PDG I
(Zentralübung)

Problem Sheet 9

Question 1

(a) Prove Theorem 47 from the lectures: suppose $g \in C^2(\mathbb{R})$, $h \in C^1(\mathbb{R})$. Define

$$u(t,x) := \tfrac{1}{2}\big[g(x+t) + g(x-t)\big] + \tfrac{1}{2}\int_{x-t}^{x+t} h(y)\,\mathrm{d}y\,, \quad (t,x) \in (0,\infty)\times\mathbb{R}\,.$$

Then

(i) $u \in C^2([0,\infty)\times\mathbb{R})$.

(ii) $u_{tt} - u_{xx} = 0$ in $(0,\infty)\times\mathbb{R}$.

(iii)
$$\lim_{\substack{(t,x)\to(0,x_0)\\ t>0}} u(t,x) = g(x_0)\,, \qquad \lim_{\substack{(t,x)\to(0,x_0)\\ t>0}} u_t(t,x) = h(x_0) \quad \text{for all } x_0 \in \mathbb{R}\,.$$

(b) Now let $f \in C^1((0,\infty)\times\mathbb{R})$. Prove that the function $v(t,x)$, defined (as in lectures) as

$$v(t,x) := \tfrac{1}{2}\big[g(x+t) + g(x-t)\big] + \tfrac{1}{2}\int_{x-t}^{x+t} h(y)\,\mathrm{d}y + \tfrac{1}{2}\int_0^t \int_{x-t+s}^{x+t-s} f(s,y)\,\mathrm{d}y\,\mathrm{d}s\,,$$

satisfies $v_{tt} - v_{xx} = f$ in $(0,\infty)\times\mathbb{R}$ (i.e. v solves the inhomogenous wave equation in one dimension).

Question 2

Let $f \in C^2((0,\infty)\times\mathbb{R})$, $g \in C^2(\mathbb{R})$, $h \in C^1(\mathbb{R})$, and consider the initial-value problem for the one-dimensional wave equation:

$$\begin{cases} u_{tt} - u_{xx} = f & \text{in } (0,\infty)\times\mathbb{R} \\ u = g \quad u_t = h & \text{on } \{t=0\}\times\mathbb{R}\,. \end{cases}$$

(a) Show that the solution for $f = g = 0$ is given by

$$u(t,x) = \int_{\mathbb{R}} K(t,x-y)h(y)\,\mathrm{d}y\,,$$

where $K(t,x) := \frac{1}{2}H(|t|-|x|)\mathrm{sgn}(t)$, and H is the characteristic function of the interval $[0,\infty)$ (the "Heaviside function").

Hint: Use d'Alembert's formula.

(b) Use Duhamel's principle to determine the solution for $g = h = 0$, f non-zero.

Question 3

Prove that the general solution $u \in C^2(\mathbb{R}^2)$ of

$$u_{xy}(x, y) = 0\,, \quad (x, y) \in \mathbb{R}^2$$

is given by $u(x, y) = \xi(x) + \zeta(y)$, where $\xi, \zeta \in C^2(\mathbb{R})$.

Question 4

Let u be the solution of the oscillating string problem on $\mathbb{R}_+$, as covered in lectures for $g \in C^2(\overline{\mathbb{R}_+})$, $h \in C^1(\overline{\mathbb{R}_+})$. Prove that if $g''(0) = g(0) = h(0) = 0$, then $u \in C^2([0, \infty) \times \overline{\mathbb{R}_+})$.

(Recall $\mathbb{R}_+ := (0, \infty)$, $\overline{\mathbb{R}_+} = [0, \infty)$.)

Deadline for handing in: 0800 Wednesday 17 December

Please put solutions in Box 17, 1st floor (near the library)

Homepage: `http://www.mathematik.uni-muenchen.de/~soneji/pde1.php`