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PDG I
(Tutorium)

Tutorial 4

(Harmonic, homogeneous polynomials)

Suppose that $u \in C^\infty(\Omega)$, $\Omega \subset \mathbb{R}^n$ open. Note that

$$\frac{\partial}{\partial x_i}(\Delta u) = \Delta\Big(\frac{\partial u}{\partial x_i}\Big)$$

and generally that $p(D)\Delta u = \Delta(p(D)u)$, where $p(D)$ is a general differential operator of the form

$$p(D)u = \sum_{|\alpha| \le m} c_\alpha D^\alpha u\,, \quad c_\alpha \in \mathbb{R}\,.$$

Hence it is easy to deduce that when u is harmonic, so are $\frac{\partial u}{\partial x_i}$, $p(D)u$.

We say that $u\colon \mathbb{R}^n \to \mathbb{R}$ is *homogeneous of order* s when $u(\lambda x) = \lambda^s u(x)$ for all $\lambda > 0$, $x \in \mathbb{R}^n$.

Question 1

(i) Prove the product rule $\Delta(uv) = (\Delta u)v + 2\nabla u \cdot \nabla v + u(\Delta v)$, where u, v are smooth.

(ii) Show that if $u \in C^1(\mathbb{R}^n)$ is homogeneous of order s then

$$\sum_{i=1}^n u_{x_i} x_i = s u(x)\,.$$

Hint: Consider $g'(t)$, where $g(t) := u(tx)$.

(iii) Let s, $t \in R$, and suppose $u\colon \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ is in C^2, and is homogeneous of order s. Show that

$$\Delta\big(|x|^t u(x)\big) = t(n + t + 2s - 2)|x|^{t-2}u(x) + |x|^t \Delta u(x)\,.$$

This shows that u is harmonic and s-homogeneous if and only if $|\cdot|^{2-n-2s}u$ is harmonic and $2 - n - s$ homogeneous.

Question 2

Let $\mathcal{P}_m(\mathbb{R}^n)$ denote the vector space of all homogeneous polynomials in $\mathbb{R}^n$ of order m, and $\mathcal{H}_m(\mathbb{R}^n)$ denote the subspace of all such polynomials that are harmonic.

(i) Show that if $p \in \mathcal{P}_m(\mathbb{R}^n)$, $\lambda > 0$ then

$$p(\lambda x) = \lambda^m p(x) \, .$$

(ii) Prove, using strong induction (without using complex numbers), that the functions u_m, $v_m \colon \mathbb{R}^2 \to \mathbb{R}$, which are given in polar co-ordinates $(x, y) = (r \cos\varphi, r \sin\varphi)$ by

$$u_m(r, \varphi) := r^m \cos(m\varphi) \, , \quad v_m(r, \varphi) := r^m \sin(m\varphi) \, , \quad m \in \mathbb{N}_0 \, ,$$

satisfy the *Cauchy-Riemann equations*

$$u_x = v_y \, , \quad u_y = -v_x \, .$$

(iii) Show, using part (ii), that $\mathcal{H}_m(\mathbb{R}^n) \neq \{0\}$ for all $m \in \mathbb{N}_0$ and $n \geq 2$.