MATHEMATISCHES INSTITUT
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– Blatt 6 –
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PDG I
(Tutorium)

Tutorial 6

(Convex sets in $\mathbb{R}^n$ and the exterior ball condition)

Question 1

Let $\Omega \subset \mathbb{R}^n$ be open and convex.

(i) Show that $\overline{\Omega}$ is also convex.

(ii) For $x \in \mathbb{R}^n$, show that there exists an element $x_\infty \in \overline{\Omega}$ such that

$$|x - x_\infty| = \inf_{z \in \overline{\Omega}} |x - z| \,.$$

(i.e. x_∞ minimises $z \mapsto |x - z|$ in $\overline{\Omega}$.)

(iii) If $x \notin \Omega$, show that $x_\infty \in \partial\Omega$.

(iv) Show that x_∞ is unique (*Hint:* Use the strict convexity of the map $y \mapsto |y|^2$).

Hence we have shown that the map $x \mapsto x_\infty =: p(x)$ is well defined on $\mathbb{R}^n$. It is called the *projection* onto $\overline{\Omega}$.

Question 2

Let Ω, $p \colon \mathbb{R}^n \to \overline{\Omega}$ be as in Question 1. For $x \in \mathbb{R}^n$, $z \in \overline{\Omega}$, and $t \in [0, 1]$, define

$$g(t) := |x - [(1-t)p(x) + tz]|^2 \,.$$

(i) By considering difference quotients, show that we must have $g'(0) \geq 0$ (where this is understood to be the right hand derivative at 0).

(ii) Calculate $g'(t)$. Deduce from part (i) that we must have

$$\langle x - p(x), z - p(x) \rangle \leq 0 \,.$$

(iii) Show that if $x, y \in \mathbb{R}^n$, then

$$|p(x) - p(y)|^2 \leq \langle p(x) - p(y), x - y \rangle \,,$$
$$|p(x) - p(y)| \leq |x - y| \,.$$

Question 3

(i) Suppose $x_0 \notin \overline{\Omega}$. Show that $a = p(x_0) - x_0 \neq 0$ satisfies

$$\langle x_0, a \rangle < \inf\{\langle z, a \rangle : z \in \overline{\Omega}\}\,.$$

(ii) Now suppose $x_0 \in \partial\Omega$. Show that there exists a vector $a \in \mathbb{R}^n$, $a \neq 0$, such that

$$\langle x_0, a \rangle \leq \langle z, a \rangle \quad \forall z \in \overline{\Omega}\,,$$

and

$$\langle x_0, a \rangle < \langle z, a \rangle \quad \forall z \in \Omega\,.$$