorem 27 (Maximum Modulus Principle). Let $f : D \rightarrow \mathbb{C}$ be holomorphic. If there exists $z_0 \in D$ and $r > 0$ such that

$$|f(z_0)| \geq |f(z)| \quad \text{for all } z \in B_r(z_0) \subset D, \quad (74)$$

then there exists $c \in \mathbb{C}$ such that $f(z) = c$ for all $z \in D$ (that is, f is constant).