Mathematisches Institut
der Universität München
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WS 2014/15
– Blatt 13 –
21.1.2015

PDG I
(Zentralübung)

Problem Sheet 13

Question 1

Let $1 \leq p \leq \infty$. Let I be an open interval in $\mathbb{R}$. We shall prove that there exists a bounded linear operator $P\colon W^{1,p}(I) \to W^{1,p}(\mathbb{R})$, called an *extension operator*, satisfying:

(i) $Pu|_I = u$ for all $u \in W^{1,p}(I)$,

(ii) $\|Pu\|_{L^p(\mathbb{R})} \leq C\|u\|_{L^p(I)}$,

(iii) $\|Pu\|_{W^{1,p}(\mathbb{R})} \leq C\|u\|_{W^{1,p}(I)}$,

where $C > 0$ depends only on I.

(a) First suppose $I = (0, \infty)$. Show that the operator P given by

$$(Pu)(x) = \begin{cases} u(x) & \text{if } x \geq 0\,, \\ u(-x) & \text{if } x < 0\,, \end{cases}$$

satisfies the required properties.

(b) Now suppose $I = (0, 1)$. Fix a function $\eta \in C^1(\mathbb{R})$, $0 \leq \eta \leq 1$, such that

$$\eta(x) = \begin{cases} 1 & \text{if } x < 1/4\,, \\ 0 & \text{if } x > 3/4\,. \end{cases}$$

Now let $u \in W^{1,p}((0,1))$, and define

$$(Qu)(x) = \begin{cases} u(x) & \text{if } 0 < x < 1\,, \\ 0 & \text{if } x > 1\,. \end{cases}$$

Show that $\eta(Qu) \in W^{1,p}((0,\infty))$, and $(\eta(Qu))' = \eta'(Qu) + \eta(Q(u'))$.

(c) Use parts (a) and (b) to define P satisfying the required properties for $u \in W^{1,p}((0,1))$.

Hint: Write $u = \eta u + (1-\eta)u$. Extend and reflect each of these two terms in different directions.

Question 2

Suppose $u \in W^{1,p}((0,\infty))$, where $1 \leq p < \infty$. Show that

$$\lim_{x\to\infty} u(x) = 0\,.$$

Hint: Use Theorems 61 and 62 from the lectures.

Question 3

Let I be an open interval in $\mathbb{R}$, and $u,\, v \in W^{1,p}(I)$, for $1 \leq p < \infty$. Show that the product $uv \in W^{1,p}(I)$, and

$$(uv)' = u'v + uv'\,.$$

(*Remark:* Note that this type of result does **not** hold in general in higher dimensions!)

Question 4

Consider the following ordinary differential equation with boundary data:

$$\begin{cases} -(pu')' + ru' + qu = f & \text{in } (0,1) \\ u(0) = u(1) = 0\,. \end{cases}$$

Here, $f, r, q \in C([0,1])$, and $p \in C^1([0,1])$. Assume furthermore that for some fixed $\alpha > 0$ we have

$$p \geq \alpha\,, \quad q \geq 1\,, \quad r^2 < 4\alpha\,, \quad \text{on } [0,1]\,.$$

Prove that there exists a unique classical solution u to this problem.

Deadline for handing in: 0800 Wednesday 28 January

Please put solutions in Box 17, 1st floor (near the library)