MATHEMATISCHES INSTITUT
DER UNIVERSITÄT MÜNCHEN
Prof. Dr. S. Bachmann / Dr. P. Soneji

WS 2014/15
– Blatt 4 –
5.11.2014

PDG I
(Zentralübung)

Problem Sheet 4

Question 1

Let $\Omega \subset \mathbb{R}^n$ be open and bounded. A function $v \in C^2(\overline{\Omega})$ is said to be *sub-harmonic* if

$$-\Delta v(x) \leq 0 \quad \text{for all } x \in \Omega\,.$$

(i) Prove that if v is sub-harmonic, then

$$v(x) \leq \fint_{B(x,r)} v(y)\,\mathrm{d}y \quad \text{for all } B(x,r) \subset\subset \Omega\,.$$

(ii) Prove therefore that for $v \in C^2(\overline{\Omega})$ sub-harmonic we have

$$\max_{x\in\overline{\Omega}} v(x) = \max_{x\in\partial\Omega} v(x)\,.$$

(iii) Suppose v is sub-harmonic and, for a ball $B = B(x,r) \subset\subset \Omega$, define $H_B v$ as in the lectures by integration against the Poisson kernel. Show that

$$v(y) \leq H_B v(y) \quad \text{for all } y \in B\,.$$

(iv) Let $\varphi\colon \mathbb{R} \to \mathbb{R}$ be smooth and convex. Suppose $u \in C^2(\overline{\Omega})$ is harmonic in Ω and define $v(x) := \varphi(u(x))$. Prove that v is sub-harmonic.

Hint: Recall that since φ is convex and C^2, $\varphi''(t) \geq 0$ for all $t \in \mathbb{R}$.

(v) Suppose $u \in C^2(\overline{\Omega})$ is harmonic. Prove that $v(x) := |\nabla u(x)|^2$ is sub-harmonic.

Question 2

(i) For a point $x \in \mathbb{R}^n \setminus \{0\}$, define
$$\bar{x} := \frac{x}{|x|^2}\,.$$
Show that
$$\nabla_x \bar{x} (\nabla_x \bar{x})^t = |\bar{x}|^4 I\,,$$
where
$$\nabla_x \bar{x} = \Big((x/|x|^2)_{x_1}, (x/|x|^2)_{x_2}, \ldots, (x/|x|^2)_{x_n} \Big)^t\,.$$

(ii) Let $\Omega \subset \mathbb{R}^n \setminus \{0\}$ be open ($n \geq 2$). The *Kelvin transform* $\mathcal{K}u = \bar{u}$ of u is defined as
$$\begin{aligned} \bar{u}(x) &:= u(\bar{x})|\bar{x}|^{n-2} \\ &= u(x/|x|^2)|x|^{2-n}\,. \end{aligned}$$
Show that if u is harmonic on Ω, then so is $\bar{u}$.

Deadline for handing in: 0800 Wednesday 12 November

Please put solutions in Box 17, 1st floor (near the library)

Homepage: `http://www.mathematik.uni-muenchen.de/~soneji/pde1.php`