MATHEMATISCHES INSTITUT
DER UNIVERSITÄT MÜNCHEN
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WS 2015/16
– Blatt 4 –
12.11.2015

Numerik I
(Zentralübung)

Problem Sheet 4

Question 1 **[4 + 2 = 6 points]**

(a) Determine a suitable algebraic system for computing the quadratic spline s of a function $f\colon [a,b] \to \mathbb{R}$ corresponding to the points $a = x_0 < x_1 < \ldots < x_n = b$. That is, determine a suitable matrix $A \in \mathbb{R}^{n\times n}$ and a vector $g = (g_0, g_1, \ldots g_{n-1}) \in \mathbb{R}^n$ (in terms of $f(x_i)$ and x_i) such that

$$A\big(s'(x_0), s'(x_1), \ldots, s'(x_{n-1})\big)^t = g\,,$$

with the additional condition that $s'(x_n) = 0$.

(b) Calculate the quadratic spline for the function $f\colon [0,3] \to \mathbb{R}$, $f(x) = x^2$, where $h = 1$ (so $x_i = i$ for $0 \leq i \leq 3$).

Question 2 **[2 + 3 + 2 = 7 points]**

Let

$$\Delta = \{a = x_0 < x_1 < \ldots < x_n = b\}$$

be a decomposition of a given interval $[a,b]$. Let $C^1_\Delta[a,b]$ denote the space of continuous functions on $[a,b]$ that are piecewise differentiable with respect to Δ (i.e. differentiable on every interval (x_{i-1}, x_i) for $1 \leq i \leq n$).

Recall that for a piecewise differentiable $g \in C^1_\Delta[a,b]$,

$$\|g'\|_2 := \Big(\sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} |g'(x)|^2 \,\mathrm{d}x\Big)^{1/2}.$$

Suppose $f \in C^1_\Delta[a,b]$, and let $s_f \in S_{\Delta,1}$ be the piecewise linear spline that interpolates f at the points x_i That is, we have $s_f(x_i) = f(x_i)$ for $0 \leq i \leq n$.

(a) Show that

$$\int_a^b \big(f'(x) - s_f'(x)\big) s_f'(x) \,\mathrm{d}x = 0\,.$$

(b) Show that

$$\|f' - s_f'\|_2^2 = \|f'\|_2^2 - \|s_f'\|_2^2 .$$

Remark. Note that this also shows that $\|s_f'\|_2 \leq \|f'\|_2$. Hence, for any given values $y_0, \dots y_n \in \mathbb{R}$, the linear spline function $s(x_i) = y_i$ solves the Variational Problem

$$\|f'\|_2 \to \min \quad \text{for } f \in C^1_\Delta[a,b] \quad \text{where } f(x_i) = y_i \text{ for } 0 \leq i \leq n .$$

(c) Show that for any arbitrary linear spline function $s \in S_{\Delta,1}$, we have

$$\|f' - s_f'\|_2 \leq \|f' - s'\|_2 .$$

Question 3 **[4 points]**

Compute the natural cubic spline function $s\colon [0,2] \to \mathbb{R}$ for the following data:

x_i	0	1	2
y_i	1	2	0

Question 4 **[3 points]**

Suppose we have a decomposition

$$\Delta = \{0 = x_0 < x_1 < \ldots < x_n = 1\}$$

of the interval $[0,1]$ such that all the points are equi-distant. So $x_i = x_{i-1} + h$ for each $1 \leq i \leq n$, where $h = 1/n$.

We wish to approximate the function

$$f\colon [0,1] \to \mathbb{R}, \quad f(x) = \sin(2\pi x)$$

on this interval with a cubic spline function $s \in S_{\Delta,3}$ with natural boundary conditions.

How large must n be, so that the difference between s and f on the entire interval $[0,1]$ is less than 10^{-2}?

Deadline for handing in: 15:30 Thursday 19 November 2015

Please put solutions in one of the designated Numerik I boxes on the 1st floor (near the library)

Homepage: `www.mathematik.uni-muenchen.de/~soneji/numerik.php`