Numerik I
(Zentralübung)

Problem Sheet 2

Question 1 **[5 points]**

Prove **Bemerkung 4.10** from the lectures: show that

$$[y_j, \dots, y_{j+k}] = \sum_{l=j}^{j+k} y_l \prod_{m=j, m\neq l}^{j+k} \frac{1}{x_l - x_m}.$$

Hence the value $[y_j, \dots, y_{j+k}]$ depends only on the data points $(x_j, y_j), \dots, (x_{j+k}, y_{j+k})$, and is independent of the order of these data points. (Refer to the lecture notes for notation.)

Question 2 **[5 points]**

Let $L_i, i \in \{0, \dots, n\}, n \in \mathbb{N}_0$, be the Lagrange basis polynomials for the points $x_0, x_1, \dots, x_n, n \in \mathbb{N}_0$, with $x_i \neq x_j$ for $i \neq j$.

(a) Show that $\sum_{i=0}^{n} L_i(x) = 1$ for all $x \in \mathbb{R}$

(b) Show that if we write $\sum_{i=0}^{n} a_i x^i := L_i(x)$, then $(a_0, a_1, \dots, a_n)^t$ is just the i-th column of the inverse of the Vandermonde Matrix for the points $x_0, x_1, \dots x_n$.

Question 3 **[2 + 2 = 4 points]**

Calculate, using Newton's method, the interpolation polynomial for the points $(0, 1)$, $(1, 5)$ and $(2, 4)$. Now calculate the interpolation polynomial using Lagrange's method.

Question 4 **[1 + 3 + 2 = 6 points]**

Let $I = [a, b] \subset \mathbb{R}$ be an interval, and $x_0 < x_1 < \dots x_n$ be distinct points in I. For a function $f\colon I \to \mathbb{R}$, let

$$p_n(x) = \sum_{i=0}^{n} f(x_i) L_i(x) \quad \text{with} \quad L_i(x) := \prod_{k=0, k\neq i}^{n} \frac{x - x_k}{x_i - x_k}$$

be the associated Lagrange interpolation polynomial.

(a) Show that for every $f \in C([a, b])$, the interpolation polynomial satisfies the inequality

$$\|p_n\|_\infty \leq \Lambda_n \|f\|_\infty \,,$$

with the Maximum-norm $\|f\|_\infty = \max_{x\in I} |f(x)|$, and the *Lebesgue Constant*

$$\Lambda_n := \max_{x\in I} \sum_{i=0}^{n} |L_i(x)| \,.$$

(b) By finding a suitable function $g \in C([a, b])$, show that Λ_n is in fact the smallest constant for which the inequality in (a) holds. Using this, what can you say about the absolute condition number of the interpolation problem for continuous functions on the interval I?

(c) Consider the affine transformation $T\colon I \to [-1, 1]$ given by

$$x \mapsto \frac{2x - a - b}{b - a}$$

and show that the Lebesgue Constant for the interpolation problem transformed by T (on $[-1, 1]$), is the same as for the original problem on $[a, b]$.

Deadline for handing in: 1400 Thursday 5 November 2015

Please put solutions in one of the designated Numerik I boxes on the 1st floor (near the library)