PDG I
(Zentralübung)

Problem Sheet 5

Question 1

Prove the classical Harnack inequality: suppose $u\colon B(0,r) \to \mathbb{R}$ is non-negative and harmonic. Then show

$$\frac{r-|x|}{(r+|x|)^{n-1}}\, r^{n-2} u(0) \le u(x) \le \frac{r+|x|}{(r-|x|)^{n-1}}\, r^{n-2} u(0)$$

for all $x \in B(0,r)$.

Hint: Estimate the denominator in the Poisson kernel from below and above, and use the mean-value property.

Question 2

Let Φ be the fundamental solution of Laplace's equation.

(a) Show that for all $r > 0$

$$\int_{\partial B(0,r)} \frac{\partial \Phi}{\partial \nu}(y-x)\,\mathrm{d}S(y) = -1\,, \quad x \in B(0,r)\,.$$

(Recall that

$$\frac{\partial \Phi}{\partial \nu}(z) := \nabla \Phi(z) \cdot \nu(z)\,,$$

where $\nu(z) = (\nu^1(z), \ldots \nu^n(z)) \in \mathbb{R}^n$ is the normal to the boundary of the surface at z.)

(b) Let $\Omega \subset \mathbb{R}^n$ be open and bounded, with C^1 boundary. Prove that

$$\int_{\partial \Omega} \frac{\partial \Phi}{\partial \nu}(y-x)\,\mathrm{d}S(y) = -1\,, \quad x \in \Omega\,.$$

(c) Let $\mathbb{R}^n_+ := \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_n > 0\}$, and define

$$K(x,y) := \frac{2x_n}{n\alpha_n} \frac{1}{|x-y|^n}\,, \quad x \in \mathbb{R}^n_+\,,\ y \in \partial\mathbb{R}^n_+\,.$$

Prove that

$$\int_{\partial \mathbb{R}^n_+} K(x,y)\,\mathrm{d}y = 1\,, \quad x \in \mathbb{R}^n_+\,.$$

Question 3

(a) Let $\Omega \subset \mathbb{R}^n$ be open and bounded, and let $u \in C^2(\Omega) \cap C(\overline{\Omega})$ be a solution to

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega\,, \end{cases} \tag{1}$$

where $f \in C(\overline{\Omega})$ and $g \in C(\partial\Omega)$. Prove that there exists a constant C, depending only on Ω, such that

$$\max_{x\in\overline{\Omega}} |u(x)| \le C\left(\max_{x\in\partial\Omega} |g(x)| + \max_{x\in\overline{\Omega}} |f(x)| \right).$$

Hint: $-\Delta\big(u(x) + \frac{|x|^2}{2n}\lambda\big) \le 0$ for $\lambda := \max_{x\in\overline{\Omega}} |f(x)|$.

(b) Prove that the solution of (1) depends continuously on the data f, g. That is, show that there exists a constant C, depending only on Ω, such that if $u_i \in C^2(\Omega) \cap C(\overline{\Omega})$ for $i = 1, 2$ solve

$$\begin{cases} -\Delta u_i = f_i & \text{in } \Omega \\ u_i = g_i & \text{on } \partial\Omega\,, \end{cases}$$

where $f_i \in C(\overline{\Omega})$ and $g_i \in C(\partial\Omega)$, then

$$\|u_1 - u_2\|_{L^\infty(\Omega)} \le C\big(\|g_1 - g_2\|_{L^\infty(\partial\Omega)} + \|f_1 - f_2\|_{L^\infty(\Omega)}\big)\,.$$

Deadline for handing in: 0800 Wednesday 19 November

Please put solutions in Box 17, 1st floor (near the library)

Homepage: `http://www.mathematik.uni-muenchen.de/~soneji/pde1.php`