PDG I
(Zentralübung)

Problem Sheet 8

Question 1

(a) For $\alpha > 0$, let $g(x) := e^{\alpha|x|^2}$, $x \in \mathbb{R}^N$, and

$$G(t,x) := \frac{1}{(1-4\alpha t)^{N/2}} \, e^{\frac{\alpha}{1-4\alpha t}|x|^2} , \quad x \in \mathbb{R}^N , \quad t \in \left(0, \frac{1}{4\alpha}\right) .$$

Prove that G solves

$$\begin{cases} G_t - \Delta G = 0 & \text{in } \left(0, \frac{1}{4\alpha}\right) \times \mathbb{R}^N \\ G(0,x) = g(x) & \text{on } \{t = 0\} \times \mathbb{R}^N . \end{cases}$$

(b) Let Φ be the fundamental solution of the heat equation. Compute $\int_{\mathbb{R}^N} \Phi(t, x-y) g(y) \, \mathrm{d}y$.

Question 2

Give an alternate (direct) proof that if $\Omega \subset \mathbb{R}^N$ is open and bounded, $T > 0$, and $u \in C^{1,2}(\Omega_T) \cap C(\overline{\Omega_T})$ solves the heat equation (for $g \in C(\partial'\Omega_T)$)

$$\begin{cases} u_t - \Delta u = 0 & \text{in } \Omega_T \\ u = g & \text{on } \partial'\Omega_T , \end{cases}$$

then

$$\max_{(t,x)\in\overline{\Omega_T}} u(t,x) = \max_{(t,x)\in\partial'\Omega_T} u(t,x) .$$

Hint: Define $u_\epsilon := u - \epsilon t$ for $\epsilon > 0$, and show that u_ϵ cannot attain its maximum over $\overline{\Omega_T}$ at a point in the interior Ω_T.

Question 3

Let $N = 1$ and Φ be the fundamental solution of the heat equation. Use properties of the convolution

$$u(t,x) = \int_{\mathbb{R}} \Phi(t, x-y) f(y) \, \mathrm{d}y$$

to prove:

Weierstrauss'approximation theorem: A function $f \in C([a,b])$ can be approximated uniformly by polynomials. That is, there exists a sequence of polynomials p_j such that

$$\max_{x\in[a,b]} |f(x) - p_j(x)| \to 0 \quad \text{as } j \to \infty \, .$$

Hint: Define $f(x) = f(b)$ for $x > b$ and $f(x) = f(a)$ for $x < a$. Then $u(t,x) \to f(x)$ as $t \to 0$ uniformly for $a \leq x \leq b$. Approximate $\Phi(t, x-y)$ by its truncated power series with respect to $x - y$.

Deadline for handing in: 0800 Wednesday 10 December

Please put solutions in Box 17, 1st floor (near the library)

Homepage: `http://www.mathematik.uni-muenchen.de/~soneji/pde1.php`