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WS 2014/15
– Blatt 1 –
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PDG I
(Zentralübung)

Problem Sheet 1

Question 1

(a) Prove the *Multinomial Theorem*:

$$(x_1 + \ldots + x_n)^k = \sum_{|\alpha|=k} \binom{|\alpha|}{\alpha} x^\alpha ,$$

where

$$\binom{|\alpha|}{\alpha} := \frac{|\alpha|!}{\alpha!}, \qquad \alpha! := \alpha_1!\alpha_2!\ldots\alpha_n!$$

and $x^\alpha = x_1^{\alpha_1}\ldots x_n^{\alpha_n}$.

The sum is taken over all multi-indices $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$ with $|\alpha| = k$.

(b) Prove *Leibniz's formula*:

$$D^\alpha(uv) = \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} D^\beta u\, D^{\alpha-\beta} v ,$$

where $u, v\colon \mathbb{R}^n \to \mathbb{R}$ are smooth,

$$\binom{\alpha}{\beta} := \frac{\alpha!}{\beta!(\alpha-\beta)!},$$

and $\beta \leq \alpha$ means $\beta_i \leq \alpha_i$ for $i = 1, \ldots, n$.

Question 2

Classify each of the following partial differential equations as *linear*, *semilinear*, *quasilinear* or *fully nonlinear*. Also determine the order of each equation. In each case we have $u\colon \Omega \to \mathbb{R}$ for some open subset Ω of $\mathbb{R}^n$.

(a) $u_{x_1x_2} + u_{x_2x_3} = 0$ linear ord 2

(b) $u|u_{x_1}|^2 + u_{x_1x_2} = 0$ semilinear : nonlinear 1st order + linear 2nd order ord 2

(c) $u|u_{x_1}|^2 u_{x_1x_2} = 0$ quasilinear nonlinear 1st order × linear second order order 2

(d) $x_1x_2^2 u_{x_1x_2} = x_2 \sin(x_1)$ linear f(x) × second order order 2

(e) $uu_{x_1} + |u_{x_1x_2}|^2 = 0$ fully non linear $\cancel{x}$ 2

(f) $-\sum_{i=1}^n (b^i u)_{x_i} = 0$, where $b = (b^1, \dots b^n) \in \mathbb{R}^n$ linear 1

(g) $-\Delta u = f(u)$ (recall $\Delta u := \sum_{i=1}^n u_{x_i x_i}$) semilinear 2

(h) $iu_t + \Delta u = f(|u|^2)u$ (here $u\colon (0,T) \times \Omega \to \mathbb{R}$) semilinear 2 , 0th order nonlinearity

(i) $\det(D^2 u) = f$ fully non linear . 2nd order terms multiplied by each other order 2

(j) $\mathrm{div}(|\nabla u|^{p-2}\nabla u) = 0.$ quasilinear

Question 3

Show that functions of the form $u(x, y) = f(x) + g(y)$, where f and g both belong to $C^1(\mathbb{R})$, are solutions to the partial differential equation

$$u_{xy}(x, y) = 0 \quad \text{on } \mathbb{R}^2 .$$

What is the order of this PDE? Is such a solution u necessarily in $C^2(\mathbb{R}^2)$?

Question 4

Write down an explicit formula for a function u solving the initial value problem

$$\begin{cases} u_t + b \cdot Du + cu = 0 & \text{in } \mathbb{R}^n \times (0, \infty) \\ u = g & \text{on } \mathbb{R}^n \times \{t = 0\}. \end{cases}$$

Here, $g\colon \mathbb{R}^n \to \mathbb{R}$ is smooth, and $c \in \mathbb{R}$, $b \in \mathbb{R}^n$ are constants.

Hint: Use the method of characteristics. Here, recognize the left hand side of the equation as the derivative of a product of u with a simple function.

$$|\nabla u|^{p-2} = (u_{x_1}^2 + \dots + u_{x_n}^2)^{\frac{p-2}{2}}$$

div : vector valued $\to$ scalar valued

p-laplace : $\mathrm{div}(|\nabla u|^{p-2}\nabla u)$

$$= \sum_{i=1}^n (|\nabla u|^{p-2} u_{x_i})_{x_i}$$

$$\boxed{\begin{aligned} v &= u_{x_1}^2 + \dots + u_{x_n}^2 \\ \frac{\partial v}{\partial x_i} &= 2u_{x_1x_i} u_{x_1} + \dots + 2u_{x_n x_i} u_{x_i} \\ \frac{d}{dv} v^{\frac{p-2}{2}} &= \frac{p-2}{2} v^{\frac{p-4}{2}}. \end{aligned}}$$

$$(|\nabla u|^{p-2} u_{x_i})_{x_i} = \underbrace{|\nabla u|^{p-2} u_{x_i x_i}}_{\text{quasilinear term}} + u_{x_i}(|\nabla u|^{p-2})_{x_i}$$

$$(|\nabla u|^{p-2})_{x_i} = \frac{\partial}{\partial x_i}(u_{x_1}^2 + \dots + u_{x_n}^2)^{\frac{p-2}{2}} \overset{\text{chain rule partial derivatives}}{=} \sum_{j=1}^n u_{x_j x_i} u_{x_j}^2 (p-2) |\nabla u|^{p-4}$$

quasilinear

Deadline for handing in: 0800 Wednesday 22 October

Please put solutions in Box 17, 1st floor (near the library)

Homepage: `http://www.mathematik.uni-muenchen.de/~soneji/pde1.php`

So $\mathrm{div}(|\nabla u|^{p-2}\nabla u)$ = sum of quasilinear terms - quasilinear

PDE 1 , Problem Sheet 1

① (a) $u : \mathbb{R}^n \to \mathbb{R}$ smooth, how many distinct partial derivatives of order 2?

These are all of the form $u_{x_i x_j}$ where ~~1≤j≤i≤n~~ $1 \le i \le j \le n$

(1) $u_{x_1 x_j}$ - n possible derivs

(2) $u_{x_2 x_j}$ exclude $j=1$ (covered in case (1))
so $n-1$ possible derivs

(3) $u_{x_3 x_j}$ exclude $j=1,2$, so $n-2$ possible derivs

⋮

(n) $u_{x_n x_j}$ only $j=n$ hasn't been covered earlier 1 possible derivs

<u>Hence</u> $1+2+\dots+n = \underline{\frac{1}{2}n(n+1)}$ distinct partial derivs of order 2.

(b) $k=3$. Distinct partial derivs of order 3.
All of form $u_{x_i x_j x_k}$ $1 \le i \le j \le k \le n$.

(1) $u_{x_1 x_j x_k}$ from part (a), we take $1 \le j \le k \le n$
$\frac{1}{2}n(n+1)$ possible

(2) $u_{x_2 x_j x_k}$ $j=1$ or $k=1$ covered in (1).
So take $2 \le j \le k \le n$: part (a) $\frac{1}{2}(n-1)n$

⋮

(n) $u_{x_n x_j x_k}$ need $n \le j \le k \le n$: $1 = \frac{1}{2}(n-(n-1))(n-(n-2))$.

So in total we have $\frac{1}{2}\left(\sum_{i=1}^{n} i(i+1)\right) = \frac{1}{2}\sum i^2 + \sum i$

$= \dfrac{n(n+1)(2n+1)}{12} + \dfrac{n(n+1)}{4}$

(2)

$$= \frac{1}{12}\left(n(n+1)(2n+1) + 3n(n+1) \right)$$

$$= \frac{1}{12}\left(n(n+1)(2n+4)\right)$$

$$= \frac{1}{6} n(n+1)(n+2)$$

or $\# = 1 + 3 + 6 + 10 + \dots$ - first n triangle # numbers.

(c) ~~k=2~~ $n = 2$, k general. $u(x) = u(x_1, x_2)$

possibilities are: all k derivatives w.r.t x_1

$k-1$ " " "

$\vdots$

1 " "

0 " " (ie all $u_{x_2 x_2 \dots x_2}$)

$(k+1)$ possibilities

In general (using results from combinatorics) we can show that $u : \mathbb{R}^n \to \mathbb{R}$ smooth has

$$\binom{n+k-1}{k} = \binom{n+k-1}{n-1}$$ distinct partial derivatives of order k.

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② Multinomial Theorem

$$(x_1 + \dots + x_n)^k = \sum_{|\alpha| = k} \binom{|\alpha|}{\alpha} x^\alpha$$

(*) $(x_1 + \dots x_n)^{k\alpha} = \underbrace{(x_1 + \dots + x_n)(x_1 + \dots + x_n) \dots (x_1 + \dots + x_n)}_{k \text{ times}}$

$= \sum_{|\alpha| = k} c_\alpha x^\alpha$ each sum is a multiplicati of k-many of the x_i.

Given fixed $\alpha = (\alpha_1, \dots, \alpha_n)$ with $|\alpha| = k$, what is c_α?

How many ways can we obtain $x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n}$ in the above product?

First note that there are $\binom{|\alpha|}{\alpha_1}$ ways of obtaining a term with $x_1^{\alpha_1}$ ^. This leaves $|\alpha| - \alpha_1$ brackets to obtain $x_2^{\alpha_2} \dots x_n^{\alpha_n}$.
ignoring sth terms

So: $c_\alpha x^\alpha = \binom{|\alpha|}{\alpha_1} \times \sum_{|(\alpha_2, \dots, \alpha_n)| = |\alpha| - \alpha_1} c_\alpha^{(2)} x_1^{\alpha_1} (x_2, \dots, x_n)^{(\alpha_2, \dots, \alpha_n)}$

How many ways can we obtain $x_2^{\alpha_2}$ from remaining brackets?

$\binom{|\alpha| - \alpha_1}{\alpha_2}$. So there are $\binom{|\alpha|}{\alpha_1} \times \binom{|\alpha| - \alpha_1}{\alpha_2}$ ways of obtaing $x_1^{\alpha_1} x_2^{\alpha_2}$ in (*) ^. This leaves $|\alpha| - \alpha_1 - \alpha_2$ brackets.
(ignoring other terms)

Then, of these, there are $\binom{|\alpha| - \alpha_1 + \alpha_2}{\alpha_3}$ ways of getting $x_3^{\alpha_3}$.

Continuing like this, we see

$$C_\alpha = \binom{|\alpha|}{\alpha_1} \times \binom{|\alpha| - \alpha_1}{\alpha_2} \times \binom{|\alpha| - \alpha_1 - \alpha_2}{\alpha_3} \times \dots \times \binom{\dots}{\dots}$$

$$= \frac{|\alpha|!}{(\cancel{|\alpha| - \alpha_1})!\,\alpha_1!} \times \frac{(\cancel{|\alpha| - \alpha_1})!}{(\cancel{|\alpha| - \alpha_1 - \alpha_2})!\,\alpha_2!} \times \frac{(|\alpha| - \alpha_1 - \alpha_2)!}{(|\alpha| - \alpha_1 - \alpha_2 - \alpha_3)!\,\alpha_3!} \times \dots$$

$$\dots \times \frac{(|\alpha| - \alpha_1 - \dots - \alpha_{n-1})!}{0!\ \alpha_n!}$$

$$= \frac{|\alpha|!}{\alpha_1!\,\alpha_2! \dots \alpha_n!} = \binom{|\alpha|}{\alpha}.$$

(7) Leibniz's formula

$u, v : \mathbb{R}^n \to \mathbb{R}$ smooth. Show:

$$D^\alpha (uv) = \sum_{\beta \le \alpha} \binom{\alpha}{\beta} D^\beta u \, D^{\alpha - \beta} v$$

$$D^\alpha (uv) = \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}} \frac{\partial^{\alpha_{n-1}}}{\partial x_{n-1}^{\alpha_{n-1}}} \cdots \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} (uv)$$

$$\frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} (uv) = \sum_{\beta_1 = 1}^{\alpha_1} \binom{\alpha_1}{\beta_1} \frac{\partial^{\beta_1} u}{\partial x_1^{\beta_1}} \frac{\partial^{\alpha_1 - \beta_1} v}{\partial x_1^{\alpha_1 - \beta_1}}$$

(standard Binomial formula for derivatives)

$= w_1$ (say)

Then $\cancel{\frac{\partial^{\alpha_2}}{\partial x_1^{\alpha_1}} w_1 = \sum_{\beta_2 = 1}^{\alpha_2} \binom{\alpha_2}{\beta_2} \frac{\partial^{\beta_2} w_1}{\partial x_2^{\beta_2}}}$

= additivity of differential operator

Then $$\frac{\partial^{\alpha_2}}{\partial x_2^{\alpha_2}} (w_1) = \sum_{\beta_1 = 1}^{\alpha_1} \binom{\alpha_1}{\beta_1} \frac{\partial^{\alpha_2}}{\partial x_2^{\alpha_2}} \left(\frac{\partial^{\beta_1} u}{\partial x_1^{\beta_1}} \frac{\partial^{\alpha_1 - \beta_1} v}{\partial x_1^{\alpha_1 - \beta_1}} \right)$$

$$= \sum_{\beta_1 = 1}^{\alpha_1} \binom{\alpha_1}{\beta_1} \sum_{\beta_2 = 1}^{\alpha_2} \binom{\alpha_2}{\beta_2} \frac{\partial^{\beta_1 + \beta_2} u}{\partial x_1^{\beta_1} \partial x_2^{\beta_2}} \frac{\partial^{\alpha_1 + \alpha_2 - \beta_1 - \beta_2} v}{\partial x_1^{\alpha_1 - \beta_1} \partial x_2^{\alpha_2 - \beta_2}}$$

Continuing in this way:

$$D^\alpha (uv) = \sum_{\beta_1 = 1}^{\alpha_1} \sum_{\beta_2 = 1}^{\alpha_2} \cdots \sum_{\beta_n = 1}^{\beta_n} \binom{\alpha_1}{\beta_1} \binom{\alpha_2}{\beta_2} \cdots \binom{\alpha_n}{\beta_n}$$

$$\frac{\partial^{\beta_1 + \beta_2 \cdots + \beta_n} u}{\partial x_1^{\beta_1} \cdots \partial x_n^{\beta_n}} \frac{\partial^{\alpha_1 + \alpha_2 + \cdots + \alpha_n - \beta_1 - \cdots - \beta_n} v}{\partial x_1^{\alpha_1 - \beta_1} \cdots \partial x_n^{\alpha_n - \beta_n}}$$

$$= \sum_{\beta \le \alpha} \underbrace{\binom{\alpha_n}{\beta_n} \cdots \binom{\alpha_1}{\beta_1}} D^{\beta} u \, D^{\alpha-\beta} v$$

$$= \frac{\alpha_n! \cdots \alpha_1!}{\beta_n! \cdots \beta_1! \, (\alpha_n-\beta_n)! \cdots (\alpha_1-\beta_1)!} = \binom{\alpha}{\beta}$$

(4) (a) $u_{x_1 x_2} + u_{x_2 x_3} = 0$ linear order 2

(b) $\underset{\text{non linear 1st order term}}{u|u_{x_1}|^2} + \underset{\text{linear 2nd order}}{u_{x_1 x_2}} = 0$ semilinear order 2

(c) $\underbrace{u|u_{x_1}|^2}_{\text{non linear 1st order}} \times \underbrace{u_{x_1 x_2}}_{\text{2nd order}}$ quasilinear order 2

(d) $u u_{x_1} + \underset{\text{non linear!}}{|u_{x_1 x_2}|^2}$ fully nonlinear order 2

(e) $-\sum_{i=1}^{n} (b^i u)_{x_i} = 0$ linear order 1

(f) $-\underset{\text{lin}}{\Delta u} = \underset{\text{non linear}}{f(u)}$ semilinear order 2

(g) $i u_t + \Delta u = \underbrace{f(|u|^2)u}_{\text{non lin order 0}}$ semilinear order 2

(h) $\det(D^2 u) = f$ Fully nonlinear

(i) $\operatorname{div}(|Du|^{p-2}Du) = 0$ $\qquad |Du|^{p-2}Du = |Du|^{p-2}\begin{pmatrix} u_{x_1} \\ \vdots \\ u_{x_n} \end{pmatrix}$

$$\overset{\shortparallel}{\sum_{i=1}^{n} \left(|Du|^{p-2} u_{x_i}\right)_{x_i}}$$

$$\left(|Du|^{p-2} u_{x_i}\right)_{x_i} = \underbrace{|Du|^{p-2} u_{x_i x_i}}_{\text{quasilinear term}} + u_{x_i} \left(|Du|^{p-2}\right)_{x_i}$$

$$\left(|Du|^{p-2}\right)_{x_i} = \cancel{(p-2)\,|Du|^{p-3}} \cdot$$

$$\frac{\partial}{\partial x_i} \left(u_{x_1}^2 + \dots u_{x_n}^2\right)^{\frac{p-2}{2}} \qquad v = u_{x_1}^2 + \dots + u_{x_n}^2$$

$$v_{x_i} = 2 u_{x_1 x_i}\, u_{x_1}^2$$

$$= \cancel{2 x_i \frac{p-2}{2} \left(x_1^2 + \dots + x_n^2\right)^{\frac{p-4}{2}}}$$

$$= \sum_{j=1}^{n} u_{x_j x_i}\, u_{x_j}^2\, (p-2) \left(u_{x_1}^2 + \dots + u_{x_n}^2\right)^{\frac{p-4}{2}}$$ quasilinear oder?

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Problem Sheet I

1a) Show that ~~solutions~~ functions of the form $u(t,x) = f(t) + g(x)$

are

What is the order

1b) Is $u(t,x)$ necessarily C^2?

—

$u \in C^1(\Omega)$ $\varphi \in C_c^{\infty}(\Omega^d)$

$u\varphi \in C^1(\overline{B})$

—

$B \subset \mathbb{R}^2$

$u(x) = |x|^{-s}$ $0 < s < 1$.

$(x_1^2 + x_2^2)^{-s/2}$ $B \setminus \{0\}$

—

Extra tutorial: $\boxed{\text{Th } 8\text{–}10}$

$\sum$ $\alpha_N = \mathcal{L}^1(B) = \omega_{N-1} \int_0^1 r^{N-1}\, dr$

$= \frac{\omega_N}{N}$

③ Suppose $u(x,y) = f(x) + g(y)$, $f, g \in C^1(\mathbb{R})$.

Then $u_x = g(y)$, $u_{xy} = 0$.

$u_x(x,y) = g(y)$ $\forall (x,y) \in \mathbb{R}^2$

$u_{xy}(x,y) = 0$ $\forall (x,y) \in \mathbb{R}^2$.

This PDE has order 2, but solution u may not be C^2!

If f or g are not C^2, then ~~u_x not C~~

e.g. $f(x) = \begin{cases} x^2 & x \le 0 \\ x^3 & x > 0 \end{cases}$ in $C^1(\mathbb{R})$ but not $C^2(\mathbb{R})$.

Moral / <u>This shows:</u> Solutions to k-th order PDEs aren't necessarily in C^k!

④ $(*)\begin{cases} u_t + b\cdot\nabla u + c\,u = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ u = g & \text{on } \mathbb{R} \times \{0\} \end{cases}$

$g: \mathbb{R}^n \to \mathbb{R}$ smooth $b \in \mathbb{R}^n$, $c \in \mathbb{R}$ constants

Idea behind Transport equation:

Find a function $z: \mathbb{R} \to \mathbb{R}$ defined in terms of a solution u of $(*)$, s.t. $\frac{dz}{ds} = 0$.
for given (x,t)

In TE, we used $z(s) := u(x+sb, t+s)$

$$\frac{dz}{ds} = u_t(x+sb, t+s) + b\cdot\nabla u(x+sb, t+s)$$

No u term here!
1st order

try $z(s) = e^{cs}\, u(x+sb, t+s)$

then $\frac{dz}{ds} = e^{cs}\big(u_t(x+sb,t+s) + \nabla u(x+sb,t+s)\cdot b + c\,u(x+sb,t+s)\big)$

$= 0$ So z is constant

~~Hence z is constant along curves of the form~~ Find $z(0)$ in terms of g..

Now note $z(-t) = e^{-ct} u(x-tb, 0) = e^{-ct} g(x-tb)$

by initial condition

Thus $\cancel{u(x,t) = z(0) = e^{-c\cdot 0} g^{c}}$

$$u(x,t) \overset{\text{def of } z}{=} \frac{z(0)}{e^{c\cdot 0}} = z(0) \underset{\text{constant}}{=} e^{-ct} g(x-tb)$$