# PDG I
# (Tutorium)

## Tutorial 3
(Integrals on the Ball)

### Question 1

Recall that

$$\alpha_n := \text{ volume of unit ball in } \mathbb{R}^n\,, \qquad \omega_n := \text{ surface area of unit ball in } \mathbb{R}^n\,.$$

Use polar coordinates to show that $\omega_n = n\alpha_n$.

### Question 2

Let $B(x,R)$ be a ball in $\mathbb{R}^n$, and let $u \in C(B(x,r))$. For $r \in (0,R)$, consider the integrals

$$\varphi(r) := \fint_{\partial B(x,r)} u(y)\,\mathrm{d}S(y) = \fint_{\partial B(0,1)} u(x+rz)\,\mathrm{d}S(z)$$

and

$$\psi(r) := \int_{B(x,r)} u(y)\,\mathrm{d}y\,.$$

(i) Show that $\varphi \in C(0,R)$.

(ii) Show that $\psi \in C^1(0,R)$ with

$$\psi'(r) = \int_{\partial B(x,r)} u(y)\,\mathrm{d}S(y) \quad \big(= \omega_n r^{n-1}\varphi(r)\big)\,.$$

(iii) Now suppose $u \in C^1(\overline{B(x,R)})$. Show that $\varphi \in C^1(0,R)$ with

$$\varphi'(r) = \fint_{\partial B(0,1)} \nabla u(x+rz)\cdot z\,\mathrm{d}S(z)\,.$$

*Hint:* Note that $\frac{d}{dr}(u(x+rz)) = \nabla u(x+rz)\cdot z$. Consider the difference quotient

$$\frac{\varphi(r+h)-\varphi(r)}{h} \qquad \big(h \neq 0\,,\ r+h \in (0,R)\big)$$

and use the Dominated Convergence Theorem.