MATHEMATISCHES INSTITUT
DER UNIVERSITÄT MÜNCHEN
Prof. Dr. S. Bachmann / Dr. P. Soneji

WS 2014/15
– Blatt 6 –
19.11.2014

PDG I
(Zentralübung)

Problem Sheet 6

Question 1

Prove Theorem 34 from the lectures: consider the half-space

$$\mathbb{R}^n_+ := \{(x_1, \ldots, x_n \in \mathbb{R}^n : x_n > 0\}\,.$$

Suppose $f \in C^2_c(\mathbb{R}^n_+)$ and $g \in C^0(\mathbb{R}^{n-1}) \cap L^\infty(\mathbb{R}^{n-1})$. As in the lectures, define the Green function for $\mathbb{R}^n_+$ by

$$G(x, y) := \Phi(x - y) - \Phi(\tilde{x} - y)$$

where $\Phi : \mathbb{R}^n \to \mathbb{R}$ is the fundamental solution to the Laplace equation and

$$\tilde{x} := (x_1, \ldots, x_{n-1}, -x_n)\,, \quad x \in \mathbb{R}^n_+\,.$$

Now define

$$v(x) := \begin{cases} \int_{\mathbb{R}^n_+} f(y)\, G(x, y)\, \mathrm{d}y - \int_{\partial\mathbb{R}^n_+} g(y)\ \frac{\partial G}{\partial \nu}(x, y)\, \mathrm{d}S(y) & x \in \mathbb{R}^n_+ \\ g(x) & x \in \partial\mathbb{R}^n_+\,. \end{cases}$$

Then

(i) $v \in C^2(\mathbb{R}^n_+)$.

(ii) v satisfies

$$\begin{cases} -\Delta v(x) = f(x) & x \in \mathbb{R}^n_+ \\ v(x) = g(x) & x \in \partial\mathbb{R}^n_+\,. \end{cases}$$

(iii) We have

$$\lim_{\substack{x \to x_0 \\ x \in \mathbb{R}^n_+}} v(x) = g(x_0)$$

for all $x_0 \in \partial\mathbb{R}^n_+$.

Question 2

Let $\Omega \subset \mathbb{R}^n$ be open and bounded, with C^1 boundary. For a function $w \in C^1(\overline{\Omega})$ the n-dimensional surface area over its graph

$$\{(x, w(x)) : x \in \overline{\Omega}\}$$

is given by the functional

$$A(w) := \int_\Omega \sqrt{1 + |Dw(x)|^2}\, \mathrm{d}x\,.$$

Let $g \in C(\partial\Omega)$ and suppose that $u \in C^2(\overline{\Omega})$ is a minimiser of A within the set

$$\{w \in C^1(\overline{\Omega}) : w = g \text{ on } \partial\Omega\}\,.$$

Prove that this minimiser u solves the boundary-value problem

$$\begin{cases} \operatorname{div}\left(\frac{Du(x)}{\sqrt{1+|Du(x)|^2}}\right) = 0 & x \in \Omega \\ u(x) = g(x) & x \in \partial\Omega\,, \end{cases}$$

which is called the *minimal surface equation*.

Deadline for handing in: 0800 Wednesday 26 November

Please put solutions in Box 17, 1st floor (near the library)

Homepage: `http://www.mathematik.uni-muenchen.de/~soneji/pde1.php`