MATHEMATISCHES INSTITUT
DER UNIVERSITÄT MÜNCHEN
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WS 2014/15
– Blatt 7 –
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# PDG I
# (Zentralübung)

## Problem Sheet 7

### Question 1

Let $v \in C^2(\mathbb{R})$ (so $N = 1$) and, for $t > 0$, $x \in \mathbb{R}$, define

$$u(t,x) := v\Big(\frac{x}{\sqrt{t}}\Big) .$$

(a) Show that

$$u_t = u_{xx}$$

if and only if

$$v''(z) + \frac{z}{2}v'(z) = 0 , \quad z \in \mathbb{R} . \tag{1}$$

Show that the general solution of (1) is

$$v(z) = c\int_0^z e^{-s^2/4}\,\mathrm{d}s + d .$$

(b) Differentiate $u(t,x) = v\big(\frac{x}{\sqrt{t}}\big)$ with respect to $x$ and select the constant $c$ properly, to obtain the fundamental solution $\Phi$ for the Heat Equation for $N = 1$. Explain why this procedure produces the fundamental solution. (*Hint:* What is the initial condition for $u$?)

**Question 2**

(a) Solve the intial value problem for the Heat Equation with convection:

$$\begin{cases} u_t - \Delta u + b \cdot \nabla u = 0 & \text{in } (0,\infty) \times \mathbb{R}^N \\ u(0,x) = g(x) & \text{on } \{t=0\} \times \mathbb{R}^N\,, \end{cases}$$

where $g\colon \mathbb{R}^N \to \mathbb{R}$ is smooth, and $b \in \mathbb{R}^N$ is a constant. Write your solution in terms of an integral involving the heat kernel $\Phi(\cdot,\cdot)$.

(b) Similarly, write down an explicit solution for the initial value problem

$$\begin{cases} u_t - \Delta u + cu = 0 & \text{in } (0,\infty) \times \mathbb{R}^N \\ u(0,x) = g(x) & \text{on } \{t=0\} \times \mathbb{R}^N\,, \end{cases}$$

where $g\colon \mathbb{R}^N \to \mathbb{R}$ is smooth, and $c \in \mathbb{R}$ is a constant.

(*Hint for (a) and (b):* Recall methods used previously for the Transport Equation)

(c) Now solve

$$\begin{cases} u_t - \Delta u + b \cdot \nabla u + cu = 0 & \text{in } (0,\infty) \times \mathbb{R}^N \\ u(0,x) = g(x) & \text{on } \{t=0\} \times \mathbb{R}^N\,, \end{cases}$$

where $g\colon \mathbb{R}^N \to \mathbb{R}$ is smooth, and $b \in \mathbb{R}^N$, $c \in \mathbb{R}$ are constants.

(Note that in the case $n = 1$, this equation is closely linked to the *Black-Scholes Model* for pricing stock options.)

**Deadline for handing in: 0800 Wednesday 3 December**

*Please put solutions in Box 17, 1st floor (near the library)*