MATHEMATISCHES INSTITUT
DER UNIVERSITÄT MÜNCHEN
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WS 2015/16
– Blatt 3 –
5.11.2015

Numerik I
(Zentralübung)

Problem Sheet 3

Question 1 **[4 points]**

Prove **Corollary 4.21** from the lectures: let $n \in \mathbb{N}_0$, and let $t_1, \ldots, t_{n+1} \in [-1, 1]$ be the zeroes of the $(n + 1)^{\text{th}}$ Tschebyscheff polynomial of the first kind, T_{n+1}. Let a, $b \in \mathbb{R}$, $a < b$, and define $\varphi\colon [-1, 1] \to [a, b]$ by

$$\varphi(t) := \frac{b-a}{2}t + \frac{a+b}{2} \quad \big(t \in [-1,1]\big)\,.$$

Then we have

$$\begin{aligned}\min_{x_0,\ldots x_n \in [a,b]} \max_{x\in[a,b]} \big|(x-x_0)\ldots(x-x_n)\big| &= \max_{x\in[a,b]} \big|(x-\varphi(t_1))\ldots(x-\varphi(t_{n+1}))\big| \\ &= \frac{(b-a)^{n+1}}{2\cdot 4^n}\,.\end{aligned}$$

Question 2 **[2 + 2 + 2 = 6 points]**

(a) Let $I = [a, b] \subset \mathbb{R}$ be an interval, and let $a = x_0 < x_1 < x_2 < \ldots < x_n = b$ be distinct points. Now define, for $x \in I$,

$$\omega_{n+1}(x) := (x-x_0)(x-x_1)(x-x_2)\ldots(x-x_n)\,.$$

Moreover, let $h_{\max}$ denote the maximal distance of two neighbouring points. That is,

$$h_{\max} := \max_{1\le i\le n} |x_i - x_{i-1}|\,.$$

Show that for any $x \in I$ we have

$$|\omega_{n+1}(x)| = \prod_{k=0}^{n} |x-x_k| \le n!\,h_{\max}^{n+1}\,.$$

(b) We now wish to approximate the function

$$f\colon [0, \frac{\pi}{2}] \to \mathbb{R} \quad , \quad f(x) = \sin(2x)$$

with an interpolation polynomial.

Compute an x-independent upper bound for the interpolation error when using the equidistant (equally-spaced) points

$$x_k = \frac{k\pi}{2n} \quad , \quad k = 0, \ldots, n$$

in the case $n = 5$.

(c) Now suppose the inner points $0 < x_1 < \ldots < x_4 < \pi/2$ are not equidistant. Determine a suitable maximal distance $h_{\max}$ of two neighbouring points, so that the interpolation error (independent of x) is no more than 10^{-2}.

Question 3 **[4 points]**

Compute the interpolation polynomial corresponding to the following table in the form $p(x) = \sum_{i=0}^{n} a_i x^i$.

x_k	-2	-1	0	1
$f(x_k)$	4	1	6	7
$f'(x_k)$		1	5	

Question 4 **[2 + 4 = 6 points]**

Let $I = [a, b] \subset \mathbb{R}$ be an interval, let $\Delta := \{x_0, \ldots, x_n\} \subset I$ be a set of distinct points in I, and $S_{\Delta,1}$ be the set of corresponding piecewise linear spline functions

$$s\colon I \to \mathbb{R}\,.$$

(That is, if $x_i < x_j$ are two neighbouring points in Δ, then s is linear on the interval $[x_i, x_j]$.)

For a function $f \in C([a, b])$, let $s_f \in S_{\Delta,1}$ be the piecewise linear spline that interpolates f at the points x_k. That is, we have

$$s_f(x_k) = f(x_k) \quad \forall 0 \le k \le n\,.$$

(a) Show that for each spline $s \in S_{\Delta,1}$ we have:

$$\|s_f - s\|_\infty \le \|f - s\|_\infty\,.$$

Here, $\|f\|_\infty = \max_{x\in I} |f(x)|$ denotes the maximum-norm for continuous functions.

(b) We define the minimal distance of the function f from the splines in $S_{\Delta,1}$ as:

$$\mathrm{dist}_{S_{\Delta,1}}(f) := \min_{s\in S_{\Delta,1}} \|f - s\|_\infty\,.$$

Show that the following inequality holds:

$$\mathrm{dist}_{S_{\Delta,1}}(f) \le \|f - s_f\|_\infty \le 2\,\mathrm{dist}_{S_{\Delta,1}}(f)\,.$$

Sketch a situation in which the left hand inequality holds strictly.

(That is, where $\mathrm{dist}_{S_{\Delta,1}}(f) < \|f - s_f\|_\infty$.)

Deadline for handing in: 15:30 Thursday 12 November 2015

Please put solutions in one of the designated Numerik I boxes on the 1st floor (near the library)

Homepage: `www.mathematik.uni-muenchen.de/~soneji/numerik.php`